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ON THE COHOMOLOGY OF CERTAIN QUOTIENTS OF THE SPECTRUM *BP*

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Abstract. The aim of this note is to present a new, elementary proof of a result of Baas and Madsen on the mod p cohomology of certain quotients of the spectrum BP.

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In 1970s, Baas and Madsen [**1**] calculated the cohomology of certain quotients of MU, the Thom spectrum of the universal bundle over BU. Recall that

$$MU_* = \pi_*(MU) \cong \mathbb{Z}[x_1, x_2, \ldots],$$

where x_i lies in degree $|x_i| = 2i$. The spectra $MU\langle n_1, \ldots, n_q\rangle$ considered by Baas and Madsen are defined for any string of integers $0 < n_1 < \cdots < n_q$ of the form $n_i = 2(p^{j_i} - 1)$, $j_i > 0$, where p is some fixed prime. They satisfy

$$\pi_*(MU\langle n_1, \ldots, n_q\rangle) \cong \mathbb{Z}[x_{n_1}, \ldots, x_{n_q}].$$

Baas and Madsen determined their mod p cohomology, by relying heavily on the Atiyah–Hirzebruch spectral sequence and previous work of Cohen on the Hurewicz homomorphism on MU [**4**].

The aim of this note is to present an alternative proof of Baas–Madsen's result without reference to any spectral sequence or to the work of Cohen. Our arguments use the techniques developed in [**5**] and thus are more transparent than the original ones. They are based on a construction of the spectra $MU\langle n_1, \ldots, n_q\rangle$ which is algebraic in nature and which does not require the use of bordism with singularities any more.

The spectrum MU is a commutative S-algebra, see [**5**]. This leads to a well-behaved homotopy theory of MU-modules. In particular, the derived or homotopy category of MU-modules is a symmetric monoidal category for the smash product over MU. Our constructions take place in this category.

The spectra $MU\langle n_1, \ldots, n_q\rangle$ can be obtained in a standard way, as regular quotients of MU [**5**]:

$$MU\langle n_1, \ldots, n_q\rangle \simeq MU/(x_k : k \neq n_1, \ldots, n_q).$$

Instead of the x_k, we may take any other regular sequence generating the kernel J of the projection $MU_* \to MU\langle n_1, \ldots, n_q\rangle_*$. So it is legitimate to write MU/J for $MU\langle n_1, \ldots, n_q\rangle$. As our interest lies in the mod p cohomology of these MU-modules, we might just as well consider their p-localisations $MU\langle n_1, n_2, \ldots, n_q\rangle_{(p)}$. These spectra admit a much more economical presentation, as quotients of BP. To see this, recall that

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots], \quad |v_i| = 2(p^i - 1),$$

where the v_i are Araki's generators [**6**]. Recall also that BP can be realised as an MU-algebra [**2**], and that the unit $\pi : MU \to BP$ induces an isomorphism

$$\overline{\pi}_* : \mathbb{Z}_{(p)} \otimes MU_*/(x_k : k \neq p^i - 1) \cong BP_*.$$

From this, we deduce that

$$MU\langle n_1, \ldots, n_q\rangle_{(p)} \simeq BP \wedge_{MU} MU/(x_{p^j - 1} : j \neq j_1, \ldots, j_q).$$

As an additional advantage, this presentation exhibits the MU-modules $MU\langle n_1, \ldots, n_q\rangle_{(p)}$ as left BP-modules. Setting $I = BP_* \cdot J$, we can unambiguously write

$$MU\langle n_1, n_2, \ldots, n_q\rangle_{(p)} \simeq BP/I.$$

Following the convention $x_0 = v_0 = p$ and extending the string of the n_i by $n_0 = 0$, we obtain the other family of spectra considered by Baas and Madsen,

$$MU_p\langle n_1, \ldots, n_q\rangle = MU\langle n_1, \ldots, n_q\rangle_{(p)}/p = BP/(I + (p)),$$

as quotients of BP as well.

Recall that the construction of regular quotients is natural in the following sense. If $I_1 \subseteq I_2 \subseteq BP_*$ are two ideals such that I_1 is generated by a regular sequence over BP_* and such that I_2/I_1 is generated by a regular sequence over BP_*/I_1, then there is a canonical map of BP-modules $BP/I_1 \to BP/I_2$ [**5**, V.1.]. Note that the Eilenberg–MacLane spectrum $H\mathbb{F}_p$ is the quotient of BP by the ideal generated by all the v_i's and will serve as a terminal object in our construction.

Let $\mathcal{A}_p$ denote the mod p Steenrod algebra and let Q_i, $i \geq 0$, be the primitive element of degree $2p^i - 1$ of Milnor's basis. Let us write $(y_1, y_2, \ldots)$ for the left ideal of $\mathcal{A}_p$ generated by elements $y_1, y_2, \ldots \in \mathcal{A}_p$. Recalling that $x_{p^k - 1} \equiv v_k$ modulo decomposables, we have the following result.

THEOREM. *Let $I \subset BP_*$ be the ideal generated by a regular sequence $w_{i_1}, w_{i_2}, \ldots$ with $w_{i_k} \equiv v_{i_k}$ modulo decomposables and $0 \leq i_1 < i_2 < \cdots$. Then the natural map $BP/I \to H\mathbb{F}_p$ induces an isomorphism of $\mathcal{A}_p$-modules*

$$H^*(BP/I; \mathbb{F}_p) \cong \mathcal{A}_p/(Q_i : i \neq i_1, i_2, \ldots).$$

As a special case, we obtain the results of Baas–Madsen.

COROLLARY. *There are canonical isomorphisms of $\mathcal{A}_p$-modules:*

$$H^*(MU\langle n_1, \dots, n_q\rangle; \mathbb{F}_p) \cong \mathcal{A}_p/(Q_0, Q_{j_1}, \dots, Q_{j_q});$$
$$H^*(MU_p\langle n_1, \dots, n_q\rangle; \mathbb{F}_p) \cong \mathcal{A}_p/(Q_{j_1}, \dots, Q_{j_q}).$$

Proof. We first prove the result in the particular case of the left BP-modules $P(n) = BP/I_n$, where I_n is the ideal of BP_* generated by the elements $v_0, \dots, v_{n-1}$. That is, $P(0) = BP$ by definition and $P(n)_* \cong \mathbb{F}_p[v_n, v_{n+1}, \dots]$ for $n \geq 1$. Let $\phi_n\colon P(n) \to H\mathbb{F}_p$ be the natural map and set $C(n) = H^*(P(n); \mathbb{F}_p)$. The cofibre sequence of left BP-modules and left BP-morphisms

$$\cdots \to P(n) \xrightarrow{v_n} P(n) \xrightarrow{\eta_n} P(n+1) \xrightarrow{\partial_n} P(n) \to \cdots$$

induces a long exact sequence of $\mathcal{A}_p$-modules in cohomology

$$\cdots \to C(n) \xrightarrow{v_n^*} C(n) \xrightarrow{\partial_n^*} C(n+1) \xrightarrow{\eta_n^*} C(n) \to \cdots .$$

The accurate reader has noticed that we have suppressed the mention of suspension coordinates. As a consequence, ∂_n^* is not a morphism of degree 0 but rather of degree $2p^n - 1$. Proposition B.5.15(b) in [7] implies that the image of v_n under the Hurewicz homomorphism $P(n)_* \to H_*(P(n); \mathbb{F}_p)$ is trivial. Therefore, $(v_n)_*\colon H_*(P(n); \mathbb{F}_p) \to H_*(P(n); \mathbb{F}_p)$ is trivial, and by duality the same holds for $v_n^*\colon C(n) \to C(n)$. As a consequence, we obtain a short exact sequence of $\mathcal{A}_p$-modules:

$$0 \to C(n) \xrightarrow{\partial_n^*} C(n+1) \xrightarrow{\eta_n^*} C(n) \to 0. \tag{1}$$

It obviously splits in the category of $\mathbb{F}_p$-vector spaces. We now inductively define elements $q^{n+1}_{j_0,\dots,j_l} \in C(n+1)$ of degree $\sum_{k=0}^{l}(2p^{j_k} - 1)$, for any $l \in \{0, \dots, n\}$:

$$\eta_n^*(q^{n+1}_{j_0,\dots,j_l}) = \begin{cases} q^n_{j_0,\dots,j_l} & \text{if } j_l < n, \\ 0 & \text{if } j_l = n. \end{cases} \tag{2}$$

For $C(0)$ we take $q^0_{\emptyset} = \phi_0\colon P(0) \to H\mathbb{F}_p$. Assume we have chosen elements $q^k_{j_0,\dots,j_l} \in C(k)$ as indicated for $k \leq n$. For degree reasons, the $q^n_{j_0,\dots,j_l}$ admit unique lifts $q^{n+1}_{j_0,\dots,j_l}$ to $C(n+1)$. For $j_l = n$, we define

$$q^{n+1}_{j_0,\dots,j_l} = \partial_n^*(q^n_{j_0,\dots,j_{l-1}}). \tag{3}$$

Observe that the product on BP induces a coalgebra structure on $C(0)$. Moreover, the action of BP on $P(n)$ gives rise to left $C(0)$-comodule structures on the $C(n)$ with respect to which equation (1) is a short exact sequence of $C(0)$-comodules. We show by induction that there are isomorphisms of $C(0)$-comodules

$$C(n) \cong C(0) \otimes_{\mathbb{F}_p} \left(\bigoplus_{0 \leq j_0 < \cdots < j_l < n} \mathbb{F}_p\, q^n_{j_0,\dots,j_l} \right), \tag{4}$$

so that under this isomorphism, η_n^* maps $y \otimes q^{n+1}_{j_0,\ldots,j_l}$ to $y \otimes \eta_n^*(q^n_{j_0,\ldots,j_l})$. This is clear for $n = 0$. For the inductive step, note that comodules of this form are relatively injective [**6**, A1]. Since equation (1) splits over $\mathbb{F}_p$ and is a sequence of $C(0)$-comodules, it splits as a sequence of $C(0)$-comodules, which proves the claim.

From equation (4), it follows that the primitives of $C(n)$ (with respect to the $C(0)$-comodule structure) are given by

$$P(C(n)) = \bigoplus_{0\le j_0<\cdots<j_l<n} \mathbb{F}_p\, q^n_{j_0,\ldots,j_l}.$$

We now show by induction that

$$C(k) \cong \mathcal{A}_p/(Q_k, Q_{k+1}, \ldots) \tag{5}$$

and that

$$\phi_k^*(Q_{j_0}\cdots Q_{j_l}) = \begin{cases} \alpha_{j_l} q^k_{j_0,\ldots,j_l} & \text{if } j_l < k, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

for some non-zero $\alpha_{j_l} \in \mathbb{F}_p$. In fact, one can show that $\alpha_{j_l} = 1$ for all l, but it will be enough to know equation (6) for our purposes.

It is well known that equation (5) holds for $k = 0$, see the original paper [**3**] or Theorem 4.1.12 in [**6**] for the mod p homology. Also, equation (6) is trivial in this case. Assume inductively that equations (5) and (6) are true for $k \le n$. For the inductive step, recall that $q^{n+1}_{j_0,\ldots,j_l}$ is uniquely determined by equation (2) for $j_l < n$. Thus, equation (6) certainly holds for $k = n+1$ in this case. Now note that $\phi_n\colon P(n) \to H\mathbb{F}_p$ induces an isomorphism in degrees $< 2(p^n - 1)$ on homotopy groups, and so $\phi_n^*\colon \mathcal{A}_p \to C(n)$ is an isomorphism in degrees $< 2p^n - 3$. This implies that ϕ^*_{n+1} sends Q_n to some non-zero primitive element in $C(n+1)$. The only primitives in $C(n+1)$ of degree $|Q_n| = 2p^n - 1$ are the scalar multiples of q_n^{n+1} (because of $\sum_{i=0}^{n-1}(2p^i - 1) < 2p^n - 1$). Therefore, $\phi^*_{n+1}(Q_n) = \alpha_n q_n^{n+1}$ for some $\alpha_n \in \mathbb{F}_p$.

Now consider the diagram of $\mathcal{A}_p$-modules

$$\begin{tikzcd} \mathcal{A}_p \arrow[r, "-\cdot Q_n"] \arrow[d, "\phi_n^*"] & \mathcal{A}_p \arrow[d, "\phi^*_{n+1}"] \\ C(n) \arrow[r, "\alpha_n \partial_n^*"] & C(n+1) \end{tikzcd} . \tag{7}$$

We have just shown that the two compositions agree on $1 \in \mathcal{A}_p$. By $\mathcal{A}_p$-linearity, the diagram therefore commutes. The inductive assumption shows that equation (6) holds for $j_l = n$ in general. Extending (7) to the right, we obtain a commutative diagram of $\mathcal{A}_p$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}_p/(Q_n, \ldots) \arrow[r, "-\cdot Q_n"] \arrow[d, "\bar\phi_n^*"] & \mathcal{A}_p/(Q_{n+1}, \ldots) \arrow[r] \arrow[d, "\bar\phi^*_{n+1}"] & \mathcal{A}_p/(Q_n, \ldots) \arrow[r] \arrow[d, "\bar\phi_n^*"] & 0 \\ 0 \arrow[r] & C(n) \arrow[r, "\alpha_n \partial_n^*"] & C(n+1) \arrow[r, "\eta_n^*"] & C(n) \arrow[r] & 0, \end{tikzcd}$$

where the unlabelled map is the natural projection and $\bar{\phi}_n^*$ and $\bar{\phi}_{n+1}^*$ are induced by ϕ_n^* and ϕ_{n+1}^*, respectively. The reader may check that the upper sequence is exact. By inductive assumption, $\bar{\phi}_n^*$ is an isomorphism, therefore $\bar{\phi}_{n+1}^*$ is an isomorphism, too. This concludes the proof of equation (5).

We now consider the general case and determine the mod p cohomology of BP/I for any satisfying the hypotheses of the theorem. We define ideals

$$J_0 = (0) \subset J_1 = (w_{i_1}) \subset J_2 = (w_{i_1}, w_{i_2}) \subset \ldots \subset I$$

and prove by induction that

$$H^*(BP/J_k; \mathbb{F}_p) \cong \mathcal{A}_p/(Q_i : i \neq i_1, \ldots, i_k). \tag{8}$$

As BP/I is the homotopy colimit of the sequence

$$BP/J_0 \to BP/J_1 \to BP/J_2 \to \ldots,$$

this implies the result:

$$H^*(BP/I; \mathbb{F}_p) \cong \lim H^*(BP/J_k; \mathbb{F}_p) \cong \mathcal{A}_p/(Q_i : i \neq i_1, i_2, \ldots).$$

For $k = 0$, equation (8) is just equation (5). Assume that equation (8) holds for $k \leq n$. By hypothesis, we have $w_{i_k} \equiv v_{i_k} \mod I_{i_k - 1}$. Hence J_k is contained in I_{i_k+1}, and therefore, there are canonical maps of left BP-modules $\widetilde{\psi}_k \colon BP/J_k \to P(i_k + 1)$. Composing them with the canonical maps $P(i_k + 1) \to P(i_{k+1})$ gives maps $\psi_k \colon BP/J_k \to P(i_{k+1})$. Now $v_{i_{k+1}}$ and $w_{i_{k+1}}$ agree as endomorphisms of $P(i_k)$, because $v_i \colon P(i_k) \to P(i_k)$ is homotopically trivial for $i < i_k$. Hence, there is a commutative diagram of cofibre sequences

$$\begin{tikzcd}
\cdots \arrow[r] & BP/J_k \arrow[r, "w_{i_{k+1}}"] \arrow[d, "\psi_k"] & BP/J_k \arrow[r] \arrow[d, "\psi_k"] & BP/J_{k+1} \arrow[r] \arrow[d, "\widetilde{\psi}_{k+1}"] \arrow[r, phantom, shift right=6, "(*)"] & BP/J_k \arrow[r] \arrow[d, "\psi_k"] & \cdots \\
\cdots \arrow[r] & P(i_{k+1}) \arrow[r, "v_{i_{k+1}}"] & P(i_{k+1}) \arrow[r] & P(i_{k+1}+1) \arrow[r] & P(i_{k+1}) \arrow[r] & \cdots .
\end{tikzcd}$$

Square $(*)$ commutes by unicity of ψ_{k+1} as a lift of ψ_k. Taking mod p cohomology, we obtain a commutative diagram of $C(0)$-comodules with exact rows, of the form

$$\begin{tikzcd}
0 \arrow[r] & H^*(BP/J_k; \mathbb{F}_p) \arrow[r] & H^*(BP/J_{k+1}; \mathbb{F}_p) \arrow[r] & H^*(BP/J_k; \mathbb{F}_p) \arrow[r] & 0 \\
0 \arrow[r] & C(i_{k+1}) \arrow[r] \arrow[u, "\psi_k^*"] & C(i_{k+1}+1) \arrow[r] \arrow[u, "\widetilde{\psi}_{k+1}^*"] & C(i_{k+1}) \arrow[r] \arrow[u, "\psi_k^*"] & 0.
\end{tikzcd}$$

For $j_0, \ldots, j_l \in \{i_1, \ldots, i_{k+1}\}$ with $j_0 < \cdots < j_l$, we define

$$\tilde{q}^{k+1}_{j_0,\ldots,j_l} = \widetilde{\psi}^*_{k+1}\big(q^{i_{k+1}+1}_{j_0,\ldots,j_l}\big) \in H^*(BP/J_{k+1}; \mathbb{F}_p).$$

Following analogous arguments as before, we show that

$$H^*(BP/J_{k+1};\mathbb{F}_p) \cong C(0) \otimes_{\mathbb{F}_p} \left(\bigoplus_{\substack{j_0,\dots,j_l \in \{i_1,\dots,i_k\} \\ j_0<\dots<j_l}} \mathbb{F}_p\, \tilde{q}^{k+1}_{j_0,\dots,j_l} \right).$$

From the fact that the natural map $BP/J_{k+1} \to H\mathbb{F}_p$ factors as

$$BP/J_{k+1} \xrightarrow{\widetilde{\psi}_{k+1}} P(i_{k+1}+1) \xrightarrow{\phi_{i_{k+1}+1}} H\mathbb{F}_p,$$

we easily deduce equation (8) for $k = n+1$, which concludes the proof. □

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