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Clifford algebras from quotient ring spectra

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Abstract. We give natural descriptions of the homology and cohomology algebras of regular quotient ring spectra of even E_∞-ring spectra. We show that the homology is a Clifford algebra with respect to a certain bilinear form naturally associated to the quotient ring spectrum F. To identify the cohomology algebra, we first determine the derivations of F and then prove that the cohomology is isomorphic to the exterior algebra on the module of derivations. We treat the example of the Morava K-theories in detail.

1. Introduction

It has long been a difficult problem to realize quotient constructions in stable homotopy theory. The situation changed completely with the introduction of point-set categories of spectra endowed with monoidal structures, for instance in [7]. Since then, the definition of a large class of quotient constructions has become a pure formality. Namely, suppose that R is an E_∞-ring spectrum and that $I \subseteq \pi_*(R) = R_*$ is an ideal of the homotopy ring of R generated by a regular sequence. Then there is a spectrum F equipped with a map $R \to F$ which induces an isomorphism $F_* \cong R_*/I$. Moreover, F is unique up to equivalence, see Remark 2.3.

Such regular quotients of R arise naturally as objects in the derived category of R-module spectra $\mathscr{D}_R$. Working in this category makes it much easier to study multiplicative structures. Partly, this is due to the fact that $\mathscr{D}_R$ is equipped with a monoidal structure, induced by the smash product $\wedge_R$. In particular, Strickland [14] showed that a regular quotient can always be realized as an R-ring spectrum, i.e. as a monoid in $\mathscr{D}_R$, if R_* forms a domain and R is even, meaning that R_* is trivial in odd degrees.

A fundamental problem is to compute the homology and cohomology algebras of regular quotients $F_*^R(F) = \pi_*(F \wedge_R F)$ and $F_R^*(F) = \mathscr{D}_R^*(F, F)$, respectively. Whereas the underlying graded F_*-modules are trivial to determine if R is even, the multiplicative structures have only been identified in special cases up to now, see [1–3, 10] and [14]. The main goal of this article is to determine the homology and

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cohomology algebras in general. Our descriptions are valid for arbitary products on F and functorial in nature. In particular, they are independent on any choices, such as the fixing of generators of I. This is important in [8], where the results proved here are used to solve the classification problem of R-ring structures on regular quotients.

We do not restrict to regular quotient rings, but consider arbitrary quotient rings of an even E_∞-ring R, i.e. R-rings F with $F_* \cong R_*/I$ for some ideal $I \subseteq R_*$. We write $F = R/I$ for such an F. We study the homology and cohomology of F with respect to any quotient R-ring spectrum k which comes with a unital map $\pi\colon F \to k$. We call (F, k, π) with these properties an admissible pair. An important example of an admissible pair is given by (F, k, π), where F is a quotient ring, $k = F$ as an R-module, but endowed with a possibly different product, and where π is the identity map 1_F.

Our arguments are based on a canonical homomorphism of k_*-modules, the *characteristic homomorphism*

$$\varphi\colon k_* \otimes_{F_*} I/I^2[1] \longrightarrow k_*^R(F).$$

Here, $I/I^2[1]$ denotes the graded F_*-module I/I^2 with degrees raised by one. We show that φ is independent of the products on F and k and functorial in both F and k. We then use φ to define the *characteristic bilinear form*

$$b\colon \Big(k_* \otimes_{F_*} I/I^2[1]\Big) \otimes_{k_*} \Big(k_* \otimes_{F_*} I/I^2[1]\Big) \longrightarrow k_*,$$

Letting $q\colon k_* \otimes_{F_*} I/I^2[1] \to k_*$ be the associated quadratic form and writing $\mathcal{C}\ell(k_* \otimes_{F_*} I/I^2[1], q)$ for the Clifford algebra with respect to q, we prove:

Theorem. *For an admissible pair $(F = R/I, k, \pi)$, the characteristic homomorphism lifts to a natural homomorphism of k_*-algebras*

$$\Phi\colon \mathcal{C}\ell(k_* \otimes_{F_*} I/I^2[1], q) \longrightarrow k_*^R(F).$$

If F is a regular quotient, then Φ is an isomorphism.

We show that the characteristic bilinear form of $(F, F^{\mathrm{op}}, 1_F)$ is trivial, where F^{op} denote the opposite ring of F. This leads to a new proof of the fact that $F_*^R(F^{\mathrm{op}}) \cong \Lambda(I/I^2[1]))$ is an exterior algebra [10]. If F is a *diagonal* regular quotient, i.e. the smash product of quotient rings of the form R/x_i with $x_i \in R_*$, the characteristic bilinear form of F is diagonal. We prove that the diagonal elements are determined by the commutativity obstructions of R/x_i introduced in [14].

In the second part of the article, we consider the cohomology modules $k_R^*(F)$, endowed with the profinite topology. We first show:

Proposition. *For an admissible pair $(F = R/I, k, \pi)$, there exists a natural continuous homomorphism of k_*-modules*

$$\Psi\colon k_R^*(F) \longrightarrow \mathrm{Hom}_{F_*}^*(\Lambda(I/I^2[1]), k_*).$$

If F is a regular quotient ring, Ψ is a homeomorphism.

For the determination of the cohomology algebra of a regular quotient ring F, we consider the group of derivations $\mathscr{D}\mathrm{er}^*_R(F, F)$. More generally, we study $\mathscr{D}\mathrm{er}^*_R(F, k)$ for any multiplicative admissible pair (F, k, π), i.e. one for which π is multiplicative. It inherits from $k^*_R(F)$ a linear topology.

Proposition. *For a multiplicative admissible pair $(F = R/I, k, \pi)$ such that F and k are regular quotients, there is a natural homeomorphism*

$$\psi\colon \mathscr{D}\mathrm{er}^*_R(F, k) \longrightarrow \mathrm{Hom}^*_{F_*}(I/I^2[1], k_*).$$

We then describe the cohomology algebra of F in terms of its derivations. Let $\widehat{\Lambda}(\mathscr{D}\mathrm{er}^*_R(F, F))$ denote the completed exterior algebra on $\mathscr{D}\mathrm{er}^*_R(F, F)$.

Theorem. *For a regular quotient ring $F = R/I$, there is a canonical homeomorphism of F^*-algebras $F^*_R(F) \cong \widehat{\Lambda}(\mathscr{D}\mathrm{er}^*_R(F, F))$.*

These two statements are generalizations of results of Strickland [14]. He considered the special case where F is a diagonal quotient ring of R.

In the last section, we discuss the case of the Morava K-theories $K(n)$. We determine explicitly the bilinear form $b_{K(n)}$. The reader will find in [8] more examples of computations of characteristic bilinear forms.

Here is an overview over the contents of this article. In Sect. 2, we recall some background material from [14], construct the characteristic homomorphism and characteristic bilinear form of admissible pairs and compute them in special cases. In Sect. 3, we consider the homology of admissible pairs. In Sect. 4, we study derivations and the cohomology of admissible pairs. Finally, in Sect. 5, we discuss the example of Morava K-theories.

1.1. Notation and conventions

For definiteness, we work in the framework of $\mathbb{S}$-modules of [7]. In this setting, E_∞-ring spectra correspond to commutative $\mathbb{S}$-algebras. Throughout the paper, R denotes an even commutative $\mathbb{S}$-algebra. We also assume that the coefficient ring R_* of R is a domain (see Remark 2.11 for an explanation). Associated to R is the homotopy category $\mathscr{D}_R$ of R-module spectra. For simplicity, we refer to its objects as R-modules. The smash product $\wedge_R$ endows $\mathscr{D}_R$ with a symmetric monoidal structure. We will abbreviate $\wedge_R$ by $\wedge$ throughout the paper.

Monoids in $\mathscr{D}_R$ are called R-ring spectra or just R-rings. Unless otherwise specified, we use the generic notation $\eta_F\colon R \to F$ and $\mu_F\colon F \wedge F \to F$ for the unit and the multiplication maps of an R-ring F. Mostly, η_F will be clear from the context, in which case we call a map $\mu_F\colon F \wedge F \to F$ which gives F the structure of an R-ring an R-product or just a product. We denote the opposite of an R-ring F by F^{op}. Its product is given by $\mu_{F^{\mathrm{op}}} = \mu_F \circ \tau$, where $\tau\colon F \wedge F \to F \wedge F$ is the switch map. An R-ring (F, μ_F, η_F) determines multiplicative homology and cohomology theories $F^R_*(-) = \pi_*(F \wedge -) = \mathscr{D}^{-*}_R(R, F \wedge -)$ and $F^*_R(-) = \mathscr{D}^*_R(-, F)$, respectively, on $\mathscr{D}_R$.

Since we are working with non-commutative R-rings, we must carefully describe the various module structures involved. For an R-ring k and an R-module M, the homology $k_*^R(M)$ is a k_*-bimodule in a natural way. Even if k_* is commutative, the left and right k_*-actions may well be different. However, if we assume that k is a quotient of R, by which we mean that the unit map $\eta_k\colon R \to k$ induces a surjection on homotopy groups (see Definition 2.1 below), the left and right k_*-actions agree. In this case, we can refer to $k_*^R(M)$ as a k_*-module without any ambiguity. A similar discussion applies to cohomology $k_R^*(M)$.

We will assume that k is a quotient of R for the rest of this section.

For R-modules M and N, we write

$$\kappa_k\colon k_*^R(M) \otimes_{k_*} k_*^R(N) \longrightarrow k_*^R(M \wedge N)$$

for the Künneth homomorphism, a homomorphism of k_*-modules. Note that k is not required to be commutative for the definition of κ_k (see [15, §2]). If $k_*^R(M)$ or $k_*^R(N)$ is k_*-flat, then κ_k is an isomorphism of k_*-modules.

Let F be a second R-ring. The composition

$$m_F^k\colon k_*^R(F) \otimes_{k_*} k_*^R(F) \xrightarrow{\kappa_k} k_*^R(F \wedge F) \xrightarrow{k_*^R(\mu_F)} k_*^R(F)$$

defines a (central) k_*-algebra structure on $k_*^R(F)$, where the unit is given by $(1_k \wedge \eta_F)_*\colon k_* \to k_*^R(F)$. In unambiguous situations, we will write $a \cdot b$ for $m_F^k(a \otimes b)$.

To relate the homology $k_*^R(M)$ and cohomology $k_R^*(M)$, we will use the Kronecker duality morphism

$$d\colon k_R^*(M) \longrightarrow \mathrm{Hom}_{k_*}^*(k_*^R(M), k_*),$$

which associates to $f\colon M \to k$ the homomorphism of k^*-modules $d(f) = (\mu_k)_* k_*^R(f)$. If $k_*^R(M)$ is k_*-free, d is an isomorphism. This implies that the Hurewicz homomorphism

$$k_*^R(-)\colon k_R^*(M) \longrightarrow \mathrm{Hom}_{k_*}^*(k_*^R(M), k_*^R(k))$$

is injective whenever $k_*^R(M)$ is k_*-free. See e.g. [15, Lemma 6.2] for a detailed discussion, which covers in particular the case where k is non-commutative.

For R-modules M and N, we write $\zeta\colon M_* \otimes_{R_*} N_* \longrightarrow (M \wedge N)_*$ for the canonical map, which is natural in the following graded sense. Two maps of R-modules $f\colon \Sigma^k M \to M'$ and $g\colon \Sigma^l N \to N'$ induce commutative diagrams

$$\begin{tikzcd} M_m \otimes N_n \arrow[r, "\zeta"] \arrow[d, "f_* \otimes g_*"'] & (M \wedge N)_{m+n} \arrow[d, "(-1)^{m \cdot l} f \wedge g"] \\ M'_{k+m} \otimes N'_{l+n} \arrow[r, "\zeta"] & (M' \wedge N')_{k+m+l+n}. \end{tikzcd} \tag{1.1}$$

We write $M_*[d]$ for the d-fold suspension of a graded abelian group M_*, so $(M_*[d])_k = M_{k-d}$. With this convention, we have $(\Sigma^d M)_* = M_*[d]$ for an R-module M. We denote the image of some element $\alpha \in M_k$ under the shift isomorphism $M_* \cong M_*[d]$ by $\alpha[d] \in (M_*[d])_{k+d}$. We use the convention $M^* = M_{-*}$. If the ground ring is clear from the context, we omit it from the tensor product symbol $\otimes$ from now on.

2. The characteristic bilinear form

2.1. Quotient modules, quotient rings

The point of this subsection is to introduce some convenient terminology and to recall some basic constructions in the category $\mathscr{D}_R$, mainly from [14].

Definition 2.1. A *quotient module* of R is an R-module F with a map of R-modules $\eta_F\colon R \to F$ which induces a surjection on homotopy groups, that is $F_* \cong R_*/I$ where $I \subseteq R_*$ is an ideal. A morphism $f\colon F \to G$ of quotient modules of R is a map of R-modules such that $f \circ \eta_F = \eta_G$.

Let F be a quotient module of R with $F_* = R_*/I$ and let X be the homotopy fibre of $\eta_F\colon R \to F$. As the canonical map $X \to R$ induces an isomorphism $X_* \cong I \subseteq R_*$, we write I for X. So we have a cofibre sequence of the form

$$I \xrightarrow{\iota} R \xrightarrow{\eta_F} F \xrightarrow{\beta} \Sigma I. \tag{2.1}$$

We will write $F = R/I$ in the sequel.

Recall that a graded R_*-module F_* is said to be a (finite) regular quotient of R_* if it is isomorphic to $R_*/(x_1, x_2, \ldots)$ for some (finite) regular sequence $(x_1, x_2, \ldots)$ in R_*. There is the following analogous topological notion.

Definition 2.2. A quotient module $F = R/I$ of R is a (finite) *regular quotient module* of R if the ideal I is generated by some (finite) regular sequence $(x_1, x_2, \ldots)$ in R_*.

We now recall the definition of the building blocks of regular quotients of R. The coefficient ring R_* may be canonically identified with the graded endomorphisms of R in $\mathscr{D}_R$. If x is a given element of R_d, we write R/x for the homotopy cofibre of $x\colon \Sigma^d R \to R$. As R_* lies in even degrees, R/x is well defined up to canonical homotopy equivalence. By construction, R/x fits into a cofibre sequence of the form

$$\Sigma^d R \xrightarrow{x} R \xrightarrow{\eta_x} R/x \xrightarrow{\beta_x} \Sigma^{d+1} R. \tag{2.2}$$

Since R_* is a domain, $(R/x)_* \cong R_*/(x)$.

Remark 2.3. If $F = R/I$ is a regular quotient and $(x_1, x_2, \ldots)$ is a regular sequence generating I, then F is isomorphic in $\mathscr{D}_R$ to

$$R/x_1 \wedge R/x_2 \wedge \cdots := \operatorname{hocolim}_k R/x_1 \wedge \cdots \wedge R/x_k.$$

Due to the lack of a specific reference, we give a brief outline of the argument underlying the proof. We construct by induction, using [7, V.1, Lemma 1.5] factorizations $R/x_1 \wedge \cdots \wedge R/x_k \to F$ of the unit $\eta\colon R \to F$ and from these a map $\bar{\eta}\colon \operatorname{hocolim}_k R/x_1 \wedge \cdots \wedge R/x_k \to F$. By construction, $\bar{\eta}$ induces an isomorphism on homotopy groups and is thus an isomorphism in $\mathscr{D}_R$.

Definition 2.4. A (regular) *quotient ring* of R is an R-ring (F, μ_F, η_F) such that (F, η_F) is a (regular) quotient module of R.

Products on regular quotients of the form R/x have been studied in [14, Sect. 3].

Proposition 2.5. *Let $x \in R_d$. If u is in R_{2d+2}/x and μ is a product on R/x, then $\mu + u \circ (\beta_x \wedge \beta_x)$ is another product. This construction gives a free transitive action of the group R_{2d+2}/x on the set of products on R/x.*

Proposition 2.6. *There is a natural map c from the set of products on R/x to R_{2d+2}/x such that $c(\mu) = 0$ if and only if μ is commutative.*

Recall that R-ring maps $f\colon A \to C$ and $g\colon B \to C$ are said to commute if $\mu_C \circ (f \wedge g) = \mu_C \circ \tau \circ (f \wedge g)\colon A \wedge B \to C$.

Remark 2.7. Let $F = R/I$ be a regular quotient of R and $(x_1, x_2, \ldots)$ a regular sequence generating the ideal I. For any products μ_i on R/x_i, $i \geqslant 1$, [14, Prop. 4.8] implies that there is a unique product on $F = R/I$ such that the natural maps $R/x_i \to F$ are multiplicative and commute. See Proposition 2.27 for a generalization.

Definition 2.8. We call F, endowed with the product described in Remark 2.7, the *smash ring spectrum* of the R/x_i. If we need to be more precise, we refer to the product map μ_F on F as the *smash ring product* of the μ_i.

For the next definition, recall that two R-ring spectra F and G are called equivalent if there is an isomorphism $f\colon F \to G$ in $\mathscr{D}_R$ which is multiplicative.

Definition 2.9. We call a regular quotient ring F of R *diagonal* if it is the smash ring spectrum of ring spectra R/x_i, where $(x_1, x_2, \ldots)$ is a regular sequence. We say that F is *diagonalizable* if it is equivalent to a diagonal regular quotient ring.

Corollary 2.10. *Any regular quotient ring of R_* can be realized as the coefficient ring of a diagonal R-ring.*

Proof. Let $F_* = R_*/(x_1, x_2, \ldots)$ be a regular quotient of R_*. By Remark 2.3, the R-module $F = R/x_1 \wedge R/x_2 \wedge \cdots$ satisfies $\pi_*(F) = F_*$. By Proposition 2.5, every R/x_i admits a product. Finally, endow F with the induced smash ring product. □

Remark 2.11. Note that the proof requires each of the elements x_k of the regular sequence to be a non-zero divisor. This is guaranteed by our assumption that R_* is a domain.

Let $(R/x, \mu, \eta)$ be a regular quotient ring, $x \in R_d$, and A an even R-ring. Clearly, there is a unital map $j\colon R/x \to A$ if and only if x maps to zero in A_*, and j is unique if it exists. We will extensively use the following fact:

Proposition 2.12. *Let A and x be as above and assume that A is a quotient ring of R. Then there exists a product on R/x such that the canonical map $j\colon R/x \to A$ is multiplicative.*

Proof. Choose an arbitrary product μ on $F = R/x$. By Proposition 2.5, any other product on F is of the form $u \cdot \mu := \mu + u(\beta \wedge \beta)$, for some $u \in R_{2d+2}/x$. By Proposition [14, Prop. 3.15], there is an obstruction $d_F(u \cdot \mu) \in A_{2d+2}$ which vanishes if and only if j is multiplicative for $u \cdot \mu$. Furthermore, $d_F(u \cdot \mu)$ is related to $d_F(\mu)$ by $d_F(u \cdot \mu) = d_F(\mu) + j_*(u)$. Thus, on choosing u with $j_*(u) = -d_F(\mu)$, j is multiplicative with respect to $u \cdot \mu$. □

Corollary 2.13. *Let $F = R/I$ be a commutative regular quotient ring of R. Then F is diagonalizable.*

Proof. Let $(x_1, x_2, \ldots)$ be a regular sequence which generates I. By Proposition 2.12, there are products μ_i on R/x_i such that the canonical maps $j_i : R/x_i \to F$ are multiplicative. By commutativity of F, the j_i commute with each other. By [14, Prop. 4.8], they therefore induce a multiplicative equivalence $j : \bigwedge_{i \geqslant 1} R/x_i \to F$. □

2.2. Admissible pairs

In this subsection we introduce the category of admissible pairs. It will play a central role in the sequel.

Definition 2.14. An *admissible pair* is a triple (F, k, π) consisting of two quotient R-rings (F, μ_F, η_F), (k, μ_k, η_k) and a unital map of R-modules $\pi : F \to k$, i.e. an R-morphism such that $\pi \circ \eta_F = \eta_k$. If π is a map of R-ring spectra, we call (F, k, π) a *multiplicative admissible pair*.

Note that $\pi_* : F_* \to k_*$ is always a ring homomorphism, even for non-multiplicative admissible pairs, as $(\eta_F)_* : R_* \to F_*$ is surjective. We may therefore view k_* as an F_*-module.

Remark 2.15. If $F = R/I$ and k are quotient R-rings, a necessary condition for the existence of a map π making (F, k, π) into an admissible pair is that $(\eta_k)_*(I) = 0$. If F is a *regular* quotient ring, this condition is sufficient, by [14, Lemma 4.7]. If $F = R/x$, the map π is unique.

Admissible pairs (F, k, π) form the objects of a category. The morphisms between two admissible pairs $(F = R/I, k, \pi)$ and $(G = R/J, l, \pi')$ are pairs of R-ring maps $(f : F \to G,\ g : k \to l)$ which make the diagram

$$\begin{tikzcd} F \arrow[r, "\pi"] \arrow[d, "f"] & k \arrow[d, "g"] \\ G \arrow[r, "\pi'"] & l \end{tikzcd} \tag{2.3}$$

commutative. Observe that in this case $I \subseteq J$. If we say that a certain construction is "natural in F and k", we mean that it is a functor on this category. Similarly, we refer to a morphism as being "natural in F and k" if it defines a natural transformation of functors defined on this category.

Example 2.16. An important example of an admissible pair (F, k, π) is the special case where k coincides with F as an R-module and $\pi = 1_F$, the identity on F, but where we distinguish two products μ and ν on F.

2.3. The characteristic homomorphism

Let $(F = R/I, k, \pi)$ be an admissible pair. We define a homomorphism of F_*-modules

$$\varphi_F^k \colon I/I^2[1] \longrightarrow k_*^R(F), \tag{2.4}$$

which is natural in F and k. Here, we view $k_*^R(F)$ as an F_*-module via the ring homomorphism $\pi_* \colon F_* \to k_*$ and the k_*-module structure on $k_*^R(F)$ as discussed in Sect. 1.1.

Applying $k \wedge -$ to the cofibre sequence (2.1) yields

$$k \xrightarrow{1\wedge\eta_F} k \wedge F \xrightarrow{1\wedge\beta} k \wedge \Sigma I \xrightarrow{1\wedge\iota} \Sigma k. \tag{2.5}$$

We consider the map $\psi \colon k \wedge F \to k$, defined as $\psi = \mu_k \circ (1 \wedge \pi)$, and we note that ψ is a retraction of $1 \wedge \eta_F$, natural in F and k. Observe that if $k = F$ as R-modules (Example 2.16), then ψ is just the second product ν of F.

The cofibre sequence (2.5) induces a short exact sequence of k_*-modules:

$$0 \longrightarrow k_* \underset{k_*^R(\eta_F)}{\overset{\psi_*}{\rightleftarrows}} k_*^R(F) \xrightarrow[k_*^R(\beta)]{} k_*^R(\Sigma I) \longrightarrow 0 \tag{2.6}$$

The retraction ψ_*, which is easily seen to be a k_*-homomorphism, induces a k_*-linear section $\sigma_* \colon k_*^R(\Sigma I) \to k_*^R(F)$, which is natural in F and k as well. So there is a natural isomorphism of k_*-modules

$$k_*^R(F) \cong k_* \oplus k_*^R(\Sigma I),$$

given by $b \longmapsto (\psi_*(b), k_*^R(\beta)(b))$, with inverse $(c, a) \longmapsto k_*^R(\eta_F)(c) + \sigma_*(a)$.

We define φ_F^k to be the composite

$$\varphi_F^k \colon I/I^2[1] \cong F_* \otimes I[1] \xrightarrow{\pi_*\otimes 1} k_* \otimes I[1] \xrightarrow{\zeta} (k \wedge \Sigma I)_* \xrightarrow{\sigma_*} k_*^R(F),$$

where ζ is the map as considered in (1.1). Observe that φ_F^k is a homomorphism of F_*-modules.

Definition 2.17. We call φ_F^k the characteristic homomorphism of the admissible pair (F, k, π). If k and F are understood, we just write φ.

For another description of φ based on a Künneth spectral sequence compare Remark 3.6.

We defer the proof of the following fact to Sect. 2.5:

Proposition 2.18. *The characteristic homomorphism φ_F^k does not depend on the products on F and k.*

Recall that any regular quotient module F of R can be realized as a regular quotient ring (Proposition 2.10). The following definition is meaningful by Proposition 2.18:

Definition 2.19. The characteristic homomorphism of a regular quotient module $F = R/I$ is the characteristic homomorphism $\varphi_F^F\colon I/I^2[1] \to F_*^R(F)$ of the multiplicative admissible pair $(F, F, 1_F)$, where F is endowed with an arbitrary product. We will denote it by φ_F or simply by φ if F is understood.

2.4. The characteristic bilinear form

Assume that (F, k, π) is an admissible pair. For brevity, we write ${}^\pi\varphi$ for the composition of the following k_*-homomorphisms

$$ {}^\pi\varphi\colon k_* \otimes_{F_*} I/I^2[1] \xrightarrow{1\otimes\varphi} k_* \otimes_{k_*} k_*^R(F) \cong k_*^R(F). $$

Recall the R-module map $\psi\colon k \wedge F \to k$ from Sect. 2.3 and the algebra structure on $k_*^R(F)$ from Sect. 1.1.

We define b_F^k to be the composite of k_*-homomorphisms

$$ b_F^k\colon (k_* \otimes_{F_*} I/I^2[1])^{\otimes 2} \xrightarrow{{}^\pi\varphi^{\otimes 2}} k_*^R(F)^{\otimes 2} \xrightarrow{m_F^k} k_*^R(F) \xrightarrow{\psi_*} k_*. $$

Definition 2.20. We call b_F^k the characteristic bilinear form associated to the admissible pair (F, k, π).

Observe that b_F^k preserves the gradings and is natural in F and k.

In the following, we write $\bar{x}$ for either of the elements $(x + I^2)[1] \in I/I^2[1]$ or $1 \otimes (x + I^2)[1] \in k_* \otimes_{F_*} I/I^2[1]$ associated to some $x \in I$. The context will make it clear which element is meant.

Associated to b_F^k is the quadratic form $q_F^k\colon k_* \otimes_{F_*} I/I^2[1] \to k_*$, defined by $q_F^k(\bar{x}) = b_F^k(\bar{x} \otimes \bar{x})$ for $x \in I$. Note that q_F^k doubles the degrees.

In the special situation of Example 2.16 ($k = F$ as R-modules and $\pi = 1_F$), we write b_μ^ν and q_μ^ν instead of b_F^F and q_F^F, to keep track of the products. If $\mu = \nu$, we simply write b_F and q_F. If $\mu = \nu^{\mathrm{op}}$, we write $b_{F^{\mathrm{op}}}^F$ and $q_{F^{\mathrm{op}}}^F$. Whenever no confusion can be caused, we simply write b and q.

If (F, k, π) is multiplicative, its bilinear form b_F^k is determined by b_F as well as by b_k. To describe the relationship, let $k_* \otimes b_F$ denote the bilinear form on the k_*-module $k_* \otimes_{F_*} I/I^2[1]$ determined by

$$ (k_* \otimes b_F)((1 \otimes \bar{x}) \otimes (1 \otimes \bar{y})) = \pi_*(b_F(\bar{x} \otimes \bar{y})). $$

Let moreover $\pi^*(b_k)$ be the bilinear form on $k_* \otimes_{F_*} I/I^2[1]$ determined by

$$ \pi^*(b_k)((1 \otimes \bar{x}) \otimes (1 \otimes \bar{y})) = b_k(\bar{\pi}_*(\bar{x}) \otimes \bar{\pi}_*(\bar{y})), $$

where $\bar{\pi}_*\colon I/I^2 \to J/J^2$ is the canonical homomorphism and where $k = R/J$.

Proposition 2.21. *The characteristic bilinear form of a multiplicative admissible pair (F, k, π) is given by $b_F^k = k_* \otimes b_F = \pi^*(b_k)$.*

Proof. This follows from naturality, by considering the admissible pairs $(F, F, 1_F)$, (F, k, π) and $(k, k, 1_k)$. □

The bilinear form b_F^k will be determined for various k and F in the next subsection. At this point, we can offer the following general statement, which will be useful in the sequel.

Proposition 2.22. *Let (F, k, π) be a multiplicative admissible pair. Then the characteristic bilinear form $b_{F^{op}}^k$ of the admissible pair (F^{op}, k, π) is trivial. In particular, $b_{F^{op}}^F = 0$ for a quotient ring F.*

Proof. We first show that $b_{F^{\mathrm{op}}}^F = 0$ for a quotient ring F. The natural left and right actions of F on $F \wedge F$ and F induce left actions of $F \wedge F^{\mathrm{op}}$. The product map $\mu\colon F \wedge F \to F$ respects these actions, and so $\mu_*\colon F_*^R(F) \to F_*$ is a map of left $F_*^R(F^{\mathrm{op}})$-modules. On $F_*^R(F^{\mathrm{op}})$, the $F_*^R(F^{\mathrm{op}})$-action is the same as the one given by left multiplication in the algebra $F_*^R(F^{\mathrm{op}})$. As a consequence, we have for any $x, y \in I$ with residue classes $\bar{x}, \bar{y} \in k_* \otimes_{F_*} I/I^2[1]$ (where $\cdot^{\mathrm{op}}$ denotes the product in $F_*^R(F^{\mathrm{op}})$):

$$\begin{aligned} b_{F^{\mathrm{op}}}^F(\bar{x} \otimes \bar{y}) &= \psi_*(\varphi(\bar{x}) \cdot^{\mathrm{op}} \varphi(\bar{y})) = \mu_*(\varphi(\bar{x}) \cdot^{\mathrm{op}} \varphi(\bar{y})) \\ &= \varphi(\bar{x}) \cdot \mu_*(\varphi(\bar{y})) = 0, \end{aligned}$$

because $\psi_* = \mu_*$ (second equality), μ_* is $F_*^R(F^{\mathrm{op}})$-linear (third equality) and F_* is concentrated in even degrees (fourth equality).

The statement for arbitrary multiplicative admissible pairs (F, k, π) now follows directly from Proposition 2.21. □

Corollary 2.23. *For a commutative quotient ring F, we have $b_F = 0$.*

2.5. The test case $F = R/x$

Assume that $(R/x, k, \pi)$ is an admissible pair, where $x \in R_d$. We will first determine its characteristic homomorphism and bilinear form.

We need some preparations. Applying $k_*^R(-)$ to the cofibre sequence (2.2) gives the short exact sequence of k_*-modules

$$0 \longrightarrow k_*^R(R) \xrightarrow{k_*^R(\eta_x)} k_*^R(R/x) \xrightarrow{k_*^R(\beta_x)} k_*^R(\Sigma^{d+1}R) \longrightarrow 0. \tag{2.7}$$

Because of $k_{\mathrm{odd}} = 0$, $k_*^R(\Sigma^{d+1}R) \cong k_*[d+1]$ and because d is even, there exists a unique class $a_x \in k_*^R(R/x)$ with $k_*^R(\beta_x)_*(a_x) = 1[d+1]$. Therefore

$$k_*^R(R/x) \cong k_* \oplus k_*[d+1], \tag{2.8}$$

where $1 \in k_*^R(R/x)$ corresponds to $(1, 0)$ and $a_x \in k_*^R(R/x)$ to $(0, 1[d+1])$.

Remark 2.24. By (2.8), the k_*-module $k_*^R(R/x)$ is k_*-free. As a consequence, $k_*^R(F)$ is k_*-free for any regular quotient F. Namely, by Remark 2.3 and a Künneth isomorphism, $k_*^R(F) \cong \operatorname{colim}_k k_*^R(R/x_1) \otimes \cdots \otimes k_*^R(R/x_k)$ is k_*-free. For another argument based on a Künneth spectral sequence, see Remark 3.6.

The k_*-module $k_* \otimes_{F_*} I/I^2[1]$ is freely generated by $\bar{x}$. Therefore, $b = b^k_{R/x}$ and $q = q^k_{R/x}$ are determined by the single element $b(\bar{x} \otimes \bar{x}) = q(\bar{x})$.

Lemma 2.25. *We have* $\varphi^k_{R/x}(\bar{x}) = a_x$ *and* $q^k_{R/x}(\bar{x}) \cdot 1 = a_x^2$.

Proof. The first equality is a direct consequence of the definition of φ. For the second one, notice that by definition of q and by the first equality, we have

$$q(\bar{x}) = \psi_*(k_*^R(\mu)(\kappa(a_x \otimes a_x))) = \psi_*(m_F^k(a_x \otimes a_x)) = \psi_*(a_x \cdot a_x).$$

This implies the statement for dimensional reasons. □

We can now prove Proposition 2.18:

Proof of Proposition 2.18. Observe first that φ_F^k is obviously independent on the product on F, since the latter does not enter into its definition.

To show independence on μ_k, we let $x \in I$ be arbitrary and show that $\varphi_F^k(\bar{x})$ can be expressed without reference to μ_k. Let $\bar{\eta}_F \colon R/x \to F$ be the unique factorization of $\eta_F \colon R \to F$. Choose a product on R/x such that $\bar{\eta}_F$ is multiplicative (Proposition 2.12). Then the pair $(\bar{\eta}_F, 1_F)$ is a morphism between the admissible pairs $(R/x, k, \pi\bar{\eta}_F)$ and (F, k, π). Therefore, by naturality of the characteristic homomorphism, the following diagram commutes:

$$\begin{tikzcd}
(x)/(x^2)[1] \arrow[r, "\varphi^k_{R/x}"] \arrow[d] & k_*^R(R/x) \arrow[d] \\
I/I^2[1] \arrow[r, "\varphi^k_F"] & k_*^R(F).
\end{tikzcd}$$

Now $\varphi^k_{R/x}(\bar{x}) = a_x$ by Lemma 2.25, which is defined independently of the product on k. Hence so is $\varphi_F^k(\bar{x})$. □

We now aim to relate $q_{R/x}$ to Strickland's commutativity obstruction $c(\mu_{R/x})$ (Proposition 2.6).

Proposition 2.26. *For a regular quotient ring* $F = R/x$ *with product* μ, *we have* $q_F(\bar{x}) = -c(\mu) \in R_*/x$.

Proof. The quadratic form q_F on $(x)/(x^2)[1] \cong R_*/x \cdot \bar{x}$ on the one hand is determined by $q = q_F(\bar{x}) = \mu_*(a_x^2)$ (we are in the situation where $\psi = \mu$). The obstruction $c = c(\mu)$ on the other hand is characterized by the identity $c(\beta \wedge \beta) = \mu - \mu\tau$, where $\beta = \beta_x : F \to \Sigma^{|x|+1} R$ is taken from the cofibre sequence (2.2). Therefore

we need to show that the maps $f_1 = -q(\beta \wedge \beta)$ and $f_2 = \mu - \mu\tau$ coincide. We prove this using the isomorphism of F_*-modules

$$d\kappa^* \colon F_R^*(F \wedge F) \longrightarrow \mathrm{Hom}_{R_*}(F_*^R(F) \otimes F_*^R(F), F_*) \tag{2.9}$$

given by composing the duality isomorphism d from Sect. 1.1 with the one induced by the Künneth isomorphism $\kappa = \kappa_\mu$.

First consider $(d\kappa^*)(f_1)$. Observe that by definition $k_*^R(\beta)(1) = 0$ and $k_*^R(\beta)(a_x) = 1$. From this, we easily deduce that

$$(d\kappa^*)(f_1)(1 \otimes 1) = (d\kappa^*)(f_1)(a_x \otimes 1) = (d\kappa^*)(f_1)(1 \otimes a_x) = 0$$

and that

$$(d\kappa^*)(f_1)(a_x \otimes a_x) = -q(k_*^R(\beta) \otimes k_*^R(\beta))(a_x \otimes a_x) = q$$

A sign is arising here according to (1.1), because we let commute an odd degree map, $k_*^R(\beta)$, with an odd degree element, a_x.

Now consider $(d\kappa^*)(f_2)$. As both μ and $\tau\mu = \mu^{\mathrm{op}}$ are products on F, we have

$$0 = (d\kappa^*)(f_2)(1 \otimes 1) = (d\kappa^*)(f_2)(a_x \otimes 1) = (d\kappa^*)(f_2)(1 \otimes a_x).$$

By definition of q, we have $(d\kappa^*)(\mu)(a_x \otimes a_x) = \mu_*(a_x \cdot a_x) = q$ and moreover, as $a_x \cdot^{\mathrm{op}} a_x = 0 \in F_*^R(F^{\mathrm{op}})$ by Lemma 2.25 and Proposition 2.22,

$$(d\kappa^*)(\mu^{\mathrm{op}})(a_x \otimes a_x) = \mu_*(a_x \cdot^{\mathrm{op}} a_x) = 0.$$

It follows that $(d\kappa^*)(f_2) = (d\kappa^*)(f_1)$, which concludes the proof. □

2.6. Diagonal ring spectra

The main aim of this subsection is to determine the characteristic bilinear form of a diagonal regular quotient ring. More generally, we consider R-rings F which are obtained by smashing together an arbitrary family of quotient R-ring spectra F_i. We specify conditions on the F_i which imply that F is a quotient ring and that the characteristic bilinear form b_F is determined by those of the F_i.

Suppose that $(F_i, \mu_i, \eta_i)_{i \geqslant 1}$ is a family of R-ring spectra. There is an obvious way to endow a finite smash product $F_1 \wedge \cdots \wedge F_n$ with a product structure, by mimicking the construction of the tensor product of finitely many algebras. We now show that this construction extends to infinitely many smash factors. Let $F = F_1 \wedge F_2 \wedge \cdots$ and let $j_i \colon F_i \to F$ be the natural maps. The following statement generalizes [14, Prop. 4.8]:

Proposition 2.27. *There is a unique R-ring structure on F such that j_k commutes with j_l if $k \neq l$.*

Proof. There is an obvious right action of F_n on $F(n) = F_1 \wedge \cdots \wedge F_n$. It extends in an evident way to compatible F_n-actions on $F(i)$ for all $i \geqslant n$, which induce an action $\psi_n\colon F \wedge F_n \to F$. We claim that the natural maps $\pi_n\colon [B \wedge F_n, F] \to [B, F]$ induced by the units $\eta_n\colon R \to F_n$ are surjective for any R-module B. In fact, we obtain a section of π_n by associating to a map $\alpha\colon B \to F$ the composition $\psi_n(\alpha \wedge 1)\colon B \wedge F_n \to F$, because the diagram

$$\begin{tikzcd}
B \wedge F_n \arrow[r, "\alpha\wedge 1"] & F \wedge F_n \arrow[r, "\psi_n"] & F \\
B \arrow[u, "1\wedge\eta_n"] \arrow[r, "\alpha"] & F \arrow[u, "1\wedge\eta_n"] \arrow[ur, equal] &
\end{tikzcd}$$

commutes. As a consequence, we find that $[F^{\wedge r}, F] \cong \lim_n [F(n)^{\wedge r}, F]$ for $r \geqslant 1$, by Milnor's exact sequence. For the rest of the argument, we follow the proof of [14, Prop. 4.8]. □

Definition 2.28. We call F with the product from Proposition 2.27 the *smash ring spectrum* of the F_i.

Suppose now that $(F_i = R/I_i, \mu_i, \eta_i)_{i \geqslant 1}$ is a family of quotient rings. Let (F, μ, η) be the smash ring spectrum of the F_i (Definition 2.28) and let $I = I_1 + I_2 + \cdots$. We aim to express b_F in terms of the b_{F_i} under conditions on the ideals I_i which guarantee that $F_* \cong R_*/I$ and that

$$I/I^2 \cong \bigoplus_i R_*/I \otimes_{R_*} I_i/I_i^2.$$

To begin with, note that the canonical homomorphisms

$$R_*/(I_1 + \cdots + I_k) \cong (F_1)_* \otimes \cdots \otimes (F_k)_* \longrightarrow (F_1 \wedge \cdots \wedge F_k)_*$$

induce on passing to colimits a map $\theta\colon R_*/I = R_*/(I_1 + I_2 + \cdots) \to F_*$. Consider the following hypotheses:

(i) θ is an isomorphism;
(ii) $(I_1 + \cdots + I_{k-1}) \cdot I_k = (I_1 + \cdots + I_{k-1}) \cap I_k$ for all $k > 1$.

Remark 2.29. It may be interesting to note that in the case where $I_k = (x_k)$ for all k, hypothesis (ii) is equivalent to the condition that $(x_1, x_2, \ldots)$ is a regular sequence. This is easy to verify. The assumption that R_* is a domain is essential here.

Proposition 2.30. *Hypotheses (i) and (ii) are both satisfied if for $k > 1$*

$$\mathrm{Tor}_{i,*}^{R_*}(R_*/(I_1 + \cdots + I_{k-1}), R_*/I_k) = 0 \quad \forall i > 0.$$

In particular, (i) and (ii) hold if I_k is generated by a sequence which is regular on $R_*/(I_1 + \cdots + I_{k-1})$, for all $k > 1$.

Proof. To show (i), we prove by induction that

$$R_*/(I_1 + \cdots + I_k) \cong (F_1 \wedge \cdots \wedge F_k)_*.$$

For the inductive step, it suffices to consider the Künneth spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{R_*}_{*,*}((F_1 \wedge \cdots \wedge F_{k-1})_*, (F_k)_*) \Longrightarrow (F_1 \wedge \cdots \wedge F_k)_*,$$

(see [7, IV.4]), which degenerates by assumption.

For (ii), recall that for ideals $J, K \subseteq R_*$, we have [6, Exercise A3.17]

$$\mathrm{Tor}^{R_*}_{1,*}(R_*/J, R_*/K) = (J \cap K)/(J \cdot K).$$

The last statement can be easily verified by using Koszul complexes. □

The following fact must be well known. For lack of a reference, we indicate its proof in Appendix A.

Proposition 2.31. *Suppose that (ii) is satisfied. Then there is a canonical isomorphism of R_*/I-modules*

$$I/I^2 \cong \bigoplus_{i \geqslant 1} R_*/I \otimes_{R_*} I_i/I_i^2. \tag{2.10}$$

We record the following immediate, well-known consequence:

Corollary 2.32. *Let $I \subseteq R_*$ be an ideal generated by a regular sequence $(x_1, x_2, \ldots)$ and let $\bar{x}_i \in I/I^2$ denote the residue classes of the x_i. Then there is an isomorphism of R_*/I-modules $I/I^2 \cong \bigoplus_i R_*/I\, \bar{x}_i$.*

The next proposition describes the characteristic bilinear form associated to a smash ring spectrum. For the definition of the bilinear forms $F_* \otimes b_{F_i}$ see the paragraph preceding Proposition 2.21.

Proposition 2.33. *Let F be the smash ring spectrum of quotient rings F_i and suppose that conditions (i) and (ii) above are satisfied. Then the bilinear form b_F is isomorphic to the direct sum of the $F_* \otimes b_{F_i}$.*

Proof. Let $V_i = I_i/I_i^2[1]$, $V = I/I^2[1]$ and let $j_i : F_i \to F$ be the natural maps. As a consequence of naturality, the diagonal terms of the bilinear form b_F with respect to the decomposition in condition (ii) are given by $F_* \otimes b_{F_i}$. Hence we need to show that the off-diagonal terms of b_F vanish. More precisely, we must have $b_F(\bar{x}_k \otimes \bar{x}_l) = 0$ for $k \neq l$, $x_k \in I_k$ and $x_l \in I_l$. By definition, this means that the composition

$$V_k \otimes V_l \to V \otimes V \xrightarrow{\varphi_F \otimes \varphi_F} F^R_*(F) \otimes F^R_*(F) \xrightarrow{m^F_F} F^R_*(F) \xrightarrow{\mu_*} F_* \tag{2.11}$$

has to be trivial, where the first map is induced by the inclusions of I_k and I_l into I. By naturality, the composition of the first two morphisms of (2.11) coincides with

$$V_k \otimes V_l \xrightarrow{\varphi^F_{F_k} \otimes \varphi^F_{F_l}} F^R_*(F_k) \otimes F^R_*(F_l) \xrightarrow{F^R_*(j_k) \otimes F^R_*(j_l)} F^R_*(F) \otimes F^R_*(F).$$

Because $j_k\colon F_k \to F$ and $j_l\colon F_l \to F$ commute, the composition of the last two morphisms of (2.11) with $F_*^R(j_k) \otimes F_*^R(j_l)$ coincides with

$$F_*^R(F_k) \otimes F_*^R(F_l) \xrightarrow{F_*^R(j_k)\otimes F_*^R(j_l)} F_*^R(F) \otimes F_*^R(F) \xrightarrow{m^F_{F^{\mathrm{op}}}} F_*^R(F) \xrightarrow{\mu_*} F_*.$$

Note that $m^F_{F^{\mathrm{op}}}$ can be viewed as the left action map of $F_*^R(F^{\mathrm{op}})$ on itself which is induced by the left action of $F \wedge F^{\mathrm{op}}$ on itself. Now $\mu_*\colon F_*^R(F) \to F_*$ is left $F_*^R(F^{\mathrm{op}})$-linear, as we have noted earlier. Because V_k and V_l are concentrated in odd degrees, an argument as in the proof of Proposition 2.22 shows that (2.11) is zero. □

We close this section by determining the characteristic bilinear form b_F of a diagonal regular quotient ring F.

Proposition 2.34. *Let $(x_1, x_2, \ldots)$ be a regular sequence in R_* generating an ideal $I \subseteq R_*$. Suppose that μ_i are products on R/x_i and let $F = R/I = R/x_1 \wedge R/x_2 \wedge \cdots$ be the induced diagonal regular quotient ring. Then the characteristic bilinear form $b_F\colon I/I^2[1] \otimes_{F_*} I/I^2[1] \to F_*$ is diagonal with respect to the basis $\bar{x}_1, \bar{x}_2, \ldots$ and $b_F(\bar{x}_i \otimes \bar{x}_i) \equiv -c(\mu_i) \mod I$.*

Proof. Combine Propositions 2.26 and 2.33. □

3. The homology algebra

The aim of this section is to study the homology algebra $k_*^R(F)$ for an admissible pair (F, k, π), with its natural product m_F^k from Sect. 2.4.

3.1. The main result and some consequences

Before stating the main result, we recall the definition and the universal property of Clifford algebras.

Let M_* be a graded quadratic module, i.e. a graded module over a graded commutative ring k_*, endowed with a quadratic form $q\colon M_* \to k_*$ which doubles degrees (for instance the quadratic form associated to a degree-preserving bilinear form). Let $\mathrm{T}(M_*)$ denote the tensor algebra over k_*, with its natural grading. The Clifford algebra $\mathcal{C}\ell(M_*, q)$ is defined as

$$\mathcal{C}\ell(M_*, q) = \mathrm{T}(M_*)/(x \otimes x - q(x) \cdot 1;\ x \in M_*).$$

As the ideal $(x \otimes x - q(x) \cdot 1;\ x \in M_*)$ is homogenous, $\mathcal{C}\ell(M_*, q)$ inherits a grading from $\mathrm{T}(M_*)$. Up to unique isomorphism, $\mathcal{C}\ell(M_*, q)$ is characterized by the following universal property: Any degree-preserving k_*-linear map $f\colon M_* \to A_*$ into a graded k_*-algebra A_* such that $f(x)^2 = q(x) \cdot 1$ for all $x \in M_*$ lifts to a unique algebra map $\mathcal{C}\ell(M_*, q) \to A_*$.

Theorem 3.1. *Let $(F = R/I, k, \pi)$ be an admissible pair. Then the characteristic homomorphism*

$$^{\pi}\varphi\colon k_* \otimes_{F_*} I/I^2[1] \longrightarrow k_*^R(F)$$

lifts to a natural homomorphism of k_*-algebras

$$\Phi\colon \mathcal{C}\ell(k_* \otimes_{F_*} I/I^2[1], q_F^k) \longrightarrow k_*^R(F).$$

If F is a regular quotient, then Φ is an isomorphism.

We will prove this result in Sect. 3.2 and draw some consequences now. Let us first spell out the following important special cases:

Corollary 3.2. *Let $F = R/I$ be a regular quotient ring. Then there is a natural F_*-algebra isomorphism*

$$F_*^R(F) \cong \mathcal{C}\ell(I/I^2[1], q_F).$$

Corollary 3.3. *Let $F = R/I$ be a regular quotient ring. Then there is an F_*-algebra isomorphism*

$$F_*^R(F^{op}) \cong \Lambda(I/I^2[1]).$$

Under this isomorphism, the homomorphism $(\mu_F)_*\colon F_*^R(F^{op}) \to F_*$ corresponds to the canonical augmentation $\varepsilon\colon \Lambda(I/I^2[1]) \to F_*$.

Proof. The first statement follows from the fact that $q_{F^{\mathrm{op}}}^F = 0$, by Proposition 2.22. For the second statement, note that the map $(\mu_F)_*$ is determined as the unique $F_*^R(F^{\mathrm{op}})$-bilinear map which is trivial on the image of φ. The augmentation ε, in turn, is a map of algebras, hence $\Lambda(I/I^2[1])$-bilinear, and it is trivial on $I/I^2[1]$. Hence the two maps coincide. □

Remark 3.4. Let (F, k, π) be a multiplicative admissible pair, with $F = R/I$ a regular quotient ring. From Corollary 3.3 and Proposition 2.21, we deduce that there is an isomorphism of k_*-algebras

$$k_*^R(F^{\mathrm{op}}) \cong \Lambda(k_* \otimes_{F_*} I/I^2[1]).$$

We can be more explicit in the case of a regular quotient ring $F = R/I$ if we fix a regular sequence $(x_1, x_2, \ldots)$ generating I. By Corollary 2.32, this choice determines an isomorphism $I/I^2 \cong \bigoplus_i F_*\bar{x}_i$, where $\bar{x}_i$ denote the residue classes, as usual. Letting $a_i = \varphi(\bar{x}_i) \in F_*^R(F)$, we have

$$F_*^R(F^{\mathrm{op}}) \cong \Lambda(a_1, a_2, \ldots). \tag{3.1}$$

Assume now that F is diagonal and let $c_i \in R_*/x_i$ be the commutativity obstruction of R/x_i of Proposition 2.6 and let $\bar{c}_i$ be its residue class in F_*. Using the explicit description of b_F (and hence q_F) from Proposition 2.34, we find:

$$F_*^R(F) \cong \mathrm{T}(a_1, a_2, \ldots)/(a_i^2 + \bar{c}_i \cdot 1,\ a_k a_l + a_l a_k;\ i \geqslant 1,\ k \neq l). \tag{3.2}$$

We add an example to illustrate the usefulness of the naturality of the isomorphism Φ in Theorem 3.1.

Example 3.5. Let $R = H\mathbb{Z}$ and p be a prime. Recall that R, $F = H\mathbb{Z}/p^4$ and $G = H\mathbb{Z}/p^3$ are commutative $\mathbb{S}$-algebras and that the canonical map $F \to G$ corresponding to the inclusion $I = (p^4) \to J = (p^3)$ is multiplicative [7, IV.2]. Multiplication by p^4 and p^3 induces isomorphisms $\mathbb{Z}/p^4 \cong I/I^2$ and $\mathbb{Z}/p^3 \cong J/J^2$, respectively. Under these identifications, the map $I/I^2 \to J/J^2$ corresponds to $p\colon \mathbb{Z}/p^4 \to \mathbb{Z}/p^3$. For any (G, k, π) admissible, the map of k_*-algebras $k_*^R(F) \to k_*^R(G)$ identifies with $\Lambda_{k_*}(a) \to \Lambda_{k_*}(b)$, $a \longmapsto p \cdot b$. If $k = H\mathbb{Z}/p$, this map is trivial, if $k = H\mathbb{Z}/p^2$, it is non-trivial.

Remark 3.6. [1] The Künneth spectral sequence (see [7, IV.4])

$$E^2_{p,q} = \mathrm{Tor}^{R_*}_{p,q}(k_*, F_*) \implies k^R_{p+q}(F) \tag{3.3}$$

is a multiplicative spectral sequence of k_*-algebras, see [3, Lemma 1.3]. By standard techniques,

$$\mathrm{Tor}^{R_*}_{*,*}(k_*, F_*) \cong \Lambda(k_* \otimes_{F_*} I/I^2[1])$$

as k_*-algebras (this follows for instance from [11, VII.6, Exercise 3]). For dimensional reasons, the elements of $k_* \otimes_{F_*} I/I^2[1]$ are permanent cycles and thus by multiplicativity, the spectral sequence collapses. As $\Lambda(k_* \otimes_{F_*} I/I^2[1])$ is a free k_*-module, there are no *additive* extensions and hence $k_*^R(F) \cong \Lambda(k_* \otimes_{F_*} I/I^2[1])$ as k_*-modules. The proof of Theorem 3.1 can be seen as resolving the *multiplicative* extensions in the spectral sequence.

The characteristic homomorphism ${}^{\pi}\varphi\colon k_* \otimes_{F_*} I/I^2[1] \to k_*^R(F)$ can also be considered from the point of view of this spectral sequence. Let $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots \subseteq k_*^R(F)$ be the filtration naturally associated to the spectral sequence. Consider the short exact sequence

$$0 \longrightarrow \mathscr{F}_0 \longrightarrow \mathscr{F}_1 \longrightarrow E^\infty_{1,*} \longrightarrow 0. \tag{3.4}$$

The retraction ψ_* from (2.6) induces a natural retraction in (3.4). Therefore we obtain a natural isomorphism $\mathscr{F}_1 \cong \mathscr{F}_0 \oplus E^\infty_{1,*}$. We can show that the composition

$$k_* \otimes_{F_*} I/I^2[1] \cong \mathrm{Tor}^{R_*}_{1,*}(k_*, F_*) = E^\infty_{1,*} \subseteq \mathscr{F}_1 \subseteq k_*^R(F)$$

coincides with ${}^{\pi}\varphi$.

3.2. *Proof of Theorem 3.1*

To begin with, suppose that $F = R/x$, for some $x \in R_*$, and let $a_x = \varphi^k_{R/x}(\bar{x})$. Then $k_*^R(F) \cong k_* \oplus k_* a_x$ by (2.8) and $a_x^2 = q_x^k(\bar{x}) \cdot 1$ by Lemma 2.25. Hence Φ is an isomorphism of algebras

$$\mathrm{T}_{k_*}(a_x)/(a_x^2 - q_x^k(\bar{x}) \cdot 1) \cong k_*^R(R/x), \tag{3.5}$$

which is exactly the statement of the theorem for $F = R/x$.

[1] This remark has been suggested by the referee.

Assume now that F is a quotient ring of R. By the universal property of Clifford algebras, the lift Φ exists if and only if $\varphi_F^k(\bar{x})^2 = q_F^k(\bar{x}) \cdot 1$ for all $x \in I$. Fix $x \in I$ and consider the natural map $j\colon R/x \to F$. There exists a product μ_x on R/x such that j is multiplicative by Proposition 2.12. Now the inclusion $(x) \subseteq I$ induces a commutative diagram of the form

$$\begin{tikzcd}
(x)/(x)^2[1] \arrow[r, "\varphi^k_{R/x}"] \arrow[d] & k_*^R(R/x) \arrow[d, "k_*^R(j)"] \\
I/I^2[1] \arrow[r, "\varphi^k_F"] & k_*^R(F).
\end{tikzcd}$$

As j is multiplicative, $k_*^R(j)$ is a map of algebras. We thus obtain

$$\varphi_F^k(\bar{x})^2 = k_*^R(j)(\varphi_{R/x}^k(\bar{x})^2) = k_*^R(j)(q_{R/x}^k(\bar{x}) \cdot 1) = q_F^k(\bar{x}) \cdot 1,$$

by Lemma 2.25 and by naturality of q. It follows that φ_F^k lifts to an algebra map $\Phi = \Phi_F^k$, as asserted.

Suppose now that F is a regular quotient ring. To show that Φ is an isomorphism, it suffices to prove this for the case where I is generated by a finite regular sequence $(x_1, \dots, x_n)$. The general case then follows easily by passing to colimits. Let $i_i\colon (x_i) \to I$ denote the inclusions and $\bar{i}_i\colon (x_i)/(x_i)^2 \to I/I^2$ the induced maps. As before, we choose products μ_i on the R/x_i such that the natural maps $j_i\colon R/x_i \to F$ are multiplicative. Consider the diagram of k_*-modules

$$\begin{tikzcd}
\bigotimes_{i=1}^n k_* \otimes \mathcal{C}\ell((x_i)/(x_i)^2, q_{R/x_i}^k) \arrow[r, "k_*\otimes\left(\bigotimes \Phi^k_{R/x_i}\right)"] \arrow[d, "\bigotimes(1\otimes\bar{i}_i)"'] & \bigotimes_{i=1}^n k_*^R(R/x_i) \arrow[d, "\bigotimes k_*(j_i)"] \\
\bigotimes_{i=1}^n k_* \otimes \mathcal{C}\ell(I/I^2, q_F^k) \arrow[r, "k_*\otimes(\bigotimes \Phi_F^k)"] \arrow[d] & \bigotimes_{i=1}^n k_*^R(F) \arrow[d] \\
k_* \otimes \mathcal{C}\ell(I/I^2, q_F^k) \arrow[r, "k_*\otimes\Phi_F^k"] & k_*^R(F).
\end{tikzcd}$$

The two lower vertical maps are given by the multiplication maps of the respective algebras. The top square commutes because Φ_F^k is natural in F and the bottom one because Φ_F^k is a morphism of algebras. The top horizontal map is an isomorphism by (3.5). As $I/I^2 \cong \bigoplus_{i=1}^n F_*\bar{x}_i$ by Corollary 2.32, the composite of the two left vertical maps is an isomorphism by [5, Chap. VI, §9.3, Cor. 3]. We easily check that the composite of the two right vertical maps is just the Künneth morphism and therefore an isomorphism. It follows that Φ_F^k is an isomorphism, as asserted. □

3.3. The antipode

Theorem 3.1 allows us to give a neat description of the antipode (or conjugation) homomorphism $\tau_*\colon F_*^R(F) \to F_*^R(F)$ induced by the switch map $\tau\colon F \wedge F \to F \wedge F$. For this, we recall a definition from the theory of Clifford algebras. Let $\mathcal{C}\ell(M_*, q)$ be the Clifford algebra on a quadratic graded module M_*. Then the *principal automorphism* α is the uniquely determined algebra automorphism of $\mathcal{C}\ell(M_*, q)$ whose restriction to M_* is given by $\alpha(m) = (-1)^{|m|}m$.

Proposition 3.7. *Let F be a regular quotient ring. Under the isomorphism from Corollary 3.2, the morphism $\tau_*\colon F_*^R(F) \to F_*^R(F)$ corresponds to the principal automorphism*

$$\alpha\colon \mathcal{C}\ell(I/I^2[1], q_F) \to \mathcal{C}\ell(I/I^2[1], q_F).$$

Proof. Since the switch map $\tau\colon F \wedge F \to F \wedge F$ is a ring isomorphism, $\tau_*\colon F_*^R(F) \to F_*^R(F)$ is an algebra isomorphism. It therefore suffices to check that $\tau_*(\varphi(\bar{x})) = -\varphi(\bar{x})$ for $x \in I$. Because there is always a product on R/x such that the natural map $R/x \to F$ is multiplicative (Proposition 2.12) and because φ is natural, we may therefore restrict to the case where $F = R/x$. We set $d = |x|$. Recall that d is even.

Let $a_x = \varphi(\bar{x})$. Then $F_*^R(F) = F_*1 \oplus F_*a_x$ by (2.8). Clearly, we have $\tau_*(1) = 1$. We therefore need to show that $\tau_*(a_x) = -a_x$.

We prove this by considering the canonical homomorphism

$$\iota\colon F_*^R(F) \cong (R \wedge F \wedge F)_* \xrightarrow{(\eta\wedge 1\wedge 1)_*} (F \wedge F \wedge F)_* = F_*^R(F \wedge F)$$

from the homotopy groups of $F \wedge F$ to its homology groups. As ι is injective (μ induces a retraction), it suffices to prove that $F_*^R(\tau)(\iota(a_x)) = -\iota(a_x)$. We do this by first identifying $\iota(a_x)$ and then computing $F_*^R(\tau)(\iota(a_x))$.

To simplify the notation, we identify $F_*^R(F \wedge F)$ with $F_*^R(F) \otimes_{F_*} F_*^R(F)$ via the Künneth isomorphism and $(R \wedge M)_*$, as well as $(M \wedge R)_*$, with M_*, for any R-module M.

To determine $\iota(a_x)$, we start by noting that for dimensional reasons and as $F_{\mathrm{odd}} = 0$, we have

$$\iota_*(a_x) = r \cdot 1 \otimes a_x + s \cdot a_x \otimes 1,$$

where $r, s \in F_0$. Consider the commutative diagram

$$\begin{tikzcd}
R \wedge F \wedge F \arrow[r, "\eta\wedge 1\wedge 1"] \arrow[d, "1\wedge 1\wedge \beta"'] & F \wedge F \wedge F \arrow[d, "1\wedge 1\wedge \beta"] \\
R \wedge F \wedge R \arrow[r, "\eta\wedge 1\wedge 1"] & F \wedge F \wedge R.
\end{tikzcd}$$

The composition of the upper and the right morphisms induces

$$\begin{aligned} F_*^R(1 \wedge \beta)(\iota_*(a_x)) &= 1 \otimes F_*^R(\beta)(r \cdot 1 \otimes a_x + s \cdot a_x \otimes 1) \\ &= r \cdot F_*^R(\beta)(a_x) = r \cdot 1, \end{aligned}$$

whereas the composition along the two other edges of the diagram induces $\eta_*(F_*^R(\beta)(a_x)) = \eta_*(1) = 1$. It follows that $r = 1$.

The computation of s requires another commutative diagram, namely

$$\begin{tikzcd} R \wedge F \wedge F \arrow[r, "\eta\wedge 1\wedge 1"] \arrow[d, "1\wedge\mu"'] & F \wedge F \wedge F \arrow[d, "1\wedge\mu"] \\ R \wedge F \arrow[r, "\eta\wedge 1"] & F \wedge F. \end{tikzcd}$$

With the same strategy as above, we obtain that

$$F_*^R(\mu)(\iota_*(a_x)) = F_*^R(\mu)(1 \otimes a_x + s \cdot a_x \otimes 1) = a_x + s \cdot a_x$$

and $\eta_*(\mu_*(a_x)) = 0$. This implies that $s = -1$.

We now consider the two maps $i = 1 \wedge \eta$, $j = \eta \wedge 1 \colon F \to F \wedge F$. The induced morphisms in homology satisfy $F_*^R(i)(a_x) = a_x \otimes 1$ and $F_*^R(j)(a_x) = 1 \otimes a_x$, respectively. Since $\tau \circ i = j$ and $\tau^2 = 1_{F\wedge F}$, we deduce that

$$F_*^R(\tau)(a_x \otimes 1) = 1 \otimes a_x, \;\; F_*^R(\tau)(1 \otimes a_x) = a_x \otimes 1. \tag{3.6}$$

Therefore, we have shown that

$$F_*^R(\tau)(\iota(a_x) = F_*(\tau)(1 \otimes a_x - a_x \otimes 1) = a_x \otimes 1 - 1 \otimes a_x = -\iota(a_x),$$

which concludes the proof. □

4. The cohomology algebra

The aims of this section are to to give a natural description of the cohomology module $k_R^*(F)$ for an admissible pair, to identify the derivations $\theta \colon F \to k$ in case the pair is multiplicative and to identify canonically the cohomology algebra $F_R^*(F)$ for a regular quotient ring F.

4.1. The cohomology of admissible pairs

Let (F, k, π) be an admissible pair. Using our identification of homology $k_*^R(F)$ and Kronecker duality, we derive an analogous expression for cohomology $k_R^*(F)$. As for homology, we aim for an isomorphism which is natural in both F and k.

For this, we need to modify the category of admissible pairs. We keep the objects, but declare a morphism $(F, k, \pi) \to (G, l, \pi')$ in the new category to be a pair of ring maps $f \colon G \to F$ and $g \colon k \to l$ such that

$$\begin{tikzcd} F \arrow[r, "\pi"] & k \arrow[d, "g"] \\ G \arrow[u, "f"] \arrow[r, "\pi'"] & l \end{tikzcd}$$

commutes. We refer to this category as the *bivariant category of admissible pairs* (this category is also named the twisted arrow category of the category of admissible pairs). Similarly, we refer to the full subcategory spanned by the multiplicative admissible pairs as the *bivariant category of multiplicative admissible pairs*. We use the expression "natural in F and k" in an analogous sense as for ordinary admissible pairs.

Cohomology $k_R^*(F)$ defines a functor on the bivariant category of admissible pairs. Ordinary Kronecker duality $d\colon k_R^*(F) \to D_{k_*}(k_*^R(F))$ (where $D_{k_*}(-) = \mathrm{Hom}_{k_*}(-, k_*)$) is not appropriate to study this functor, because $D_{k_*}(k_*^R(F))$ is not functorial on the bivariant category. We can get around this inconvenience by defining a modified version of Kronecker duality, which is of the form

$$d'\colon k_R^*(F) \longrightarrow \mathrm{Hom}_{F_*}^*(F_*^R(F), k_*). \tag{4.1}$$

It associates to a map $f\colon F \to k$ the homomorphism

$$F_*^R(F) \xrightarrow{F_*^R(f)} F_*^R(k) \xrightarrow{(\pi\wedge 1)_*} k_*^R(k) \xrightarrow{(\mu_k)_*} k_*.$$

We leave the easy verification of the fact that d' is a natural transformation between functors defined on the bivariant category to the reader.

We will need to work with the profinite topology on $k_R^*(M)$ for R-modules M. This is discussed in detail in [15, §2], following ideas of [4]. Recall that for any graded k_*-module N_*, $D_{k_*}(N_*)$ carries a natural linear topology, the dual-finite topology [4, Def. 4.8], which is complete and Hausdorff.

We endow $\mathrm{Hom}_{F_*}(M_*, k_*)$, for a graded F_*-module M_*, with the linear topology inherited from the dual-finite topology on $D_{k_*}(k_* \otimes_{F_*} M_*)$ under the adjunction isomorphism

$$\mathrm{Hom}_{F_*}^*(M_*, k_*) \cong D_{k_*}(k_* \otimes_{F_*} M_*). \tag{4.2}$$

By a slight abuse of terminology, we refer to this topology as the dual-finite topology, too. By naturality (in the variable M_*) of (4.2), the function $M_* \longmapsto \mathrm{Hom}_{F_*}^*(M_*, k_*)$ gives rise to a functor from the category of F_*-modules to the category of complete Hausdorff k_*-modules. As d' agrees with the following composition (the unlabelled maps are the canonical ones)

$$k_R^*(F) \xrightarrow{d} D_{k_*}(k_*^R(F)) \longrightarrow D_{k_*}(k_* \otimes_{F_*} F_*^R(F)) \cong \mathrm{Hom}_{F_*}^*(F_*^R(F), k_*), \tag{4.3}$$

it is continuous, since the Kronecker homomorphism d is continuous.

Proposition 4.1. *Let $(F = R/I, k, \pi)$ be an admissible pair. Then there exists a natural continuous homomorphism of k_*-modules*

$$\Psi\colon k_R^*(F) \longrightarrow \mathrm{Hom}_{F_*}^*(\Lambda(I/I^2[1]), k_*).$$

If F is a regular quotient ring, Ψ is a homeomorphism.

Proof. We define Ψ as the composition of continuous homomorphisms

$$k_R^*(F) \xrightarrow{d'} \mathrm{Hom}^*_{F_*}(F_*^R(F^{\mathrm{op}}), k_*) \xrightarrow{\Phi^*} \mathrm{Hom}^*_{F_*}(\Lambda(I/I^2[1]), k_*),$$

where Φ^* is induced by the homomorphism $\Phi\colon \Lambda(I/I^2) \to F_*^R(F^{\mathrm{op}})$ (Theorem 3.1, Proposition 2.22). If F is a regular quotient ring, the Kronecker duality homomorphism d is a homeomorphism, as $k_*^R(F)$ is free over k_* see [15, Prop. 2.5]. Furthermore, it follows from Theorem 3.1 that the canonical homomorphism $k_* \otimes_{F_*} F_*^R(F) \to k_*^R(F)$ is an isomorphism. Thus all the maps in (4.3) are homeomorphisms and hence d' as well. Moreover, Φ is an isomorphism by Theorem 3.1 and hence Φ^* is a homeomorphism by functoriality. It follows that Ψ is an homeomorphism. □

4.2. Derivations of regular quotient rings

Let F be an R-ring and M an F-bimodule. Recall that a map $\theta\colon F \to \Sigma^i M$ in $\mathscr{D}_R$ is called a (homotopy) derivation if the diagram

$$\begin{tikzcd} F \wedge F \arrow[r, "1\wedge\theta\vee\theta\wedge 1"] \arrow[d, "\mu_F"] & (F \wedge \Sigma^i M) \vee (\Sigma^i M \wedge F) \arrow[d] \\ F \arrow[r, "\theta"] & \Sigma^i M \end{tikzcd} \tag{4.4}$$

commutes, where the unlabelled map is induced by the left and right actions of F on M. We write $\mathscr{D}\mathrm{er}_R^i(F, M)$ for the set of all such derivations and $\mathscr{D}\mathrm{er}_R^i(F)$ for $\mathscr{D}\mathrm{er}_R^i(F, F)$.

Suppose that $(F = R/I, k, \pi)$ is a multiplicative admissible pair. Then k is an F-bimodule in a natural way, and so we may consider $\mathscr{D}\mathrm{er}_R^*(F, k)$. We endow $\mathscr{D}\mathrm{er}_R^*(F, k)$ with the subspace topology induced by the profinite topology on $k_R^*(F)$.

We now define a natural transformation

$$\psi\colon \mathscr{D}\mathrm{er}_R^*(F, k) \longrightarrow \mathrm{Hom}^*_{F_*}(I/I^2[1], k_*)$$

between functors on the bivariant category of multiplicative admissible pairs with values in the category of topological k_*-modules. We set ψ to be the composition

$$\mathscr{D}\mathrm{er}_R^*(F, k) \subseteq k_R^*(F) \xrightarrow{\Psi} \mathrm{Hom}^*_{F_*}(\Lambda(I/I^2[1]), k_*) \xrightarrow{\iota^*} \mathrm{Hom}^*_{F_*}(I/I^2[1], k_*),$$

where Ψ is the homomorphism from Proposition 4.1 and where ι denotes the canonical injection $I/I^2[1] \to \Lambda(I/I^2[1])$.

Proposition 4.2. *Suppose that (F, k, π) is a multiplicative admissible pair and that both $F = R/I$ and k are regular quotient rings.*

(i) *The homomorphism*

$$\psi\colon \mathscr{D}\mathrm{er}_R^*(F, k) \to \mathrm{Hom}^*_{F_*}(I/I^2[1], k_*)$$

is a natural homeomorphism.

(ii) *The composition*

$$\mathrm{Hom}^*_{F_*}(I/I^2[1], k_*) \xrightarrow{\psi^{-1}} \mathscr{D}\mathrm{er}^*_R(F, k) \subseteq k^*_R(F).$$

is independent of the products on F and k.

The proof of Proposition 4.2 requires some preparations and will be given at the end of this subsection.

To be able to detect derivations, we now relate homotopy derivations with algebraic ones. We denote by $\mathrm{Der}^*_{k_*}(A_*, M_*)$ the derivations from a k_*-algebra A_* to an A_*-bimodule M_* and write $\mathrm{Der}^*_{k_*}(A_*)$ for $\mathrm{Der}^*_{k_*}(A_*, A_*)$. The grading convention is that $\partial \in \mathrm{Der}^*_{k_*}(A_*, M_*)$ satisfies

$$\partial(a \cdot b) = \partial(a) \cdot b + (-1)^{|\partial|\cdot|a|} a \cdot \partial(b).$$

Lemma 4.3. *Let (F, k, π) be a multiplicative admissible pair, where $F = R/I$ is a regular quotient, and let $\bar{k}$ be k, endowed with a second (not necessarily different) product. Then the Hurewicz homomorphism*

$$h = k^R_*(-)\colon k^*_R(F) \longrightarrow \mathrm{Hom}^*_{k_*}(\bar{k}^R_*(F), \bar{k}^R_*(k)) \tag{4.5}$$

restricts to a monomorphism

$$\bar{h}\colon \mathscr{D}\mathrm{er}^*_R(F, k) \longrightarrow \mathrm{Der}^*_{k_*}(\bar{k}^R_*(F), \bar{k}^R_*(k)).$$

The induced commutative diagram

$$\begin{tikzcd}
\mathscr{D}\mathrm{er}^*_R(F, k) \arrow[r, "\bar{h}"] \arrow[d, "incl"'] & \mathrm{Der}^*_{k_*}(\bar{k}^R_*(F), \bar{k}^R_*(k)) \arrow[d, "incl"] \\
k^*_R(F) \arrow[r, "h"] & \mathrm{Hom}^*_{k_*}(\bar{k}^R_*(F), \bar{k}^R_*(k))
\end{tikzcd}$$

is a pullback diagram. Explicitly, this means that the derivations are precisely those maps in $k^*_R(F)$ which induce derivations on applying $\bar{k}^R_*(-)$.

Proof. Applying the functor $h = \bar{k}^R_*(-)$ to the diagram (4.4) and precomposing with the Künneth map $\kappa_{\bar{k}}\colon \bar{k}^R_*(F) \otimes_{k_*} \bar{k}^R_*(F) \to \bar{k}^R_*(F \wedge F)$ shows that a derivation $\theta\colon F \to \Sigma^i k$ induces a derivation $h(\theta)$ on the homology algebra $\bar{k}^R_*(F)$. Hence h, which is monomorphic (see Sect. 1.1), restricts to a monomorphism $\bar{h}$, as asserted.

For the second statement, we need to verify, for $\theta \in k^*_R(F)$, the equivalence

$$\theta \in \mathscr{D}\mathrm{er}^*_R(F, k) \Longleftrightarrow h(\theta) \in \mathrm{Der}^*_{k_*}(\bar{k}^R_*(F), \bar{k}^R_*(k)). \tag{4.6}$$

We have shown "$\Rightarrow$" above and now prove "$\Leftarrow$". By definition of $\bar{k}^R_*(F)$ and $\bar{k}^R_*(k)$, $h(\theta)$ is a derivation if the diagram obtained by applying $\bar{k}^R_*(-)$ to (4.4) and precomposing with κ_{ν_k} commutes. This implies that (4.4) commutes (see Sect. 1.1), i.e. that θ is a derivation. □

Using Lemma 4.3, we now construct certain derivations in $\mathscr{D}\mathrm{er}^*_R(F)$. We first consider the case $F = R/x$. Recall the maps β_x, η_x from (2.2). We refer to the composition

$$Q_x\colon R/x \xrightarrow{\beta_x} \Sigma^{d+1}R \xrightarrow{\eta_x} \Sigma^{d+1}R/x \tag{4.7}$$

as the *Bockstein operation* associated to x.

The following lemma is already known from Strickland [14]. Let $y \in D_{R_*/(x)}\big((x)/(x^2)[1]\big)$ denote the dual of $\bar{x} \in (x)/(x)^2[1]$.

Lemma 4.4. *The Bockstein operation $Q_x\colon R/x \to \Sigma^{|x|+1}R/x$ is a derivation for any product on R/x. It satisfies $\psi(Q_x) = y$.*

Proof. We have $(R/x^{\mathrm{op}})^R_*(R/x) \cong \Lambda(a)$ with $a = \varphi(\bar{x})$, by Corollary 3.3. Applying $(R/x^{\mathrm{op}})^R_*(-)$ to the cofibre sequence (2.2), we find that under this isomorphism, $(R/x^{\mathrm{op}})^R_*(Q_x)$ corresponds to $\frac{\partial}{\partial a}\colon \Lambda(a) \to \Lambda(a)$. Therefore, by Lemma 4.3, Q_x is a derivation, with $\psi(Q_x) = y$. □

Remark 4.5. The proof shows that $F^R_*(Q_x)$ corresponds to $\frac{\partial}{\partial a}$ under the isomorphism $F^R_*(R/x) \cong \Lambda(a)$, where $a = \varphi(\bar{x})$.

Next, we construct derivations in $\mathscr{D}\mathrm{er}^*_R(F)$ for an arbitrary regular quotient ring $F = R/I$. Let $(x_1, x_2, \ldots)$ be a regular sequence generating the ideal I and $y_i \in D_{F_*}(I/I^2[1])$ be the dual of $\bar{x}_i \in I/I^2[1]$.

Consider the R_*-algebra homomorphisms

$$\chi_i\colon (R/x_i)^*_R(R/x_i) \longrightarrow F^*_R(F)$$

defined by $f \longmapsto f \wedge 1$, where 1 denotes the identity map on $F'_i = \wedge_{j\neq i} R/x_j$ and where we identify F with $R/x_i \wedge F'_i$.

Lemma 4.6. *Let $F = R/I$ be a regular quotient ring and let $(x_1, x_2, \ldots)$ be a regular sequence generating the ideal I. For any products on R/x_i, χ_i restricts to an R_*-homomorphism*

$$\bar{\chi}_i\colon \mathscr{D}\mathrm{er}_R(R/x_i) \longrightarrow \mathscr{D}\mathrm{er}^*_R(F).$$

The derivations $Q_i = \bar{\chi}_i(Q_{x_i})$ satisfy $\psi(Q_i) = y_i$.

Proof. Fix a product on F. To prove the first statement, it suffices to verify that $\chi_i(\theta) \in \mathscr{D}\mathrm{er}^*_R(F)$ for $\theta \in \mathscr{D}\mathrm{er}^*_R(R/x_i)$. Choose a product ν on R/x_i such that the canonical map $j\colon R/x_i \to F$ is multiplicative (Proposition 2.12). By Lemma 4.4, θ is also a derivation with respect to ν. The diagram

$$\begin{tikzcd}
(F^{\mathrm{op}})^R_*(R/x_i) \arrow[r, "F^R_*(\theta)"] \arrow[d, "F^R_*(j)"] & (F^{\mathrm{op}})^R_*(R/x_i) \arrow[d, "F^R_*(j)"] \\
(F^{\mathrm{op}})^R_*(F) \arrow[r, "F^R_*(\chi_i(\theta))"] & (F^{\mathrm{op}})^R_*(F)
\end{tikzcd}$$

commutes by definition of $\chi_i(\theta)$. Since $\theta \in \mathscr{D}\mathrm{er}^*_R(R/x_i)$, Lemma 4.3 implies that $F^R_*(\theta) \in \mathrm{Der}_{k_*}((F^{\mathrm{op}})^R_*(R/x_i))$. We set as usual $a_{x_i} = \varphi^{F^{\mathrm{op}}}_{R/x_i}(\bar{x}_i) \in (F^{\mathrm{op}})^R_*(R/x_i)$ and $a_j = \varphi^{F^{\mathrm{op}}}_F(\bar{x}_j) \in (F^{\mathrm{op}})^R_*(F)$. Then $(F^{\mathrm{op}})^R_*(R/x_i) \cong \Lambda_{F_*}(a_{x_i})$ and $(F^{\mathrm{op}})^R_*(F) \cong \Lambda_{F_*}(a_1, a_2, \ldots)$. Since j is multiplicative and the characteristic homomorphism φ is natural, $F^R_*(j)$ is an algebra morphism such that $F^R_*(j)(a_{x_i}) = a_i$. Via the isomorphism

$$(F^{\mathrm{op}})^R_*(F) \cong \left(\bigotimes_{k \neq i} \Lambda_{F_*}(a_k)\right) \otimes \Lambda_{F_*}(a_i),$$

$F^R_*(\chi_i(\theta))$ corresponds to $1 \otimes F^R_*(\theta)$. It follows that $F^R_*(\chi_i(\theta))$ is a derivation. By Lemma 4.3, $\chi_i(\theta)$ is a derivation as well. In addition, we have $\psi(Q_i) = y_i$, by naturality of ψ and by Lemma 4.4. □

Definition 4.7. Let $F = R/I$ be a regular quotient ring. The Bockstein operation $Q_\alpha \in \mathscr{D}\mathrm{er}^*_R(F)$ associated to $\alpha \in \mathrm{Hom}^*_{F_*}(I/I^2[1], F_*)$ is defined to be $\psi^{-1}(\alpha)$. We write Q_i for Q_{y_i}, where $(x_1, x_2, \ldots)$ is a regular sequence generating I and where y_i is dual to $\bar{x}_i$.

Remark 4.8. Strickland defines in [14] for $F = R/I$ a regular quotient ring a homomorphism $d\colon \mathscr{D}\mathrm{er}^*_R(F) \to \mathrm{Hom}^*_{R^*}(I/I^2, F_*)$ and shows that d is injective. Moreover, he proves that d is an isomorphism for diagonal F. The homomorphism d coincides with our ψ, as $d(Q_i) = y_i$ [14, Corollary 4.19].

Proof of Proposition 4.2. (i) We first show that ψ is surjective. Choose a regular sequence $(x_1, x_2, \ldots)$ generating I. Let Q_i and y_i be as above. By Lemma 4.6 and by naturality of ψ, we have $\psi(\pi \circ Q_i) = \pi_* \circ y_i$. Because $\mathrm{Hom}^*_{F_*}(I/I^2[1], k_*)$ is generated by the elements $\pi_* \circ y_i$, ψ is surjective.

To show that ψ is injective, suppose that $\theta \in \mathscr{D}\mathrm{er}^*_R(F, k)$ satisfies $\psi(\theta) = 0$. By Corollary 3.3 $(\mu_k)_*\colon (k^{\mathrm{op}})^R_*(k) \to k_*$ is the augmentation of an exterior algebra and hence an algebra homomorphism. Therefore, the composition

$$\Lambda(I/I^2[1]) \underset{\cong}{\xrightarrow{\Phi}} (F^{\mathrm{op}})^R_*(F) \xrightarrow{F^R_*(\theta)} (F^{\mathrm{op}})^R_*(k) \xrightarrow{(\pi \wedge 1)_*} (k^{\mathrm{op}})^R_*(k) \xrightarrow{(\mu_k)_*} k_*, \quad (4.8)$$

where Φ is the isomorphism from Corollary 3.3, is a derivation. By assumption, its restriction to $I/I^2[1]$ is zero. This implies that (4.8) is zero. By duality (see Sect. 1.1), it follows that θ is trivial.

It remains to prove that ψ is open. By definition of the topology on $\mathscr{D}\mathrm{er}^*_R(F, k)$ and the fact that Ψ is a homeomorphism (Proposition 4.1), it suffices to show that

$$\iota^*\colon \mathrm{Hom}^*_{F_*}(\Lambda(I/I^2[1]), k_*) \longrightarrow \mathrm{Hom}^*_{F_*}(I/I^2[1], k_*)$$

is open. By definition of the topologies involved here, this is a consequence of the fact that an injection of k_*-modules $V_* \to W_*$ induces an open map on the duals with respect to the dual-finite topologies.

(ii) This is clear, because $\psi(\pi \circ Q_i) = \pi_* \circ y_i$ and because $Q_i = \chi_i(Q_{x_i})$ is defined independently on any products. □

Remark 4.9. It is a consequence of Proposition 4.2 that the $\bar{\chi}_i$ from Lemma 4.6 induce an isomorphism

$$\prod_{i \geqslant 1} F^* \otimes_{R^*/x_i} \mathscr{D}\mathrm{er}^*_R(R/x_i) \cong \mathscr{D}\mathrm{er}^*_R(F).$$

We close this section by giving two properties of derivations which we will need later on.

Lemma 4.10. *Any derivation $\theta \in \mathscr{D}\mathrm{er}^*_R(F)$ satisfies $\theta^2 = 0$.*

Proof. By Proposition 4.2, we may assume that $\theta = Q_i$. By Lemma 4.6, we have $\theta = \bar{\chi}_i(Q_{x_i})$. Hence θ^2 is given by smashing $Q^2_{x_i}$ with the identities on the other smash factors. But $Q^2_{x_i}$ is trivial, by definition. □

Lemma 4.11. *For any $\theta \in \mathscr{D}\mathrm{er}^*_R(F)$, the diagram below commutes:*

$$\begin{tikzcd} I/I^2[1] \arrow[r, "\psi(\theta)"] \arrow[d, "\varphi"'] & k_* \arrow[d, "(1\wedge\eta)_*"] \\ k^R_*(F) \arrow[r, "k^R_*(\theta)"] & k^R_*(F). \end{tikzcd}$$

Proof. Let $\bar{x} \in I/I^2[1]$. Choose a product on R/x such that the canonical map $j\colon R/x \to F$ is multiplicative (Proposition 2.12). Consider the maps

$$\mathscr{D}\mathrm{er}^*_R(F) \longrightarrow \mathscr{D}\mathrm{er}^*_R(R/x, F) \longleftarrow F_* \otimes_{R_*/x} \mathscr{D}\mathrm{er}^*_R(R/x)$$

induced by j. It follows from Proposition 4.2(i) that the second map is an isomorphism. Therefore, there is a derivation $\theta_x \in \mathscr{D}\mathrm{er}^*_R(R/x)$ such that

$$\begin{tikzcd} R/x \arrow[r, "\theta_x"] \arrow[d, "j"] & R/x \arrow[d, "j"] \\ F \arrow[r, "\theta"] & F \end{tikzcd}$$

commutes. By naturality of φ and ψ, we may therefore assume that $F = R/x$. By Proposition 4.2(i) and Lemma 4.4, we can restrict to $\theta = Q_x$. But then, the statement comes down to the statement in Remark 4.5. □

4.3. Cohomology of regular quotients

We now determine the cohomology algebra $F^*_R(F)$ for a regular quotient F.

We need a notation. Let M_* be a module over a graded ring F_*. The dual-finite filtration on $D(M_*) = D_{F_*}(M_*)$ induces a filtration of the exterior algebra $\Lambda(D(M_*))$. We write $\widehat{\Lambda}(D(M_*))$ for the completion of $\Lambda(D(M_*))$ with respect to this filtration.

An isomorphism of the form below was constructed by Strickland for diagonal F [14, Cor. 4.19]. His construction relies upon the choice of a regular sequence generating I. We show that there is an isomorphism which is independent on any choices, for any regular quotient ring F.

Theorem 4.12. *For a regular quotient ring $F = R/I$, there is a canonical homeomorphism of F^*-algebras*

$$\Theta\colon \widehat{\Lambda}(\mathscr{D}\mathrm{er}_R^*(F)) \cong F_R^*(F).$$

Remark 4.13. Proposition 4.2 and Theorem 4.12 imply that if $F = R/I$ is a regular quotient module, then

$$\widehat{\Lambda}(D(I/I^2[1])) \cong F_R^*(F).$$

Note that on fixing a regular sequence $(x_1, x_2, \ldots)$ generating I, we obtain

$$\widehat{\Lambda}(Q_1, Q_2, \ldots) \cong F_R^*(F),$$

where the Q_i are defined according to Definition 4.7.

Proof. Set $V = I/I^2[1]$ and recall that V is a free F_*-module with basis $\bar{x}_1, \bar{x}_2, \ldots$, where $(x_1, x_2, \ldots)$ is a regular sequence generating I. We define

$$\delta\colon D(V) \longrightarrow D(\Lambda(V))$$

by $\delta(y_i) = \varepsilon \circ \frac{\partial}{\partial \bar{x}_i}$ where ε is the canonical augmentation of $\Lambda(V)$ and y_i is dual to $\bar{x}_i$. We easily check that δ lifts to a homeomorphism

$$\Delta\colon \widehat{\Lambda}(D(V)) \longrightarrow D(\Lambda(V))$$

with $\Delta(y_{i_1} \wedge \cdots \wedge y_{i_n}) = \varepsilon \circ \frac{\partial}{\partial \bar{x}_{i_1}} \circ \cdots \circ \frac{\partial}{\partial \bar{x}_{i_1}}$ (for the proof, consider first the case where V is finitely generated and then pass to limits).

The homeomorphism $\Psi\colon F_R^*(F) \longrightarrow D(\Lambda(V))$ from Proposition 4.1 is, in the case we are considering, just the composition of the usual Kronecker homomorphism with the dual of the isomorphism $\Phi\colon \Lambda(V) \cong F_*^R(F^{\mathrm{op}})$. The Kronecker homomorphism is a homeomorphism, since $F_*^R(F^{\mathrm{op}})$ is F_*-free.

Lemma 4.6 implies that $F_*^R(Q_i)$ is a derivation of the algebra $F_*^R(F^{\mathrm{op}})$. Using Remark 4.5 and the isomorphism $F_*^R(F^{\mathrm{op}}) \cong \Lambda(V)$, we easily check that $\Psi(Q_i) = \varepsilon \circ \frac{\partial}{\partial \bar{x}_i}$. Since $\psi(Q_i) = y_i$ (Lemma 4.6), we have that $\delta\psi(Q_i) = \varepsilon \circ \frac{\partial}{\partial \bar{x}_i}$. Therefore the following diagram commutes:

$$\begin{tikzcd}
F_R^*(F) \arrow[r, "\Psi"] & D(\Lambda(V)) \\
\mathscr{D}\mathrm{er}_R^*(F) \arrow[u, "\subseteq"] \arrow[r, "\psi"] & D(V). \arrow[u, "\delta"']
\end{tikzcd}$$

As any derivation squares to 0 (Lemma 4.10) and as $F_R^*(F)$ is complete, the injection $\mathscr{D}\mathrm{er}_R^*(F) \hookrightarrow F_R^*(F)$ lifts to a continuous F_*-algebra homomorphism

$$\Theta\colon \widehat{\Lambda}(\mathscr{D}\mathrm{er}_R^*(F)) \longrightarrow F_R^*(F).$$

Explicitly, Θ is given by $\Theta(Q_{i_1} \wedge \cdots \wedge Q_{i_n}) = Q_{i_1} \circ \cdots \circ Q_{i_n}$. Because of $\Psi(Q_{i_1} \circ \cdots \circ Q_{i_n}) = \varepsilon \circ \frac{\partial}{\partial \bar{x}_{i_1}} \circ \cdots \circ \frac{\partial}{\partial \bar{x}_{i_1}}$, the diagram below commutes, too:

$$\begin{tikzcd} F_R^*(F) \arrow[r, "\Psi"] & D(\Lambda(V)) \\ \widehat{\Lambda}(\mathscr{D}\mathrm{er}_R^*(F)) \arrow[u, "\Theta"] \arrow[r, "\widehat{\Lambda}(\psi)"] & \widehat{\Lambda}(D(V)). \arrow[u, "\Delta"'] \end{tikzcd}$$

Together with Ψ, Δ and $\widehat{\Lambda}(\psi)$, Θ is therefore a homeomorphism, too. □

5. Examples

In this section we discuss the example of the Morava K-theories $K(n)$. Their 2-periodic versions K_n can be treated similarly. They are discussed in detail in [8].

5.1. Definition of Morava K-theory

We fix a prime number p. Recall that the p-localization $MU_{(p)}$ of the spectrum associated to the complex cobordism MU is a commutative $\mathbb{S}$-algebra (see [7]) satisfying:

$$(MU_{(p)})_* \cong \mathbb{Z}_{(p)}[x_1, x_2, \ldots], \ |x_i| = 2i.$$

The Hopkins–Miller theorem [9] has as a consequence that for $n \geqslant 0$, there exists an $MU_{(p)}$-algebra $\widehat{E}(n)$ with

$$\widehat{E}(n)_* \cong \lim_k \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}][v_n, v_n^{-1}]/I_n^k,$$

where I_n is the ideal generated by the regular sequence $(v_0 = p, v_1, \ldots, v_{n-1})$. Details can be found in [13, Theorem 1.5] and in his unpublished correction "A not necessarily commutative map", available on the author's home page.

The n-th Morava K-theory may be defined as the regular quotient of $\widehat{E}(n)$ by I_n:

$$K(n) = \widehat{E}(n)/I_n \cong \widehat{E}(n)/v_0 \wedge_{\widehat{E}(n)} \cdots \wedge_{\widehat{E}(n)} \widehat{E}(n)/v_{n-1}.$$

Its coefficient ring satisfies $K(n)_* \cong \mathbb{F}_p[v_n, v_n^{-1}]$.

5.2. The case p odd

We first consider the case where p is an odd prime. According to Strickland [14, Cor. 3.12], there is a commutative $\widehat{E}(n)$-product μ_k on $\widehat{E}(n)/v_k$ for $0 \leqslant k \leqslant n-1$. Let μ be the smash ring product of the μ_k on $K(n)$. Since μ is commutative, we have $b_{K(n)} = 0$ by Corollary 2.23. Therefore if $K(n)$ is endowed with this product μ, then

$$K(n)_*^{\widehat{E}(n)}(K(n)) \cong \Lambda(I_n/I_n^2[1]) \cong \Lambda(a_0, \ldots, a_{n-1})$$

where $a_i = \varphi(\bar{v}_i)$, as in Sect. 3.1.

5.3. The case $p = 2$

The case of the prime $p = 2$ is much more interesting. We use some arguments and notation from [14, Sect. 7] in the following.

Let $w_k \in MU_{2(2^k-1)}$ denote the bordism class of a smooth hypersurface W_{2^k} of degree 2 in $\mathbb{C}P^{2^k}$ and let $J_k \subseteq (MU_{(2)})_*$ be the ideal $(w_0, \dots, w_{k-1})$, where $w_0 = 2$. The sequence of the w_i is regular, and the image of J_k in $\widehat{E}(n)_*$ is the ideal $I_k = (v_0, \dots, v_{k-1})$, for $k = 0, \dots, n$ (see [14]). To simplify the notation, we write again w_k for the image of $w_k \in (MU_{(2)})_*$ in $\widehat{E}(n)_*$.

Proposition 5.1. *There is a product μ_k on $\widehat{E}(n)/w_k$ with $c(\mu_k) \equiv w_{k+1} \mod I_k$ for $k \geqslant 0$.*

Proof. As $\widehat{E}(n)$ is an $MU_{(2)}$-algebra, the functor $\mathscr{E}\colon \mathscr{D}_{MU_{(2)}} \to \mathscr{D}_{\widehat{E}(n)}$ defined as $\mathscr{E}(M) = M \wedge_{MU_{(2)}} \widehat{E}(n)$ is strictly monoidal. This can be seen as follows: For $MU_{(2)}$-modules M and N, $M \wedge_{MU_{(2)}} \widehat{E}(n) \cong \widehat{E}(n) \wedge_{MU_{(2)}} M$ is an $(\widehat{E}(n), \widehat{E}(n))$-bimodule and exactly as in [7, III. 3] there is a natural isomorphism

$$(M \wedge_{MU_{(2)}} \widehat{E}(n)) \wedge_{\widehat{E}(n)} (\widehat{E}(n) \wedge_{MU_{(2)}} N) \cong \widehat{E}(n) \wedge_{MU_{(2)}} (M \wedge_{MU_{(2)}} N)$$

of $(\widehat{E}(n), \widehat{E}(n))$-bimodules. As a consequence, the functor $\mathscr{E}$ maps $MU_{(2)}$-rings to $\widehat{E}(n)$-rings. Strickland constructs a $MU_{(2)}$-product $\tilde{\mu}_k$ on $MU_{(2)}/w_k$ with $c(\tilde{\mu}_k) \equiv w_{k+1} \mod J_k$ for $k \geqslant 0$ [14, Sect. 7]. Via the functor $\mathscr{E}$, $\tilde{\mu}_k$ induces an $\widehat{E}(n)$-product μ_k on $\widehat{E}(n)/w_k$. By definition of the obstruction c, we check that $c(\mu_k) \equiv w_{k+1} \mod I_k$. □

We endow $K(n)$ with the diagonal product μ, defined as the smash ring product of the μ_k. As $v_n \equiv w_n \mod I_n$, Propositions 2.34 and 5.1 imply that $b_{K(n)} = v_n \cdot y_{n-1} \otimes y_{n-1}$, where $y_{n-1} \in D_{\widehat{E}(n)_*}(I_n/I_n^2[1])$ is dual to $\bar{v}_{n-1} \in I_n/I_n^2[1]$. Therefore, μ is not commutative, see Corollary 2.23.

The opposite product μ^{op} is the smash ring product of the μ_k^{op}. It follows from [14, Prop. 3.1 and Lemma 3.11] that $c(\mu_k^{\mathrm{op}}) \equiv w_{k+1} \mod I_k$, as $2 \in I_k$.

Let $1 \leqslant k \leqslant n$. For dimensional reasons, we have $(\widehat{E}(n)/w_{k-1})_{2|w_{k-1}|+2} = \{0, v_k\}$. Therefore, Proposition 2.5 implies that

$$\mu_{k-1}^{\mathrm{op}} = \mu_{k-1} \circ (1 + v_k \cdot Q_{w_{k-1}} \wedge Q_{w_{k-1}}).$$

The elements $\bar{w}_{k-1}, \bar{v}_{k-1} \in I_k/I_k^2[1]$ coincide, hence their duals are the same and so Proposition 4.2 implies that $Q_{w_{k-1}} = Q_{v_{k-1}} \in \mathscr{D}\mathrm{er}^*_{\widehat{E}(n)}(\widehat{E}(n)/w_{k-1})$. As a consequence, we recover the well known formula:

$$\mu^{\mathrm{op}} = \mu \circ (1 + v_n \cdot Q_{n-1} \wedge Q_{n-1}),$$

where Q_{n-1} is defined as in Definition 4.7. Observe that $b_{K(n)} = b_{K(n)^{\mathrm{op}}}$ although $\mu \neq \mu^{\mathrm{op}}$. We now compute

$$K(n)_*^{\widehat{E}(n)}(K(n)) \cong \Lambda(a_0, \dots, a_{n-2}) \otimes \mathrm{T}(a_{n-1})/(a_{n-1}^2 - v_n \cdot 1),$$

where $K(n)$ is endowed with the product μ described above and the a_i are defined as in the case where p is odd.

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Appendix A: An algebraic fact

Proposition A.1. *Suppose that $I_1, I_2, \ldots \subseteq R_*$ are ideals which satisfy*

$$(I_1 + \cdots + I_{k-1}) \cdot I_k = (I_1 + \cdots + I_{k-1}) \cap I_k$$

for all $k > 1$. Let $I = I_1 + I_2 + \cdots$. Then there is a canonical isomorphism of R_*/I-modules

$$I/I^2 \cong \bigoplus_{i \geqslant 1} R/I_* \otimes_{R_*} I_i/I_i^2.$$

Proof. We prove the statement only for $I = I_1 + I_2$. The argument needed for the inductive step is similar and therefore left to the reader. For infinitely many I_i, the statement follows by passing to colimits.
We begin by showing that $I^2 = (I_1 + I_2^2) \cap (I_1^2 + I_2)$. The inclusion $\subseteq$ is trivial. To show $\supseteq$, suppose that $\alpha \in (I_1 + I_2^2) \cap (I_1^2 + I_2)$. Write α as $\alpha = x + w = v + y$, where $x \in I_1$, $w \in I_2^2$, $y \in I_2$ and $v \in I_1^2$. It follows that $x - v = y - w \in I_1 \cap I_2$. By hypothesis, we have $I_1 \cap I_2 = I_1 \cdot I_2$, and therefore $\alpha = (x - v) + v + w \in I_1 \cdot I_2 + I_1^2 + I_2^2 = I^2$.
It follows that the canonical homomorphism

$$I/I^2 \longrightarrow I/(I_1^2 + I_2) \oplus I/(I_1 + I_2^2) \tag{A.1}$$

is an isomorphism. Moreover, the canonical map

$$I_1/(I_1 \cap I_2 + I_1^2) \longrightarrow I/(I_1^2 + I_2) \tag{A.2}$$

and its symmetric analogue are easily seen to be isomorphisms. Finally, there is a natural isomorphism

$$R_*/I \otimes_{R_*} I_1/I_1^2 \cong I_1/(I_1 \cap I_2 + I_1^2), \tag{A.3}$$

given by the following composition:

$$\begin{aligned} R_*/I \otimes_{R_*} I_1/I_1^2 &\cong R_*/I_2 \otimes_{R_*} I_1/I_1^2 \cong (I_1/I_1^2)/\left(I_2 \cdot (I_1/I_1^2)\right) \\ &\cong I/(I_1 \cdot I_2 + I_1^2) \cong I_1/(I_1 \cap I_2 + I_1^2). \end{aligned}$$

Combining (A.1), (A.2) and (A.3) implies the result. □

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