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Algebraic prolongation and rigidity of Carnot groups

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Abstract We discuss the known results on rigidity of Carnot groups using Tanaka's prolongation theory. We also apply Tanaka's theory to study rigidity of an extended class of H-type groups which we call J-type groups. In particular we obtain a rigidity criterion giving rise to a rigid class of J-type groups which includes the H-type groups, and thus extends the results of H.M. Reimann. We also construct a noncomplex J-type group which is nonrigid and does not satisfy the rank 1 condition over the reals.

Keywords Carnot group · H-type group · Fundamental graded algebra · Prolongation · Contact map · Differential system

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Contents

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1 Introduction

A nilpotent Lie algebra $\mathfrak{n}$ is said to admit an s-step stratification if it decomposes as the direct sum of subspaces $\mathfrak{n} = \mathfrak{g}_{-s} \oplus \cdots \oplus \mathfrak{g}_{-1}$, such that $\mathfrak{g}_{j-1} = [\mathfrak{g}_{-1}, \mathfrak{g}_j]$, where $j = -1, \ldots, -s+1$, and $\mathfrak{g}_{-s}$ is contained in the centre $\mathfrak{z}(\mathfrak{n})$. In the prolongation theory of Tanaka, such a Lie algebra is said to be a fundamental graded algebra of depth s.

Let N denote the connected, simply connected nilpotent Lie group with stratified Lie algebra $\mathfrak{n}$. The left-invariant sub-bundle $\mathcal{H} \subseteq TN$ corresponding to $\mathfrak{g}_{-1}$ is called the horizontal bundle. In Tanaka's theory, the Lie group N together with the horizontal bundle form the standard differential system of type $\mathfrak{n}$. Furthermore, when $\mathfrak{n}$ is equipped with an inner product $\langle\,,\,\rangle$ such that $\mathfrak{g}_i \perp \mathfrak{g}_j$ when $i \neq j$, then a subriemannian metric on N is defined and N is called a Carnot group.

Local diffeomorphisms of N which preserve the horizontal bundle are called contact maps, and N is said to be rigid when the space of contact maps is finite dimensional. The simplest examples of contact maps are left translations and dilations. A dilation by t of an element $X \in \mathfrak{n}$ is defined by $\delta_t(X) = \sum_{k=1}^{s} t^k X_{-k}$ and the corresponding map of N induced through the exponential defines dilation on N.

The notion of rigidity makes sense with low regularity assumptions. For example in the theory of quasiconformal mapping on Carnot groups the notion of contact map arises in the class of homeomorphisms with coordinate functions in the horizontal Sobolev space $HW^{1,1}_{loc}(N, \mathbb{R})$. However, there is reason to conjecture that such contact homeomorphisms might in fact be smooth, and in the case the group is rigid, which is more often the case, this class of contact maps is in some sense trivial since it forms a finite dimensional Lie group. Furthermore there appears to be no example where the group is rigid in the class of C^∞ maps but nonrigid in some weaker regularity class. This situation provides significant motivation for studying rigidity.

For the class of C^∞ maps, it is sufficient to consider the rigidity problem for contact diffeomorphisms which are induced by vector fields. In particular we have:

Theorem [7, Theorem 3.1, page 13] *Let $\mathfrak{G}$ be a group of C^∞ transformations of a manifold M. Let S be the set of all vector fields V on M which generate global 1-parameter groups $\varphi_t = \exp(tV)$ of transformations of M such that $\varphi_t \in \mathfrak{G}$. If the set S generates a finite dimensional Lie algebra $\mathfrak{g}$ of vector fields on M, then $\mathfrak{G}$ is a Lie transformation group and S is the Lie algebra of $\mathfrak{G}$.*

A vector field V is said to be a contact vector field if it induces a flow of contact maps and can be characterised by the integrability condition $[V, \mathcal{H}] \subseteq \mathcal{H}$. It follows that in the class of C^∞ mappings the direct approach to the rigidity question is by constructing contact vector fields, however this often requires solving a large system of nonlinear PDE's which is usually a difficult task, even with software such as MAPLE.

In [22,23], Tanaka developed an algebraic prolongation, which shows that if we restrict to the class of contact diffeomorphisms which are induced by vector fields, then N is rigid if and only if the prolongation is finite. Contact maps are considered in a number of papers, e.g. [4,5,9,14,15,17,18,24]. In particular, Tanaka's theory is explicitly applied to the study of rigidity in [26] for the Iwasawa nilpotent Lie algebras, in [10] for the nilpotent Lie algebras up to dimension 6 and in [25] for the free nilpotent Lie algebras.

In [23], Tanaka not only proved that rigidity is characterised by his notion of prolongation, but that rigidity is determined by a reduction of his prolongation which turns out to be the usual prolongation of linear algebras. In this paper we recover, and in some cases extend some rigidity results for some well known classes of Carnot groups using this reduction.

A trivial case of nonrigidity occurs when $\mathfrak{n}$ is degenerate in the following sense. We say that $\mathfrak{n}$ is degenerate if $\mathfrak{g}_{-1}$ contains nontrivial degenerate elements, that is elements $X \in \mathfrak{g}_{-1}$ such that $[X, \mathfrak{g}_{-1}] = \{0\}$. When $\mathfrak{n}$ is degenerate, the corresponding Carnot group is always the direct product with $\mathbb{R}^n$, where n is the dimension of the degenerate space. It follows that the set of local contact mappings contains the local diffeomorphisms of $\mathbb{R}^n$ implying that the corresponding Carnot group is nonrigid. To avoid this triviality we will, unless otherwise stated, assume that $\mathfrak{n}$ is nondegenerate.

2 Prolongation of a linear algebra

In this section we introduce the prolongation of a linear algebra which, following Tanaka, we will refer to as the usual prolongation. The definition is as follows: let $\mathfrak{gl}(V)$ denote the Lie algebra of linear endomorphisms of a vector space V over $\mathbb{R}$, and let $\mathfrak{g}^{(0)}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. For each nonnegative integer k the k-th prolongation $\mathfrak{g}^{(k)}$ of $\mathfrak{g}^{(0)}$ is by definition the vector space consisting of all symmetric $(k+1)$-linear maps $T : V^{k+1} \to V$ such that for each k-tuple $(X_1, \ldots, X_k) \in V^k$ the endomorphism of V given by $X_{k+1} \to T(X_1, \ldots, X_k, X_{k+1})$ is an element of $\mathfrak{g}^{(0)}$. The Lie algebra $\mathfrak{g}^{(0)}$ is said to be of finite type if $\mathfrak{g}^{(k)} = \{0\}$ for some k and infinite type otherwise.

Example 1 Take $\mathfrak{g}^{(0)}$ to be a skew symmetric subalgebra of $\mathfrak{gl}(V)$. By definition, each $T \in \mathfrak{g}^{(1)}$ is of the form $T(X, Y) = D(X)Y$ where $D(X) \in \mathfrak{g}^{(0)}$. Let $\{X_1, \ldots, X_n\}$ be a basis for V and set $D^k = D(X_k)$, then

$$T(X, Y) = \sum_{i,j,k=1}^{n} x_k D^k_{ij} y_j X_i.$$

The symmetry of T implies $D^k_{ij} = D^j_{ik}$ and the skew symmetry of each D^k implies $D^k_{ij} = -D^k_{ji}$. It follows that

$$D_{ij}^k = -D_{ji}^k = -D_{jk}^i = D_{kj}^i = D_{ki}^j = -D_{ik}^j = -D_{ij}^k$$

and we conclude that $D_{ij}^k = 0$. Hence $\mathfrak{g}^{(1)} = \{0\}$ and $\mathfrak{g}^{(0)}$ is of finite type.

Example 2 Suppose $\mathfrak{g}^{(0)}$ contains a linear endomorphism A of rank 1, then $A = W \otimes \omega$ where $W \in V$ and $\omega \in V^*$. For any positive integer k we can define a symmetric $(k+1)$-linear map $T : V^{k+1} \to V$ by setting

$$T(X_1, \ldots, X_k, X_{k+1}) = \omega(X_1) \ldots \omega(X_k)\omega(X_{k+1})W$$

and the map

$$X_{k+1} \to T(X_1, \ldots, X_k, X_{k+1})$$

is precisely $\omega(X_1) \ldots \omega(X_k)A$ and thus a nontrivial element of $\mathfrak{g}^{(0)}$. Hence $\mathfrak{g}^{(k)} \neq \{0\}$ and $\mathfrak{g}^{(0)}$ is of infinite type.

A Lie subalgebra $\mathfrak{g}^{(0)} \subset \mathfrak{gl}(V)$ is said to be elliptic if it does not contain any linear endomorphism of rank 1 hence finite type subalgebras are elliptic. For more examples of prolonging linear algebras see [7, Chapter 1].

3 Tanaka prolongation

Let $\mathfrak{n} = \mathfrak{g}_{-s} \oplus \cdots \oplus \mathfrak{g}_{-1}$ be a stratified nilpotent Lie algebra. The Tanaka prolongation of $\mathfrak{n}$ is the graded Lie algebra $\mathfrak{g}(\mathfrak{n})$ given by the direct sum $\mathfrak{g}(\mathfrak{n}) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k(\mathfrak{n})$, where $\mathfrak{g}_k(\mathfrak{n}) = \{0\}$ for $k < -s$, $\mathfrak{g}_k(\mathfrak{n}) = \mathfrak{g}_k$ for $-s \leq k \leq -1$, and for each $k \geq 0$, $\mathfrak{g}_k(\mathfrak{n})$ is inductively defined by

$$\mathfrak{g}_k(\mathfrak{n}) = \left\{ u \in \bigoplus_{p<0} \mathfrak{g}_{p+k}(\mathfrak{n}) \otimes \mathfrak{g}_p(\mathfrak{n})^* \mid u([X, Y]) = [u(X), Y] + [X, u(Y)] \right\},$$

with $\mathfrak{g}_0(\mathfrak{n})$ consisting of the strata preserving derivations of $\mathfrak{n}$. If $u \in \mathfrak{g}_k(\mathfrak{n})$, where $k \geq 0$, then the condition in the definition becomes the Jacobi identity upon setting $[u, X] = u(X)$ when $X \in \mathfrak{n}$. Furthermore, if $u \in \mathfrak{g}_k(\mathfrak{n})$ and $v \in \mathfrak{g}_\ell(\mathfrak{n})$, where $k, \ell \geq 0$, then $[u, v] \in \mathfrak{g}_{k+\ell}(\mathfrak{n})$ is defined inductively according to the Jacobi identity, that is

$$[u, v](X) = [u, [v, X]] - [v, [u, X]].$$

In [23], Tanaka shows that $\mathfrak{g}(\mathfrak{n})$ determines the structure of the contact vector fields on the group N with Lie algebra $\mathfrak{n}$. In particular there is an isomorphism between the set of contact vector fields and $\mathfrak{g}(\mathfrak{n})$ when the latter is finite dimensional, moreover $\mathfrak{g}(\mathfrak{n})$ is finite dimensional if and only if $\mathfrak{g}_k(\mathfrak{n}) = \{0\}$ for some $k \geq 0$ since $\mathfrak{g}_k(\mathfrak{n}) = \{0\}$ implies $\mathfrak{g}_{k+\ell}(\mathfrak{n}) = \{0\}$ for all $\ell > 0$. Hence the rigidity of a stratified nilpotent Lie group can be determined by studying the Tanaka prolongation of the Lie algebra.

For a given subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}_0(\mathfrak{n})$, the prolongation of the pair $(\mathfrak{n}, \mathfrak{g}_0)$ is defined as the graded subalgebra

$$\mathrm{Prol}(\mathfrak{n}, \mathfrak{g}_0) = \mathfrak{n} \oplus \bigoplus_{k \geq 0} \mathfrak{g}_k \subset \mathfrak{g}(\mathfrak{n}),$$

where for each $k \geq 1$, $\mathfrak{g}_k$ is inductively defined as the subspace of $\mathfrak{g}_k(\mathfrak{n})$ satisfying $[\mathfrak{g}_k, \mathfrak{g}_{-1}] \subseteq \mathfrak{g}_{k-1}$. The pair $(\mathfrak{n}, \mathfrak{g}_0)$ is said to be of finite type if $\mathfrak{g}_k = \{0\}$ for some k, otherwise it is of infinite type and $\mathfrak{g}(\mathfrak{n})$ is infinite dimensional.

The type of $(\mathfrak{n}, \mathfrak{g}_0)$ is determined by the subalgebra

$$\mathfrak{h} = \bigoplus_{k \geq -1} \mathfrak{h}_k \subset \mathrm{Prol}(\mathfrak{n}, \mathfrak{g}_0)$$

where the subspaces $\mathfrak{h}_k \subset \mathfrak{g}_k$ are defined as follows: let

$$\hat{\mathfrak{n}} = \mathfrak{g}_{-s} \oplus \cdots \oplus \mathfrak{g}_{-2}$$

and for $k \geq -1$ define

$$\mathfrak{h}_k = \left\{u \in \mathfrak{g}_k \mid [u, \hat{\mathfrak{n}}] = \{0\}\right\}.$$

It follows that $[\mathfrak{h}_k, \mathfrak{g}_{-1}] \subset \mathfrak{h}_{k-1}$ for $k \geq 0$. The bracket generating property shows that $\mathfrak{h}_0$ identifies with a subalgebra $\mathfrak{h}^{(0)} \subseteq \mathfrak{gl}(\mathfrak{g}_{-1})$, moreover $\mathfrak{h}_k$ identifies with $\mathfrak{h}^{(k)}$ and by [23, Corollary 2, page 76] $(\mathfrak{n}, \mathfrak{g}_0)$ is of infinite type if and only if $\mathfrak{h}^{(0)}$ is of infinite type in the usual sense.

We extend the notation by writing $\mathfrak{h}_k(\mathfrak{n})$ and $\mathfrak{h}^{(k)}(\mathfrak{n})$ when $\mathfrak{g}_0 = \mathfrak{g}_0(\mathfrak{n})$, and summarise Tanaka's result as follows: $\mathfrak{n}$ is rigid if and only if $\mathfrak{h}^{(0)}(\mathfrak{n})$ is of finite type in the usual sense. In other words, $\mathfrak{n}$ is nonrigid if and only if there is a nontrivial subalgebra $\mathfrak{h}^{(0)} \subseteq \mathfrak{h}^{(0)}(\mathfrak{n})$ of infinite type in the usual sense.

Tanaka's theory provides some relatively simple tests for rigidity, at least at the levels $\mathfrak{h}_{-1}(\mathfrak{n})$, $\mathfrak{h}_0(\mathfrak{n})$, and $\mathfrak{h}_1(\mathfrak{n})$, moreover it also provides a definition of rigidity type as the smallest integer k greater or equal to -1 such that $\mathfrak{h}_k(\mathfrak{n}) = \{0\}$.

4 Rigidity in dimensions ≤ 6

According to the classification given by Kuzmich [10] there are 79 fundamental graded algebras of dimension less or equal than 7 and the application of Tanaka's Corollary to classify the rigidity of the 29 fundamental graded algebras of dimension ≤ 6 yields precisely 8 rigid cases. Using the notation of Kuzmich, these rigid cases are:

$$m5\_3\_1,\ m6\_4\_1,\ m6\_4\_2,\ m6\_4\_2r,\ m6\_2\_1,\ m6\_3\_5,\ m6\_3\_5r,\ m6\_5\_2.$$

In particular the first four algebras are type -1 and the remaining algebras are type 0. Indeed, $m5\_3\_1$ is the step three free nilpotent Lie algebra with two generators and is thus type -1 since every three step free nilpotent Lie algebra satisfies $\mathfrak{h}_{-1}(\mathfrak{n}) = \{0\}$. Furthermore, the algebras $m6\_4\_1$, $m6\_4\_2$ and $m6\_4\_2r$ constitute the three possible Lie algebras where the strata dimensions are $\dim \mathfrak{g}_{-1} = 2$, $\dim \mathfrak{g}_{-2} = 1$, $\dim \mathfrak{g}_{-3} = 2$ and $\dim \mathfrak{g}_{-4} = 1$. Clearly $\mathfrak{h}_{-1}(\mathfrak{n}) = \{0\}$ for all three algebras since $\dim \mathfrak{g}_{-3} = 2$.

Next we note that $m6\_2\_1$ is the step two free nilpotent Lie algebra with three generators and is thus type 0 since, with the exception of the Heisenberg algebra $\mathbb{H}^1$, every two step free nilpotent Lie algebra satisfies $\mathfrak{h}_{-1}(\mathfrak{n}) = \mathfrak{g}_{-1}$ and $\mathfrak{h}_0(\mathfrak{n}) = \{0\}$.

The final three cases are Iwasawa nilpotent Lie algebras. In particular $m6\_3\_5$ is the set of real 4×4 nilpotent upper triangular matrices which forms the nilpotent part of the Iwasawa decomposition of $\mathfrak{sl}(4, \mathbb{R})$. Moreover, $m6\_3\_5r$ is the nilpotent part of the Iwasawa decomposition of $\mathfrak{su}(2, 2)$, and $m6\_5\_2$ is the nilpotent part of the Iwasawa decomposition of the exceptional algebra $\mathfrak{g}_2$, corresponding to the root system G_2. In all these cases we have $\mathfrak{h}_{-1}(\mathfrak{n}) \neq \{0\}$ and $\mathfrak{h}_0(\mathfrak{n}) = \{0\}$ hence they are type 0.

The remaining 21 infinite type algebras are the complexified Heisenberg algebra, the degenerate algebras and the algebras satisfying the rank one condition, which is described in Sect. 8.

5 Iwasawa Lie algebras

Every simple Lie algebra admits an Iwasawa decomposition which is unique up to inner automorphism. Within this decomposition there is a stratified nilpotent subalgebra $\mathfrak{n}$. The rigidity of these algebras has been already considered in [4,26], so we restrict ourselves to a short overview using Tanaka's prolongation theory (see [16] for an insight). For the theory and the classification of semisimple Lie algebras, we refer the reader to [1,6].

Simple Lie algebras are classified by means of their root space decompositions and so are the corresponding nilpotent algebras $\mathfrak{n}$. To each simple Lie algebra we can associate a root system, whereas to each root system may correspond several (real) simple Lie algebras. Non isomorphic root systems are labeled by $A_n, n \geq 1$, $B_n, n \geq 2$, $C_n, n \geq 3$, $D_n, n \geq 4$, by E_6, E_7, E_8, F_4, G_2 which are called the exceptional systems and by $BC_r, r \geq 1$, the reducible system. By inspection of the root systems, it is easy to see that $\mathfrak{h}_{-1}(\mathfrak{n}) = 0$ when $\mathfrak{n}$ are the nilpotent Lie algebras that come from the Iwasawa decomposition of the simple Lie algebras with root system $A_n, n \geq 4$, $B_n, C_n, n \geq 3$, $D_n, n \geq 4$, $BC_r, r \geq 2$, and all the exceptional cases. In fact, it suffices to check that every positive simple root can always be added to some positive root of height at least two.

Since $D\mathfrak{g}_{-1} \subset \mathfrak{h}_{-1}(\mathfrak{n})$ for every $D \in \mathfrak{h}_0(\mathfrak{n})$, it is not difficult to see that $\mathfrak{h}_0(\mathfrak{n}) = \{0\}$ in the cases corresponding to A_3. Furthermore, nilpotent algebras associated to A_1, BC_1, A_2 are either commutative algebras for which the rigidity problem is trivial or H-type algebras [2,3], which display rigidity types 1 and ∞, as we shall see in Sect. 6.

Finally, corresponding to B_2 there are three simple Lie algebras: $\mathfrak{so}(2, 2+n)$, $\mathfrak{so}(5, \mathbb{C})$ and $\mathfrak{sp}(2, 2)$. The nilpotent Lie algebra in $\mathfrak{so}(2, 2+n)$ is of type ∞ if $n = 1$ and it corresponds to the Engel algebra, whereas it is of type 0 if $n \geq 2$. The nilpotent

Lie algebra in $\mathfrak{so}(5, \mathbb{C})$ is the complexified Engel algebra, which is of infinite type, as we will show in Lemma 5. For the nilpotent Lie algebra corresponding to $\mathfrak{sp}(2, 2)$, one computes explicitly a basis of vectors and their bracket relations [Ciatti, private communication]. The nilpotent algebra has step three and dimension 14. A long but straightforward calculation shows that $\mathfrak{h}_0 = \{0\}$.

6 J-type and H-type algebras

A complex structure on $\mathbb{R}^{2m}$ is an element $J \in GL(2m, \mathbb{R})$ such that $J^2 = -\mathrm{Id}$. The canonical complex structure is defined by the matrix

$$J_0 = \begin{bmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}$$

The subgroup of $GL(2m, \mathbb{R})$ which commutes with J_0 is isomorphic to $GL(m, \mathbb{C})$ via the map

$$A + iB \to \begin{bmatrix} A & B \\ -B & A \end{bmatrix},$$

and the set of all complex structures is given by the homogeneous space $GL(2m, \mathbb{R})/GL(m, \mathbb{C})$. A coset of $S \in GL(2m, \mathbb{R})$ identifies with the complex structure SJ_0S^{-1}.

If $\mathfrak{n}$ is a two step Lie algebra of a Carnot group, then we say it is J-type if the structure constants are given by antisymmetric complex structures by setting $C_{ij}^k = \langle J_k X_i, X_j \rangle$. Such algebras were first considered by Métivier [12] in his study of analytic hypoellipticity. They have also been studied by Levstein and Tiraboschi in [11] as well as Müller and Seeger in [13]. The rigidity problem for this class of algebras remains open.

To construct a J-type algebra we begin with a vector space V of dimension $2m$ with inner product $\langle\, , \,\rangle$ and orthonormal basis $\{X_1, \ldots, X_{2m}\}$. Let $\{J_1, \ldots, J_n\}$ be a linearly independent set of antisymmetric complex structures with respect to $\langle\, , \,\rangle$ and let W be a vector space of dimension n with basis $\{Z_1, \ldots, Z_n\}$. If we define structure constants by setting $C_{ij}^k = \langle J_k X_i, X_j \rangle$, then $W \oplus V$ becomes a two step fundamental graded algebra $\mathfrak{n} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ such that $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_{-2} = W$ where

$$[X, Y] = \sum \langle J_k X, Y \rangle Z_k$$

for every $X, Y \in \mathfrak{g}_{-1}$. We extend the inner product to $\mathfrak{n}$ by declaring $\{X_1, \ldots, X_{2m}, Z_1, \ldots, Z_n\}$ to be orthonormal. It follows that $\langle Z_k, [X, Y] \rangle = \langle J_k X, Y \rangle$ and the linear map $J : \mathfrak{g}_{-2} \to \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$ given by $J(Z) = \sum_k \langle Z, Z_k \rangle J_k$ has the following properties:

(i) $\langle J(Z)X, Y \rangle = \langle Z, [X, Y] \rangle$
(ii) $J(Z)^{tr} = -J(Z)$
(iii) $\mathrm{Ker}\, J = \{0\}$.

If $D \in \mathfrak{h}_0(\mathfrak{n})$ then property (i) implies that

$$\langle J(Z)DX, Y\rangle = \langle Z, [DX, Y]\rangle = -\langle Z, [X, DY]\rangle = \langle J(Z)DY, X\rangle$$

and we conclude that $J(Z)D$ is symmetric. It then follows using property (ii) that

$$J(Z)D = (J(Z)D)^{tr} = D^{tr}J(Z)^{tr} = -D^{tr}J(Z)$$

and

$$\mathfrak{h}_0(\mathfrak{n}) = \{D \in \mathfrak{gl}(\mathfrak{g}_{-1}) \mid J(Z)D + D^{tr}J(Z) = 0 \quad \text{for all } Z \in \mathfrak{g}_{-2}\}.$$

A J-type algebra is said to be Heisenberg type or H-type for short, if the following H-type condition holds:

$$J(Z)^2 = -\|Z\|^2\mathrm{Id}. \tag{1}$$

The H-type condition is equivalent to the assumption that the (antisymmetric) complex structures satisfy

$$J_k J_\ell + J_\ell J_k = 0 \tag{2}$$

for all pairs (k, ℓ) with $k \neq \ell$. Indeed, if $z_k = \langle Z, Z_k\rangle$ then

$$\begin{aligned} J(Z)^2 &= \sum_{k,\ell} z_k z_\ell J_k J_\ell \\ &= \sum_{k<\ell} z_k z_\ell (J_k J_\ell + J_\ell J_k) + \sum_k z_k^2 J_k^2 \\ &= \sum_{k<\ell} z_k z_\ell (J_k J_\ell + J_\ell J_k) - ||Z||^2\mathrm{Id}, \end{aligned} \tag{3}$$

thus showing that (1) and (2) are equivalent. In [19], the author classifies the (strata preserving) automorphism group of H-type Lie algebras. In particular, the automorphism group is given as the semidirect product of an abelian subgroup and the group of those automorphisms acting trivially on the center. In the case by case study, the latter turns out to be always nontrivial [19, Proposition 2.1] and this implies $\mathfrak{h}^{(0)}(\mathfrak{n}) \neq \{0\}$ for H-type structures.

Theorem 3 *A J-type Lie algebra is rigid with rigidity type* 1 *if the defining set of anti-symmetric complex structures contains at least three distinct elements which satisfy the H-type condition. In particular H-type Lie algebras with centre of dimension at least three are rigid of type* 1.

Proof By definition, each $T \in \mathfrak{h}^{(1)}(\mathfrak{n})$ is of the form $T(X, Y) = D(X)Y$ where $D(X) \in \mathfrak{h}^{(0)}(\mathfrak{n})$. If $D^k = D(X_k)$, then the symmetry of T implies $D^k X_j = D^j X_k$.

Since the center has dimension at least three, there exist linearly independent orthonormal vectors $Z_\alpha, Z_\beta, Z_\gamma \in \mathfrak{z}(\mathfrak{n})$, such that the matrices $J_\alpha = J(Z_\alpha)$, $J_\beta = J(Z_\beta)$ and $J_\gamma = J(Z_\gamma)$ are linearly independent. If $D \in \mathfrak{h}_0(\mathfrak{n})$ then $J(Z)D = -D^{tr}J(Z)$, hence $D^{tr} = J_\alpha D J_\alpha = J_\beta D J_\beta$ which implies $D J_\alpha J_\beta = J_\alpha J_\beta D$. It follows that

$$D^k J_\alpha J_\beta X_\ell = J_\alpha J_\beta D^k X_\ell = J_\alpha J_\beta D^\ell X_k = D^\ell J_\alpha J_\beta X_k$$

which implies

$$J_\gamma D^k J_\alpha J_\beta X_\ell = J_\gamma D^\ell J_\alpha J_\beta X_k$$

and

$$(D^k)^{tr} J_\gamma J_\alpha J_\beta X_\ell = (D^\ell)^{tr} J_\gamma J_\alpha J_\beta X_k.$$

We multiply both sides of the previous equation on the left by X_j^{tr} and use the identities: $X_j^{tr}(D^k)^{tr} = X_k^{tr}(D^j)^{tr}$ and $X_j^{tr}(D^\ell)^{tr} = X_\ell^{tr}(D^j)^{tr}$ to obtain

$$X_k^{tr}(D^j)^{tr} J_\gamma J_\alpha J_\beta X_\ell = X_\ell^{tr}(D^j)^{tr} J_\gamma J_\alpha J_\beta X_k. \tag{4}$$

We substitute the left hand side of (4) using

$$\begin{aligned} X_k^{tr}(D^j)^{tr} J_\gamma J_\alpha J_\beta X_\ell &= (X_k^{tr}(D^j)^{tr} J_\gamma J_\alpha J_\beta X_\ell)^{tr} \\ &= -X_\ell^{tr} J_\beta J_\alpha J_\gamma D^j X_k, \end{aligned}$$

and we substitute the right hand side of (4) using:

$$(D^j)^{tr} J_\gamma J_\alpha J_\beta = -J_\gamma D^j J_\alpha J_\beta = -J_\gamma J_\alpha J_\beta D^j,$$

and thus obtain

$$X_\ell^{tr} J_\beta J_\alpha J_\gamma D^j X_k = X_\ell^{tr} J_\gamma J_\alpha J_\beta D^j X_k.$$

It follows that $J_\beta J_\alpha J_\gamma D^j = J_\gamma J_\alpha J_\beta D^j$ and if $\{J_\alpha, J_\beta, J_\gamma\}$ satisfies condition (2), then

$$J_\gamma J_\alpha J_\beta = -J_\alpha J_\gamma J_\beta = J_\alpha J_\beta J_\gamma = -J_\beta J_\alpha J_\gamma.$$

So we conclude that $2J_\beta J_\alpha J_\gamma D^j = 0$, which implies $D^j = 0$. Hence we have proved $\mathfrak{h}_1(\mathfrak{n}) = \{0\}$. □

We remark that there exist J-type algebras that satisfy the hypotheses of the theorem above which are not of H-type, as the following example shows. Consider the complex structures J_1, J_2, J_3, J_4 defined as

$$J_k = \begin{bmatrix} 0 & -Q_k^{tr} \\ Q_k & 0 \end{bmatrix}, \tag{5}$$

with $Q_1 = \mathrm{Id}$,

$$Q_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The full set $\{J_1, \dots, J_4\}$ does not satisfy the H-type condition since

$$J_1 J_2 + J_2 J_1 = - \begin{bmatrix} Q_2 + Q_2^{tr} & 0 \\ 0 & Q_2 + Q_2^{tr} \end{bmatrix} \neq 0,$$

however the subset $\{J_2, J_3, J_4\}$ does satisfy the H-type condition and so by Theorem 3 the J-type algebra defined by $\{J_1, \dots, J_4\}$ is rigid with rigidity type 1. To compute the algebra we assume $\{X_1, \dots, X_8, Z_1, \dots, Z_4\}$ is an orthonormal basis and define

$$[X_i, X_j] = \sum_{k=1}^{4} \langle J_k X_i, X_j \rangle Z_k.$$

The bracket table is:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & Z_1 - Z_3 & Z_4 & 0 & -Z_2 \\ 0 & 0 & 0 & 0 & Z_2 & Z_1 & -Z_4 & -Z_3 \\ 0 & 0 & 0 & 0 & 0 & Z_2 & Z_1 - Z_3 & Z_4 \\ 0 & 0 & 0 & 0 & Z_4 & Z_3 & Z_2 & Z_1 \\ -Z_1 + Z_3 & -Z_2 & 0 & -Z_4 & 0 & 0 & 0 & 0 \\ -Z_4 & -Z_1 & -Z_2 & -Z_3 & 0 & 0 & 0 & 0 \\ 0 & Z_4 & -Z_1 + Z_3 & -Z_2 & 0 & 0 & 0 & 0 \\ Z_2 & Z_3 & -Z_4 & -Z_1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where the (i, j) element reads the value of $[X_i, X_j]$. Direct calculation using MAPLE shows that the J-type algebra given by the set $\{J_1, J_2, J_3\}$ and the orthonormal basis $\{X_1, \dots, X_8, Z_1, Z_2, Z_3\}$ is rigid with rigidity type 1.

7 Remarks

As we have seen, all known rigid algebras have rigidity type -1, 0 or 1. An example of a type 2 algebra in the usual sense is given in [7] by the conformal algebra:

$$\mathfrak{co}(n) = \mathfrak{o}(n) + \mathbb{R}\,\mathrm{Id} = \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^{tr} + A = \lambda\,\mathrm{Id},\ \lambda \in \mathbb{R}\},$$

where

$$\mathfrak{o}(n) = \{D \in \mathfrak{gl}(n, \mathbb{R}) \mid D^{tr} + D = 0\}$$

and $n > 2$. These algebras cannot be $\mathfrak{h}_0(\mathfrak{n})$ for some noncommutative $\mathfrak{n}$. Indeed, suppose $X, Y \in \mathfrak{g}_{-1} \simeq \mathbb{R}^n$ satisfy $[X, Y] \neq 0$. If we want each $A \in \mathfrak{co}(n)$ to identify with an element of $\mathfrak{h}_0(\mathfrak{n})$, then the extended action of A as a derivation must satisfy $A[X, Y] = 0$, and so necessarily

$$[DX, Y] + [X, DY] = 2\lambda[X, Y]$$

for all $D \in \mathfrak{o}(n)$ and all $\lambda \in \mathbb{R}$ which is impossible. Thus we are left with the question: Are there examples of algebras $\mathfrak{n}$ with rigidity type k where $1 < k < \infty$?

8 The rank one condition and nonrigidity

In [14], it is shown that given a stratified nilpotent Lie algebra $\mathfrak{n}$, if there exists $X \in \mathfrak{g}_{-1}$ such that rank $\mathrm{ad}_X \leq 1$, then $\mathfrak{n}$ in nonrigid. The following lemma reads this result in terms of prolongations.

Lemma 4 *There exists* $X \in \mathfrak{g}_{-1}$ *such that* rank $\mathrm{ad}_X \leq 1$ *if and only if* $\mathfrak{h}^{(0)}(\mathfrak{n})$ *is nonelliptic.*

Proof If $\mathfrak{n}$ contains a vector X of rank zero, then $\mathfrak{n}$ is degenerate and it suffices to take the derivation D defined by $D(X) = X$ and zero otherwise. Suppose now that $\mathfrak{n}$ contains an element $X \in \mathfrak{g}_{-1}$ such that rank $\mathrm{ad}_X = 1$. Following [14], it is possible to complete X to a basis of $\mathfrak{n}$, say $\mathcal{B}$, in such a way that there exists a unique $Y \in \mathcal{B}$ for which $[X, Y] \neq 0$, moreover $Y \in \mathfrak{g}_{-1}$. It follows directly that the endomorphism $D : \mathfrak{g} \to \mathfrak{g}$ defined by $D(Y) = X$ and zero elsewhere is an element of $\mathfrak{h}_0(\mathfrak{n})$, which identifies with an element of rank 1 in $\mathfrak{h}^{(0)}(\mathfrak{n})$.

Now, assume that $\mathfrak{h}^{(0)}(\mathfrak{n})$ is nonelliptic. In this case there exists $D \in \mathfrak{h}_0(\mathfrak{n})$ which identifies with an element of rank 1 in $\mathfrak{h}^{(0)}(\mathfrak{n})$. Corresponding to D there exists vectors $X, Y \in \mathfrak{g}_{-1}$ (not necessarily distinct), unique up to a scale, such that $D(Y) = X$ and $D(\mathfrak{g}_{-1}/\operatorname{span}\{Y\}) = \{0\}$. If $W \in \mathfrak{g}_{-1}/\operatorname{span}\{Y\}$, then

$$0 = D[Y, W] = [DY, W] + [Y, DW] = [X, W]$$

We conclude that rank $\mathrm{ad}_X \leq 1$. □

It was recently brought to our attention that the complete characterization of the behavior of the prolongation in the usual sense is related to that of a differential operator and thus leads to a complete characterisation of rigidity in terms of the existence of a rank 1 type condition. A result of Spencer [20] shows that the prolongation of a differential operator is of infinite type if and only if there exists for its symbol a complex nontrivial characteristic element. We will discuss this result in a forthcoming paper and observe that the condition of Spencer is equivalent to the fact that there exists a rank one element in the complexification of $\mathfrak{h}^{(0)}(\mathfrak{n})$. By a repetition of the argument above in the complex settings, this is equivalent to the existence of a vector X in the complexification of $\mathfrak{g}_{-1}$ such that rank $\mathrm{ad}_X \leq 1$, where ad_X here has to be viewed as a linear map on the complexification of $\mathfrak{n}$.

9 Nonrigid J-type algebras

Lemma 5 *The complexification of a Lie algebra of infinite type is of infinite type.*

Proof Let $\mathfrak{n}$ be a real stratified nilpotent Lie algebra such that $\mathfrak{h}_k(\mathfrak{n}) \neq 0$ for every $k \geq -1$. Denote by $\mathfrak{n}^{\mathbb{C}}$ the complexification of $\mathfrak{n}$. If $\{X_1, \ldots, X_n\}$ is a basis of $\mathfrak{n}_{-1}$, then $\{X_1, \ldots, X_n, iX_1, \ldots, iX_n\}$ is a basis of $\mathfrak{n}^{\mathbb{C}}_{-1}$. It is easy to see that given a nontrivial $D \in \mathfrak{h}^{(0)}(\mathfrak{n})$, then $D^{\mathbb{C}} : \mathfrak{n}^{\mathbb{C}}_{-1} \to \mathfrak{n}^{\mathbb{C}}_{-1}$ defined by $D^{\mathbb{C}}(X_j) = D(X_j)$ and $D^{\mathbb{C}}(iX_j) = iD(X_j)$ for every $j = 1, \ldots, n$ is a nontrivial element in the complexification of $\mathfrak{h}_0(\mathfrak{n})$. Inductively we then define nontrivial elements $u^{\mathbb{C}}$ in the complexification of $\mathfrak{h}_k(\mathfrak{n})$ by the assignments $u^{\mathbb{C}}(X_j) = u(X_j)$ and $u^{\mathbb{C}}(iX_j) = iu(X_j)$ for every $j = 1, \ldots, n$, where $u \in \mathfrak{h}_k(\mathfrak{n})$. □

The complexified Heisenberg algebras satisfy the rank one condition over $\mathbb{C}$ and are thus nonrigid by Lemma 5. Moreover, J-type algebras with complex structure and two dimensional centre are complexified Heisenberg algebras as demonstrated by the following lemma. We recall that a Lie algebra $\mathfrak{n}$ admits a complex structure if there exists a complex structure $J : \mathfrak{n} \to \mathfrak{n}$ such that

$$J[V, W] = [JV, W] = [V, JW], \tag{6}$$

for every $V, W \in \mathfrak{n}$.

Lemma 6 *Let* $\mathfrak{n} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ *be a two step stratified Lie algebra with* $\dim \mathfrak{g}_{-2} = 2$. *If* $\mathfrak{n}$ *admits a complex structure then it is a complexified Heisenberg algebra.*

Proof We prove by induction that

$$\mathfrak{n} = \mathfrak{f}_1 \oplus \cdots \oplus \mathfrak{f}_m \oplus \mathfrak{g}_{-2}$$

where the spaces $\mathfrak{f}_i$ satisfy

(i) $\dim \mathfrak{f}_i = 4$
(ii) $\mathfrak{f}_i \oplus \mathfrak{g}_{-2}$ is a complexified Heisenberg algebra
(iii) $[\mathfrak{f}_i, \mathfrak{f}_j] = 0$

for every $i, j = 1, \ldots, m$ and $i \neq j$.

For any nonzero $Z_1 \in \mathfrak{g}_{-2}$, the set $\{Z_1, JZ_1\}$ is a basis for $\mathfrak{g}_{-2}$. The linearly independence follows by observing that if $JZ_1 = aZ_1$ then $J^2 Z_1 = J(aZ_1) = a^2 Z_1$ which contradicts $J^2 = -I$. Write $Z_2 = JZ_1$ and choose any pair of distinct elements $X_1, Y_1 \in \mathfrak{g}_{-1}$ such that $[X_1, Y_1] = Z_1$. It follows from (6) that the additional nontrivial brackets for the set

$$\{X_1, Y_1, JX_1, JY_1, Z_1, Z_2\}$$

are

$$[X_1, JY_1] = Z_2, \quad [JX_1, Y_1] = Z_2, \quad \text{and} \quad [JX_1, JY_1] = -Z_1.$$

Therefore, if we set $\mathfrak{f}_1 = \text{span}\{X_1, Y_1, JX_1, JY_1\}$, then $\mathfrak{f}_1 \oplus \mathfrak{g}_{-2}$ is the complexified Heisenberg algebra.

Suppose now that there exist subspaces $\mathfrak{f}_1, \ldots, \mathfrak{f}_r$ such that (i), (ii) and (iii) hold. If $\mathfrak{f}_1 \oplus \cdots \oplus \mathfrak{f}_r \neq \mathfrak{g}_{-1}$ then there exists an $X \in \mathfrak{g}_{-1}$ such that $X \neq 0 \bmod \mathfrak{f}_1 \oplus \cdots \oplus \mathfrak{f}_r$ and $[X, \mathfrak{f}_1 \oplus \cdots \oplus \mathfrak{f}_r] = \{0\}$. Indeed, if X does not satisfy the latter condition then we replace it with the vector

$$\bar{X} = X + \sum_{i=1}^{r} (a_i X_i + b_i Y_i + c_i JX_i + d_i JY_i),$$

where $\{X_i, Y_i, JX_i, JY_i\}$ is the basis of $\mathfrak{f}_i$, and the coefficients solve the equations $[\bar{X}, \mathfrak{f}_s] = 0$ for every $s = 1, \ldots, r$. Direct calculation shows that these equations form a linear system in the coefficient variables which admits a unique solution. Since $\mathfrak{n}$ is nondegenerate, there exists $Y \in \mathfrak{g}_{-1}$ for which $[X, Y] \neq 0$. It follows that $Y \neq 0 \bmod \mathfrak{f}_1 \oplus \cdots \oplus \mathfrak{f}_r$ and by the modification procedure above, we can assume that $[Y, \mathfrak{f}_s] = 0$ for every $s = 1, \ldots, r$. Letting $\mathfrak{f}_{r+1} = \text{span}\{X, Y, JX, JY\}$ we see that the family $\{\mathfrak{f}_i : i = 1, \ldots, r+1\}$ satisfies (i), (ii) and (iii). □

We construct a nonrigid J-type algebra which does not satisfy the rank one condition over the reals. Although for what we said in the previous section we know that we can show nonrigidity by giving a rank one element in the complexification of $\mathfrak{h}_0(\mathfrak{n})$, we choose here to explicitly construct a nonzero sequence of elements in $\mathfrak{h}_k(\mathfrak{n})$ for every non negative k. We consider an example which appeared in [13], that is the J-type algebra given by the orthonormal basis $\{X_1, \ldots, X_8, Z_1, Z_2\}$ and the set $\{J_1, J_2\}$, where J_1 and J_2 are given by (5). The bracket table for this algebra is given by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & Z_1 & 0 & 0 & -Z_2 \\ 0 & 0 & 0 & 0 & Z_2 & Z_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_2 & Z_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Z_2 & Z_1 \\ -Z_1 & -Z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -Z_1 & -Z_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -Z_1 & -Z_2 & 0 & 0 & 0 & 0 \\ Z_2 & 0 & 0 & -Z_1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We note that there is no $V \in \mathfrak{n}$ such that $\operatorname{rank} \operatorname{ad}_V = 1$, indeed let $V = \sum_{i=1}^{8} v_i X_i + v_9 Z_1 + v_{10} Z_{10}$, then

$$\begin{aligned} [V, X_1] &= -v_5 Z_1 + v_8 Z_2 & [V, X_5] &= v_1 Z_1 + v_2 Z_2 \\ [V, X_2] &= -v_6 Z_1 - v_5 Z_2 & [V, X_6] &= v_2 Z_1 + v_3 Z_2 \\ [V, X_3] &= -v_7 Z_1 - v_6 Z_2 & [V, X_7] &= v_3 Z_1 + v_4 Z_2 \\ [V, X_4] &= -v_8 Z_1 - v_7 Z_2 & [V, X_8] &= v_4 Z_1 - v_1 Z_2, \end{aligned}$$

which by inspection shows that rank $\mathrm{ad}_V \geq 2$. Furthermore, Lemma 6 shows that $\mathfrak{n}$ cannot be a complex Lie algebra since it is not a complexified Heisenberg algebra.

Since $\langle J(Z)X, Y\rangle = \langle Z, [X, Y]\rangle$ it follows that

$$\begin{aligned}\mathfrak{h}_0(\mathfrak{n}) &= \{D \in \mathfrak{gl}(\mathfrak{g}_{-1}) \mid J(Z)D + D^{tr}J(Z) = 0 \text{ for all } Z \in \mathfrak{g}_{-2}\}\\ &= \{D \in \mathfrak{gl}(\mathfrak{g}_{-1}) \mid J_k D + D^{tr} J_k = 0 \text{ for every } k = 1, 2\}.\end{aligned}$$

If we set $\mathfrak{g}_{-1} = \mathfrak{a} \oplus \mathfrak{b}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are the abelian subalgebras spanned by $\{X_i : i = 1, \ldots, 4\}$ and $\{X_j : j = 5, \ldots, 8\}$ respectively, then $J(Z_k)\mathfrak{a} = \mathfrak{b}$ and $J(Z_k)\mathfrak{b} = \mathfrak{a}$ for $k = 1$ and 2. If we set

$$A = \begin{bmatrix} \alpha_1 & -\alpha_4 & -\alpha_3 & -\alpha_2 \\ \alpha_2 & \alpha_1 & -\alpha_4 & -\alpha_3 \\ \alpha_3 & \alpha_2 & \alpha_1 & -\alpha_4 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{bmatrix}, \quad C = \begin{bmatrix} \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\ \alpha_6 & \alpha_7 & \alpha_8 & -\alpha_5 \\ \alpha_7 & \alpha_8 & -\alpha_5 & -\alpha_6 \\ \alpha_8 & -\alpha_5 & -\alpha_6 & -\alpha_7 \end{bmatrix}$$

and

$$B = \begin{bmatrix} \alpha_9 & \alpha_{10} & \alpha_{11} & \alpha_{12} \\ \alpha_{10} & \alpha_{11} & \alpha_{12} & -\alpha_9 \\ \alpha_{11} & \alpha_{12} & -\alpha_9 & -\alpha_{10} \\ \alpha_{12} & -\alpha_9 & -\alpha_{10} & -\alpha_{11} \end{bmatrix},$$

then the elements of $\mathfrak{h}_0(\mathfrak{n})$ take the form

$$\begin{bmatrix} A & B & 0 \\ C & -A^{tr} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\mathfrak{h}^{(0)}(\mathfrak{n})$ consists of matrices of the form

$$\begin{bmatrix} A & B \\ C & -A^{tr} \end{bmatrix}.$$

We prove that the subalgebra $\mathfrak{h}^{(0)} \subseteq \mathfrak{h}^{(0)}(\mathfrak{n})$ consisting of matrices of the form

$$\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$$

is of infinite type in the usual sense thus implying nonrigidity of $\mathfrak{n}$. To this end we define a C valued map $\psi : \mathbb{R}^4 \to \mathfrak{gl}(4, \mathbb{R})$ by

$$\psi(X) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & -x_1 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x_1 & -x_2 & -x_3 \end{bmatrix},$$

and we set $\sigma_i = \psi(e_i)$ where $\{e_1, \ldots, e_4\}$ is the canonical basis for $\mathbb{R}^4$.

Lemma 7 *The map ψ has the following properties*:

(i) $\psi(\sigma_i e_j)e_k = \psi(\sigma_i e_k)e_j$ *for all* 3*-tuples* $(i, j, k) \in \{1, \dots, 4\}^3$.
(ii) *For every* $k \geq 2$ *and every multi-index* $I_k = (i_1, i_2, \dots, i_k)$*, there exist a positive integer* t *and* $s \in \{1, \dots, 4\}$*, both depending on* I_k*, such that*

$$\psi(\cdots(\psi(\sigma_{i_1} e_{i_2})e_{i_3})\cdots e_{i_k}) = (-1)^t \psi(e_s),$$

where ψ *appears at the left hand side* $k-1$ *times.*

Proof Part (i) follows by a direct verification. We prove (ii) by induction on k. If $k = 2$, then (ii) follows by observing that $\sigma_{i_1} e_{i_2}$ is again a canonical vector e_s carrying a sign $(-1)^t$, with s and t both depending on i_1 and i_2. Assume now that (ii) holds for every $2 \leq j \leq k-1$ and for every multi-index I_j. This means in particular

$$\psi(\cdots(\psi(\sigma_{i_1} e_{i_2})e_{i_3})\cdots e_{i_{k-1}}) = (-1)^t \psi(e_s),$$

where ψ appears $k-2$ times and t and s depend on $e_{i_1}, \dots, e_{i_{k-1}}$. Then,

$$\begin{aligned}\psi(\psi(\cdots(\psi(\sigma_{i_1} e_{i_2})e_{i_3})\cdots e_{i_{k-1}})e_{i_k}) &= \psi((-1)^t \psi(e_s)e_{i_k}) \\ &= (-1)^t \psi(\sigma_s e_{i_k}) \\ &= (-1)^t (-1)^{t'} \psi(e_{s'}),\end{aligned}$$

where t' and s' depend on s and i_k. Therefore (2) follows for $j = k$. □

Theorem 8 $\mathfrak{g}(\mathfrak{n})$ *is infinite dimensional.*

Proof It is enough to show that $\mathfrak{h}^{(0)}$ has an infinite prolongation in the usual sense. For each natural number k, we define

$$\begin{aligned}T_k(V_1, \dots, V_k, V_{k+1}) &= \begin{bmatrix} 0 & 0 \\ \psi(\cdots(\psi(\sigma_1 V_1^{\mathfrak{a}})V_2^{\mathfrak{a}})\cdots V_k^{\mathfrak{a}}) & 0 \end{bmatrix} \begin{bmatrix} V_{k+1}^{\mathfrak{a}} \\ V_{k+1}^{\mathfrak{b}} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \psi(\cdots(\psi(\sigma_1 V_1^{\mathfrak{a}})V_2^{\mathfrak{a}})\cdots V_k^{\mathfrak{a}})V_{k+1}^{\mathfrak{a}} \end{bmatrix}\end{aligned}$$

where $V^{\mathfrak{a}}$ and $V^{\mathfrak{b}}$ denote the projections of $V \in \mathfrak{g}_{-1}$ along $\mathfrak{a}$ and $\mathfrak{b}$. We claim that T_k defines a nonzero element of $\mathfrak{h}^{(k)}$. Indeed T_k is nonzero since $T_k((e_1, 0), \dots, (e_1, 0)) = (0, e_1)$, and so by construction the mapping $V_{k+1} \to T_k(V_1, \dots, V_k, V_{k+1})$ is an element of $\mathfrak{h}^{(0)}$, so by Sect. 2 it remains to check that T_k is symmetric.

We prove symmetry by induction. Consider $k = 1$, then item (i) of Lemma 7 shows that

$$T_1(V_1, V_2) = \begin{bmatrix} 0 & 0 \\ \psi(\sigma_1 V_1^{\mathfrak{a}}) & 0 \end{bmatrix} \begin{bmatrix} V_2^{\mathfrak{a}} \\ V_2^{\mathfrak{b}} \end{bmatrix} = \begin{bmatrix} 0 \\ \psi(\sigma_1 V_1^{\mathfrak{a}})V_2^{\mathfrak{a}} \end{bmatrix}$$

is symmetric.

Next we assume that T_{k-1} is symmetric and prove that T_k is symmetric. To this end we define recursively a family of maps $\Psi_k : \mathfrak{a}^k \to \mathfrak{a}$ by setting $\Psi_1(V_1) = \sigma_1 V_1$ and

$$\Psi_k(V_1, \ldots, V_{k-1}, V_k) = \psi(\Psi_{k-1}(V_1, \ldots, V_{k-1}))V_k$$

for $k \geq 2$. By definition, T_k will be symmetric if the expression

$$\psi(\Psi(V_1, V_2, \ldots, V_{k-1}))V_k$$

is symmetric in V_{k-1} and V_k. By linearity it is enough that we prove this for a basis. By (ii) of Lemma 7 we have

$$\psi(\cdots(\psi(\sigma_1 e_{i_1})e_{i_2})\cdots e_{i_{k-1}}) = (-1)^t \psi(e_s)$$

for some positive integer t and an $s \in \{1, \ldots, 4\}$ which both depend upon $e_1, e_{i_1}, \ldots, e_{i_{k-1}}$. It follows that

$$\begin{aligned} \psi(\Psi(e_{i_1}, e_{i_2}, \ldots, e_{i_{k-1}}))e_{i_k} &= (-1)^t \psi(\psi(e_s)e_{i_{k-1}})e_{i_k} \\ &= (-1)^t \psi(\psi(e_s)e_{i_k})e_{i_{k-1}} \\ &= \psi(\Psi(e_{i_1}, \ldots, e_{i_{k-2}}, e_{i_k}))e_{i_{k-1}}, \end{aligned}$$

where at the second line we used (i) of Lemma 7 after noticing that $\psi(e_s) = \sigma_s$. This concludes the induction. □

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