# ON THE COHOMOLOGY OF CERTAIN QUOTIENTS OF THE SPECTRUM $B P$ 

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#### Abstract

The aim of this note is to present a new, elementary proof of a result of Baas and Madsen on the $\bmod p$ cohomology of certain quotients of the spectrum $B P$.

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In 1970s, Baas and Madsen [1] calculated the cohomology of certain quotients of $M U$, the Thom spectrum of the universal bundle over $B U$. Recall that

$$
M U_{*}=\pi_{*}(M U) \cong \mathbb{Z}\left[x_{1}, x_{2}, \ldots\right],
$$

where $x_{i}$ lies in degree $\left|x_{i}\right|=2 i$. The spectra $M U\left\langle n_{1}, \ldots, n_{q}\right\rangle$ considered by Baas and Madsen are defined for any string of integers $0<n_{1}<\cdots<n_{q}$ of the form $n_{i}=$ $2\left(p^{j_{i}}-1\right), j_{i}>0$, where $p$ is some fixed prime. They satisfy

$$
\pi_{*}\left(M U\left\langle n_{1}, \ldots, n_{q}\right\rangle\right) \cong \mathbb{Z}\left[x_{n_{1}}, \ldots, x_{n_{q}}\right] .
$$

Baas and Madsen determined their mod $p$ cohomology, by relying heavily on the Atiyah-Hirzebruch spectral sequence and previous work of Cohen on the Hurewicz homomorphism on $M U$ [4].

The aim of this note is to present an alternative proof of Baas-Madsen's result without reference to any spectral sequence or to the work of Cohen. Our arguments use the techniques developed in $[\mathbf{5}]$ and thus are more transparent than the original ones. They are based on a construction of the spectra $M U\left\langle n_{1}, \ldots, n_{q}\right\rangle$ which is algebraic in nature and which does not require the use of bordism with singularities any more.

The spectrum $M U$ is a commutative $S$-algebra, see [5]. This leads to a well-behaved homotopy theory of $M U$-modules. In particular, the derived or homotopy category of $M U$-modules is a symmetric monoidal category for the smash product over $M U$. Our constructions take place in this category.

The spectra $M U\left\langle n_{1}, \ldots, n_{q}\right\rangle$ can be obtained in a standard way, as regular quotients of $M U$ [5]:

$$
M U\left\langle n_{1}, \ldots, n_{q}\right\rangle \simeq M U /\left(x_{k}: k \neq n_{1}, \ldots, n_{q}\right) .
$$

Instead of the $x_{k}$, we may take any other regular sequence generating the kernel $J$ of the projection $M U_{*} \rightarrow M U\left\langle n_{1}, \ldots, n_{q}\right\rangle_{*}$. So it is legitimate to write $M U / J$ for $M U\left\langle n_{1}, \ldots, n_{q}\right\rangle$. As our interest lies in the $\bmod p$ cohomology of these $M U$-modules, we might just as well consider their $p$-localisations $M U\left\langle n_{1}, n_{2}, \ldots, n_{q}\right\rangle_{(p)}$. These spectra admit a much more economical presentation, as quotients of $B P$. To see this, recall that

$$
B P_{*}=\mathbb{Z}_{(p)}\left[v_{1}, v_{2}, \ldots\right], \quad\left|v_{i}\right|=2\left(p^{i}-1\right)
$$

where the $v_{i}$ are Araki's generators [6]. Recall also that $B P$ can be realised as an $M U$-algebra [2], and that the unit $\pi: M U \rightarrow B P$ induces an isomorphism

$$
\bar{\pi}_{*}: \mathbb{Z}_{(p)} \otimes M U_{*} /\left(x_{k}: k \neq p^{i}-1\right) \cong B P_{*} .
$$

From this, we deduce that

$$
M U\left\langle n_{1}, \ldots, n_{q}\right\rangle_{(p)} \simeq B P \wedge_{M U} M U /\left(x_{p^{j}-1}: j \neq j_{1}, \ldots, j_{q}\right) .
$$

As an additional advantage, this presentation exhibits the $M U$-modules $M U\left\langle n_{1}, \ldots, n_{q}\right\rangle_{(p)}$ as left $B P$-modules. Setting $I=B P_{*} \cdot J$, we can unambiguously write

$$
M U\left\langle n_{1}, n_{2}, \ldots, n_{q}\right\rangle_{(p)} \simeq B P / I
$$

Following the convention $x_{0}=v_{0}=p$ and extending the string of the $n_{i}$ by $n_{0}=0$, we obtain the other family of spectra considered by Baas and Madsen,

$$
M U_{p}\left\langle n_{1}, \ldots, n_{q}\right\rangle=M U\left\langle n_{1}, \ldots, n_{q}\right\rangle_{(p)} / p=B P /(I+(p)),
$$

as quotients of $B P$ as well.
Recall that the construction of regular quotients is natural in the following sense. If $I_{1} \subseteq I_{2} \subseteq B P_{*}$ are two ideals such that $I_{1}$ is generated by a regular sequence over $B P_{*}$ and such that $I_{2} / I_{1}$ is generated by a regular sequence over $B P_{*} / I_{1}$, then there is a canonical map of $B P$-modules $B P / I_{1} \rightarrow B P / I_{2}$ [5, V.1.]. Note that the EilenbergMacLane spectrum $H \mathbb{F}_{p}$ is the quotient of $B P$ by the ideal generated by all the $v_{i}$ 's and will serve as a terminal object in our construction.

Let $\mathcal{A}_{p}$ denote the $\bmod p$ Steenrod algebra and let $Q_{i}, i \geq 0$, be the primitive element of degree $2 p^{i}-1$ of Milnor's basis. Let us write ( $y_{1}, y_{2}, \ldots$ ) for the left ideal of $\mathcal{A}_{p}$ generated by elements $y_{1}, y_{2}, \ldots \in \mathcal{A}_{p}$. Recalling that $x_{p^{k}-1} \equiv v_{k}$ modulo decomposables, we have the following result.

Theorem. Let $I \subset B P_{*}$ be the ideal generated by a regular sequence $w_{i_{1}}, w_{i_{2}}, \ldots$ with $w_{i_{k}} \equiv v_{i_{k}}$ modulo decomposables and $0 \leq i_{1}<i_{2}<\cdots$. Then the natural map $B P / I \rightarrow$ $H \mathbb{F}_{p}$ induces an isomorphism of $\mathcal{A}_{p}$-modules

$$
H^{*}\left(B P / I ; \mathbb{F}_{p}\right) \cong \mathcal{A}_{p} /\left(Q_{i}: i \neq i_{1}, i_{2}, \ldots\right) .
$$

As a special case, we obtain the results of Baas-Madsen.

Corollary. There are canonical isomorphisms of $\mathcal{A}_{p}$-modules:

$$
\begin{aligned}
H^{*}\left(M U\left\langle n_{1}, \ldots, n_{q}\right\rangle ; \mathbb{F}_{p}\right) & \cong \mathcal{A}_{p} /\left(Q_{0}, Q_{j_{1}}, \ldots, Q_{j_{q}}\right) \\
H^{*}\left(M U_{p}\left\langle n_{1}, \ldots, n_{q}\right\rangle ; \mathbb{F}_{p}\right) & \cong \mathcal{A}_{p} /\left(Q_{j_{1}}, \ldots, Q_{j_{q}}\right)
\end{aligned}
$$

Proof. We first prove the result in the particular case of the left $B P$-modules $P(n)=B P / I_{n}$, where $I_{n}$ is the ideal of $B P_{*}$ generated by the elements $v_{0}, \ldots, v_{n-1}$. That is, $P(0)=B P$ by definition and $P(n)_{*} \cong \mathbb{F}_{p}\left[v_{n}, v_{n+1}, \ldots\right]$ for $n \geq 1$. Let $\phi_{n}: P(n) \rightarrow H \mathbb{F}_{p}$ be the natural map and set $C(n)=H^{*}\left(P(n) ; \mathbb{F}_{p}\right)$. The cofibre sequence of left $B P$ modules and left $B P$-morphisms

$$
\cdots \rightarrow P(n) \xrightarrow{v_{n}} P(n) \xrightarrow{\eta_{n}} P(n+1) \xrightarrow{\partial_{n}} P(n) \rightarrow \cdots
$$

induces a long exact sequence of $\mathcal{A}_{p}$-modules in cohomology

$$
\cdots \rightarrow C(n) \xrightarrow{v_{n}^{*}} C(n) \xrightarrow{\partial_{n}^{*}} C(n+1) \xrightarrow{\eta_{n}^{*}} C(n) \rightarrow \cdots
$$

The accurate reader has noticed that we have suppressed the mention of suspension coordinates. As a consequence, $\partial_{n}^{*}$ is not a morphism of degree 0 but rather of degree $2 p^{n}-1$. Proposition B.5.15(b) in [7] implies that the image of $v_{n}$ under the Hurewicz homomorphism $P(n)_{*} \rightarrow H_{*}\left(P(n) ; \mathbb{F}_{p}\right)$ is trivial. Therefore, $\left(v_{n}\right)_{*}: H_{*}\left(P(n) ; \mathbb{F}_{p}\right) \rightarrow$ $H_{*}\left(P(n) ; \mathbb{F}_{p}\right)$ is trivial, and by duality the same holds for $v_{n}^{*}: C(n) \rightarrow C(n)$. As a consequence, we obtain a short exact sequence of $\mathcal{A}_{p}$-modules:

$$
\begin{equation*}
0 \rightarrow C(n) \xrightarrow{\partial_{n}^{*}} C(n+1) \xrightarrow{\eta_{n}^{*}} C(n) \rightarrow 0 . \tag{1}
\end{equation*}
$$

It obviously splits in the category of $\mathbb{F}_{p}$-vector spaces. We now inductively define elements $q_{j_{0}, \ldots, j_{l}}^{n+1} \in C(n+1)$ of degree $\sum_{k=0}^{l}\left(2 p^{j_{k}}-1\right)$, for any $l \in\{0, \ldots, n\}$ :

$$
\eta_{n}^{*}\left(q_{j_{0}, \ldots, j_{l}}^{n+1}\right)= \begin{cases}q_{j_{0}, \ldots, j_{l}}^{n} & \text { if } j_{l}<n  \tag{2}\\ 0 & \text { if } j_{l}=n\end{cases}
$$

For $C(0)$ we take $q_{\emptyset}^{0}=\phi_{0}: P(0) \rightarrow H \mathbb{F}_{p}$. Assume we have chosen elements $q_{j_{0}, \ldots, j_{l}}^{k} \in$ $C(k)$ as indicated for $k \leq n$. For degree reasons, the $q_{j_{0}, \ldots, j_{l}}^{n}$ admit unique lifts $q_{j_{0}, \ldots, j_{l}}^{n+1}$ to $C(n+1)$. For $j_{l}=n$, we define

$$
\begin{equation*}
q_{j_{0}, \ldots, j_{l}}^{n+1}=\partial_{n}^{*}\left(q_{j_{0}, \ldots, j_{l-1}}^{n}\right) . \tag{3}
\end{equation*}
$$

Observe that the product on $B P$ induces a coalgebra structure on $C(0)$. Moreover, the action of $B P$ on $P(n)$ gives rise to left $C(0)$-comodule structures on the $C(n)$ with respect to which equation (1) is a short exact sequence of $C(0)$-comodules. We show by induction that there are isomorphisms of $C(0)$-comodules

$$
\begin{equation*}
C(n) \cong C(0) \otimes_{\mathbb{F}_{p}}\left(\bigoplus_{0 \leq j_{0}<\cdots<j_{l}<n} \mathbb{F}_{p} q_{j_{0}, \ldots, j_{l}}^{n}\right) \tag{4}
\end{equation*}
$$

so that under this isomorphism, $\eta_{n}^{*}$ maps $y \otimes q_{j_{0}, \ldots, j_{l}}^{n+1}$ to $y \otimes \eta_{n}^{*}\left(q_{j_{0}, \ldots, j_{l}}^{n}\right)$. This is clear for $n=0$. For the inductive step, note that comodules of this form are relatively injective [6, A1]. Since equation (1) splits over $\mathbb{F}_{p}$ and is a sequence of $C(0)$-comodules, it splits as a sequence of $C(0)$-comodules, which proves the claim.

From equation (4), it follows that the primitives of $C(n)$ (with respect to the $C(0)$ comodule structure) are given by

$$
P(C(n))=\bigoplus_{0 \leq j_{0}<\cdots<j_{l}<n} \mathbb{F}_{p} q_{j_{0}, \ldots, j_{l}}^{n}
$$

We now show by induction that

$$
\begin{equation*}
C(k) \cong \mathcal{A}_{p} /\left(Q_{k}, Q_{k+1}, \ldots\right) \tag{5}
\end{equation*}
$$

and that

$$
\phi_{k}^{*}\left(Q_{j_{0}} \cdots Q_{j_{l}}\right)= \begin{cases}\alpha_{j l} q_{j_{0}, \ldots, j_{l}}^{k} & \text { if } j_{l}<k  \tag{6}\\ 0 & \text { otherwise }\end{cases}
$$

for some non-zero $\alpha_{j_{l}} \in \mathbb{F}_{p}$. In fact, one can show that $\alpha_{j_{l}}=1$ for all $l$, but it will be enough to know equation (6) for our purposes.

It is well known that equation (5) holds for $k=0$, see the original paper [3] or Theorem 4.1.12 in [6] for the $\bmod p$ homology. Also, equation (6) is trivial in this case. Assume inductively that equations (5) and (6) are true for $k \leq n$. For the inductive step, recall that $q_{j_{0}, \ldots, j_{l}}^{n+1}$ is uniquely determined by equation (2) for $j_{l}<n$. Thus, equation (6) certainly holds for $k=n+1$ in this case. Now note that $\phi_{n}: P(n) \rightarrow H \mathbb{F}_{p}$ induces an isomorphism in degrees $<2\left(p^{n}-1\right)$ on homotopy groups, and so $\phi_{n}^{*}: \mathcal{A}_{p} \rightarrow C(n)$ is an isomorphism in degrees $<2 p^{n}-3$. This implies that $\phi_{n+1}^{*}$ sends $Q_{n}$ to some non-zero primitive element in $C(n+1)$. The only primitives in $C(n+1)$ of degree $\left|Q_{n}\right|=2 p^{n}-1$ are the scalar multiples of $q_{n}^{n+1}$ (because of $\sum_{i=0}^{n-1}\left(2 p^{i}-1\right)<2 p^{n}-1$ ). Therefore, $\phi_{n+1}^{*}\left(Q_{n}\right)=\alpha_{n} q_{n}^{n+1}$ for some $\alpha_{n} \in \mathbb{F}_{p}$.

Now consider the diagram of $\mathcal{A}_{p}$-modules


We have just shown that the two compositions agree on $1 \in \mathcal{A}_{p}$. By $\mathcal{A}_{p}$-linearity, the diagram therefore commutes. The inductive assumption shows that equation (6) holds for $j_{l}=n$ in general. Extending (7) to the right, we obtain a commutative diagram of $\mathcal{A}_{p}$-modules

where the unlabelled map is the natural projection and $\bar{\phi}_{n}^{*}$ and $\bar{\phi}_{n+1}^{*}$ are induced by $\phi_{n}^{*}$ and $\phi_{n+1}^{*}$, respectively. The reader may check that the upper sequence is exact. By inductive assumption, $\bar{\phi}_{n}^{*}$ is an isomorphism, therefore $\bar{\phi}_{n+1}^{*}$ is an isomorphism, too. This concludes the proof of equation (5).

We now consider the general case and determine the $\bmod p$ cohomology of $B P / I$ for any satisfying the hypotheses of the theorem. We define ideals

$$
J_{0}=(0) \subset J_{1}=\left(w_{i_{1}}\right) \subset J_{2}=\left(w_{i_{1}}, w_{i_{2}}\right) \subset \ldots \subset I
$$

and prove by induction that

$$
\begin{equation*}
H^{*}\left(B P / J_{k} ; \mathbb{F}_{p}\right) \cong \mathcal{A}_{p} /\left(Q_{i}: i \neq i_{1}, \ldots, i_{k}\right) \tag{8}
\end{equation*}
$$

As $B P / I$ is the homotopy colimit of the sequence

$$
B P / J_{0} \rightarrow B P / J_{1} \rightarrow B P / J_{2} \rightarrow \ldots,
$$

this implies the result:

$$
H^{*}\left(B P / I ; \mathbb{F}_{p}\right) \cong \lim H^{*}\left(B P / J_{k} ; \mathbb{F}_{p}\right) \cong \mathcal{A}_{p} /\left(Q_{i}: i \neq i_{1}, i_{2}, \ldots\right)
$$

For $k=0$, equation (8) is just equation (5). Assume that equation (8) holds for $k \leq n$. By hypothesis, we have $w_{i_{k}} \equiv v_{i_{k}} \bmod I_{i_{k}-1}$. Hence $J_{k}$ is contained in $I_{i_{k}+1}$, and therefore, there are canonical maps of left $B P$-modules $\widetilde{\psi}_{k}: B P / J_{k} \rightarrow$ $P\left(i_{k}+1\right)$. Composing them with the canonical maps $P\left(i_{k}+1\right) \rightarrow P\left(i_{k+1}\right)$ gives maps $\psi_{k}: B P / J_{k} \rightarrow P\left(i_{k+1}\right)$. Now $v_{i_{k+1}}$ and $w_{i_{k+1}}$ agree as endomorphisms of $P\left(i_{k}\right)$, because $v_{i}: P\left(i_{k}\right) \rightarrow P\left(i_{k}\right)$ is homotopically trivial for $i<i_{k}$. Hence, there is a commutative diagram of cofibre sequences

Square $(*)$ commutes by unicity of $\psi_{k+1}$ as a lift of $\psi_{k}$. Taking $\bmod p$ cohomology, we obtain a commutative diagram of $C(0)$-comodules with exact rows, of the form


For $j_{0}, \ldots, j_{l} \in\left\{i_{1}, \ldots, i_{k+1}\right\}$ with $j_{0}<\cdots<j_{l}$, we define

$$
\tilde{q}_{j_{0}, \ldots, j_{l}}^{k+1}=\widetilde{\psi}_{k+1}^{*}\left(q_{j_{0}, \ldots, j_{l}}^{i_{k+1}+1}\right) \in H^{*}\left(B P / J_{k+1} ; \mathbb{F}_{p}\right)
$$

Following analogous arguments as before, we show that

$$
H^{*}\left(B P / J_{k+1} ; \mathbb{F}_{p}\right) \cong C(0) \otimes_{\mathbb{F}_{p}}\left(\bigoplus_{\substack{\left.j_{0}, \ldots, j_{l} \in\left\{i_{1}, \ldots, i_{k}\right\} \\ j_{0}<\cdots<j_{l}\right\}}} \mathbb{F}_{p} \tilde{q}_{j_{0}, \ldots, j_{l}}^{k+1}\right) .
$$

From the fact that the natural map $B P / J_{k+1} \rightarrow H \mathbb{F}_{p}$ factors as

$$
B P / J_{k+1} \xrightarrow{\tilde{\psi}_{k+1}} P\left(i_{k+1}+1\right) \xrightarrow{\phi_{i_{k+1}+1}} H \mathbb{F}_{p},
$$

we easily deduce equation (8) for $k=n+1$, which concludes the proof.

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