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SPACES WITH NOETHERIAN COHOMOLOGY

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Abstract Is the cohomology of the classifying space of a p-compact group, with Noetherian twisted coefficients, a Noetherian module? In this paper we provide, over the ring of p-adic integers, such a generalization to p-compact groups of the Evens–Venkov Theorem. We consider the cohomology of a space with coefficients in a module, and we compare Noetherianity over the field with p elements with Noetherianity over the p-adic integers, in the case when the fundamental group is a finite p-group.

Keywords: Noetherian; cohomology; universal coefficients; classifying space; p-compact groups; p-local finite groups

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Introduction

The main theorem of Dwyer and Wilkerson in [**13**] states that the mod p cohomology of the classifying space of a p-compact group is a finitely generated algebra. This generalizes to p-compact groups the Evens–Venkov Theorem [**14**] on the cohomology of a finite group G. There are, however, two main differences between these two results. Evens's statements allow a general base ring (any Noetherian ring is allowed) and they include the case of general twisted coefficients (this is contrary to the early work by Golod [**16**] or Venkov [**25**]) as follows: if M is Noetherian over a ring R, then $H^*(G; M)$ is Noetherian over $H^*(G; R)$. Beautiful finite generation statements on cohomology have since been proved in numerous situations. For statements as general as Evens's, however, proofs have been surprisingly elusive.

This paper is concerned with these generalizations for p-compact groups and p-local finite groups, as defined by Broto, Levi and Oliver [**9**]. More generally, we ask when Noetherianity of the mod p cohomology algebra $H^*(Y;\mathbb{F}_p)$ of a space Y implies that the cohomology with coefficients in a $R[\pi_1(Y)]$-module M, $H^*(Y;M)$, is a Noetherian module over the algebra $H^*(Y;R)$. Because the classifying space BX of a p-compact group is p-complete by definition, we work over p-complete rings (for example, $H^*((BS^3)^\wedge_p;\mathbb{Z})$ is not Noetherian).

Theorem 2.4. *Let Y be a connected space with finite fundamental group. The graded $\mathbb{Z}^\wedge_p$-algebra $H^*(Y;\mathbb{Z}^\wedge_p)$ is then Noetherian if and only if the graded $\mathbb{F}_p$-algebra $H^*(Y;\mathbb{F}_p)$ is Noetherian and the torsion in $H^*(Y;\mathbb{Z}^\wedge_p)$ is bounded.*

Theorem 3.6. *Let Y be a connected space such that $\pi_1 Y$ is a finite p-group. Let M be a $\mathbb{Z}^\wedge_p[\pi_1 Y]$-module that is finitely generated over $\mathbb{Z}^\wedge_p$. If the graded $\mathbb{Z}^\wedge_p$-algebra $H^*(Y;\mathbb{Z}^\wedge_p)$ is Noetherian, then $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;\mathbb{Z}^\wedge_p)$.*

This applies to p-compact group and to p-local finite groups to show that their p-adic cohomology algebra is Noetherian (see Theorems 4.2 and 4.5). Note that our proof makes no use of the recent classification of p-compact groups by Andersen and Grodal [**3**], Andersen, Grodal, Møller and Viruel [**4**] and Møller [**22**]. Even in the case of a compact Lie group G, our theorem provides a general finiteness theorem for the cohomology of BG with twisted coefficients. One of the few explicit computations available in the literature is for the case of $O(n)$ and is due to Čadek [**11**] (see also [**17**]).

1. The cohomology as a graded module

Before considering the mod p or p-adic cohomology as an algebra, we first make explicit the relationship between two standard milder finiteness assumptions. When the graded vector space $H^*(Y;\mathbb{F}_p)$ is of finite type, i.e. $H^n(Y;\mathbb{F}_p)$ is a finite-dimensional vector space in each degree n, is $H^n(Y;\mathbb{Z}^\wedge_p)$ a finitely generated $\mathbb{Z}^\wedge_p$-module in each degree n as well? This is clearly a necessary condition for the cohomology algebra to be finitely generated. We show that it holds when $\pi_1(Y)$ is finite.

The main tool to relate the mod p cohomology to the p-adic cohomology is the universal coefficient exact sequence (see, for example, [**24**, Theorem 5.5.10] for spaces and [**1**, Part III, Proposition 6.6] for spectra)

$$0 \to H^*(Y;\mathbb{Z}^\wedge_p)\otimes\mathbb{F}_p \xrightarrow{\rho} H^*(Y;\mathbb{F}_p) \xrightarrow{\partial} \mathrm{Tor}(H^{*+1}(Y;\mathbb{Z}^\wedge_p);\mathbb{Z}/p) \to 0, \tag{1.1}$$

which holds, since $\mathbb{Z}^\wedge_p$ is a principal ideal domain and $\mathbb{Z}/p$ is a finitely generated $\mathbb{Z}^\wedge_p$-module.

Remark 1.1. The morphism ρ in (1.1) is a ring homomorphism that makes the middle term $H^*(Y;\mathbb{F}_p)$ an $H^*(Y;\mathbb{Z}^\wedge_p)\otimes\mathbb{F}_p$-module. Evens observed in [**15**, p. 272] that ∂ is also a homomorphism of $H^*(Y;\mathbb{Z}^\wedge_p)\otimes\mathbb{F}_p$-modules, where $\mathrm{Tor}(H^*(Y;\mathbb{Z}^\wedge_p);\mathbb{Z}/p)$ has the natural module structure that Evens introduced in [**15**, Lemma 2].

Lemma 1.2. *Let G be a finite p-group, let K be a field of characteristic p and let V be a KG-module. If V^G is a finite-dimensional K-vector space, then so is V.*

Proof. Let $n = \dim_K V^G$ and let $F = (KG)^n$ be a free KG-module of rank n. Note that $\dim_K F^G = n$, so there is an isomorphism of KG-modules $\alpha \colon V^G \to F^G$. Since F is an injective KG-module, α extends to a homomorphism $\alpha' \colon V \to F$ of KG-modules, which we now prove is injective. Clearly, $(\ker \alpha')^G = \ker \alpha' \cap V^G = \ker \alpha = 0$. Since G is a finite p-group, it follows that $\ker \alpha' = 0$. Hence V embeds in F, so V is finite dimensional. $\square$

Proposition 1.3. *Let Y be a connected space with finite fundamental group. The group $H^n(Y;\mathbb{F}_p)$ is finite for every positive integer n if and only if the $\mathbb{Z}_p^\wedge$-module $H^n(Y;\mathbb{Z}_p^\wedge)$ is finitely generated for every n. Under this condition, the $\mathbb{Z}_p^\wedge$-module $H^n(Y;M)$ is finitely generated for any n and every $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module M that is finitely generated over $\mathbb{Z}_p^\wedge$.*

Proof. If $H^n(Y;\mathbb{Z}_p^\wedge)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for any n, the universal coefficient exact sequence (1.1) implies that $H^n(Y;\mathbb{F}_p)$ is finite for any n.

Conversely, assume that $H^n(Y;\mathbb{F}_p)$ is finite for every n. Since the fundamental group of Y is finite, the space Y is p-good by [**8**, Proposition VII.5.1] and therefore $H^n(Y_p^\wedge;\mathbb{F}_p) \cong H^n(Y;\mathbb{F}_p)$. Likewise, since cohomology with p-adic coefficients is represented by Eilenberg–MacLane spaces $K(\mathbb{Z}_p^\wedge, n)$, which are p-complete, $H^n(Y_p^\wedge;\mathbb{Z}_p^\wedge) \cong H^n(Y;\mathbb{Z}_p^\wedge)$ [**8**, Proposition II.2.8]. We may therefore assume that Y is p-complete and that $G = \pi_1 Y$ is a finite p-group (see [**13**, Proposition 11.14] or [**7**, § 5]).

If Y is 1-connected, then [**2**, Proposition 5.7] applies and $H^n(Y;\mathbb{Z}_p^\wedge)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for every n. For the general situation, let us consider the universal cover fibration for Y, $\tilde{Y} \to Y \to BG$. We prove by induction that $H^n(\tilde{Y};\mathbb{F}_p)$ is finite dimensional for any n. The induction starts with the trivial case $n = 0$. Assume thus that $H^m(\tilde{Y};\mathbb{F}_p)$ is finite for all $m < n$. Then, in the second page of the Serre spectral sequence in mod p cohomology, all groups $E_2^{i,j} = H^i(BG, H^j(\tilde{Y};\mathbb{F}_p))$ on the lines $j = 0, \ldots, n-1$ are finite. As $E_\infty^{0,n}$ is finite as well, it follows that $E_2^{0,n} = H^n(\tilde{Y};\mathbb{F}_p)^G$ is finite dimensional. Since G is a finite p-group, finiteness of $H^n(\tilde{Y};\mathbb{F}_p)^G$ implies finiteness of $H^n(\tilde{Y};\mathbb{F}_p)$ by Lemma 1.2.

We can now apply the 1-connected case to conclude that $H^n(\tilde{Y};\mathbb{Z}_p^\wedge)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for any n. The Evens–Venkov Theorem [**14**, Theorem 8.1] now shows that the E_2-term of the Serre spectral sequence with p-adic coefficients consists of finitely generated $\mathbb{Z}_p^\wedge$-modules. Therefore, $H^n(Y;\mathbb{Z}_p^\wedge)$ must also consist of finitely generated $\mathbb{Z}_p^\wedge$-modules for any n.

The second part of the assertion now follows easily. The first part of the proposition and the universal coefficient formula imply that $H^n(\tilde{Y};M)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module for every n. We then use the Serre spectral sequence for cohomology with twisted coefficients. The only reference we know is [**21**, Theorem 3.2], where the spectral sequence is established equivariantly; we need the case of the trivial group action. $\square$

2. Cohomology with trivial coefficients

We now turn to finite generation of the cohomology algebras $H^*(Y;\mathbb{Z}_p^\wedge)$ and $H^*(Y;\mathbb{F}_p)$, where trivial coefficients are understood. The universal coefficient theorem suggests that control of torsion is the key of this case.

Let R be either the ring $\mathbb{Z}_p^\wedge$ or the field $\mathbb{F}_p$, and note that both are Noetherian rings. The cohomology $H^*(Y;R)$ of any connected space is a commutative graded algebra, which is a Noetherian R-algebra if and only if it is finitely generated as an R-algebra [**19**, Theorem 13.1].

Lemma 2.1. *Let Y be a connected space. If the $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ is Noetherian, then $H^*(Y;\mathbb{F}_p)$ is a finitely generated module over the algebra $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$.*

Proof. The ideal $\mathrm{Tor}(H^*(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p)$ of elements annihilated by p is a finitely generated ideal of $H^*(Y;\mathbb{Z}_p^\wedge)$ by assumption. It is therefore also finitely generated as an $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$-module. The conclusion follows from Remark 1.1 on the universal coefficient exact sequence. $\square$

To be able to compare Noetherianity of the mod p cohomology and the p-adic cohomology, we need to analyse the p-torsion in $H^*(Y;\mathbb{Z}_p^\wedge)$. Let us denote by $\mathrm{T}_pH^*(Y;\mathbb{Z}_p^\wedge)$ the graded submodule of p-torsion elements. The key assumption in the main theorem of this section is that the order of the p-torsion is bounded. This implies that ρ is 'uniformly power surjective': a strong form of integrality.

Lemma 2.2. *Let Y be a connected space and let d be an integer such that the p-torsion $T_pH(Y;\mathbb{Z}_p^\wedge)$ is annihilated by p^d. If $u\in H^*(Y;\mathbb{F}_p)$, then u^{p^d} belongs to the image of $\rho\colon H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p\to H^*(Y;\mathbb{F}_p)$.*

Proof. Following the elementary proof of [**6**, Lemma 4.4], we start with the observation that for any element $x\in H^*(Y;\mathbb{Z}/p^k)$ the pth power x^p lies in the image of the reduction map $H^*(Y;\mathbb{Z}/p^{k+1})\to H^*(Y;\mathbb{Z}/p^k)$. The argument is as follows. If p is odd and the degree of x is odd, $x^p=0$ and the conclusion follows. Otherwise, $\delta(x^p)=p\delta(x)\cdot x^{p-1}=0$, because the Bockstein δ coming from the short exact sequence $\mathbb{Z}/p\to\mathbb{Z}/p^{k+1}\to\mathbb{Z}/p^k$ is a derivation with respect to the cup product pairing

$$H^*(Y;\mathbb{Z}/p^k)\otimes H^*(Y;\mathbb{Z}/p)\to H^*(Y;\mathbb{Z}/p).$$

Therefore, u^{p^d} lies in the image of the reduction $H^*(Y;\mathbb{Z}/p^{d+1})\to H^*(Y;\mathbb{F}_p)$.

The diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p^{d+1}"] \arrow[d, "\cdot p^d"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z}/p^{d+1} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/p \arrow[r] & 0
\end{tikzcd}$$

induces the commutative diagram of exact rows:

$$\begin{tikzcd}
0 \to \mathrm{Tor}(H^{*+1}(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p^{d+1}) \arrow[r] \arrow[d] & H^{*+1}(Y;\mathbb{Z}_p^\wedge) \arrow[r, "\cdot p^{d+1}"] \arrow[d, "\cdot p^d"] & H^{*+1}(Y;\mathbb{Z}_p^\wedge) \arrow[d, equal] \\
0 \to \mathrm{Tor}(H^{*+1}(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p) \arrow[r] & H^{*+1}(Y;\mathbb{Z}_p^\wedge) \arrow[r, "\cdot p"] & H^{*+1}(Y;\mathbb{Z}_p^\wedge)
\end{tikzcd}$$

Since $p^d \cdot \mathrm{T}_p H^*(Y;\mathbb{Z}_p^\wedge) = 0$, the left vertical morphism is zero. Consider now the two universal coefficient sequences respectively relating the cohomology of Y with coefficients in $\mathbb{F}_p$ and in $\mathbb{Z}/p^{d+1}$ to the cohomology of Y with coefficients in $\mathbb{Z}_p^\wedge$:

$$\begin{tikzcd}
0 \to H^*(Y;\mathbb{Z}_p^\wedge) \otimes \mathbb{Z}/p^{d+1} \arrow[r] \arrow[d] & H^*(Y;\mathbb{Z}/p^{d+1}) \arrow[r] \arrow[d] & \mathrm{Tor}(H^{*+1}(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p^{d+1}) \to 0 \arrow[d] \\
0 \to H^*(Y;\mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p \arrow[r, "\rho"] & H^*(Y;\mathbb{F}_p) \arrow[r, "\partial"] & \mathrm{Tor}(H^{*+1}(Y;\mathbb{Z}_p^\wedge);\mathbb{Z}/p) \to 0
\end{tikzcd}$$

where the vertical morphisms are induced by the mod p reduction $\mathbb{Z}/p^{d+1} \to \mathbb{Z}/p$. The element u^{p^d} lies in the image of the mod p reduction and we have shown that the morphism between the torsion groups on the right is zero. Therefore, $\partial(u^{p^d}) = 0$, which implies that u^{p^d} is in $\mathrm{Im}\,\rho$. $\square$

Lemma 2.3. *Let Y be a connected space. If the graded $\mathbb{F}_p$-algebra $H^*(Y;\mathbb{F}_p)$ is Noetherian and if $H^*(Y;\mathbb{Z}_p^\wedge)$ has bounded torsion, then $H^*(Y;\mathbb{F}_p)$ is a finitely generated module over $H^*(Y;\mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$.*

Proof. This is clear since Lemma 2.2 implies that $H^*(Y;\mathbb{F}_p)$ is integral over $\mathrm{Im}\,\rho$. Explicitly, let us choose homogeneous generators $w_1, \ldots, w_n$ of the graded algebra $H^*(Y;\mathbb{F}_p)$ and consider the finite set W of monomials of the form $w_1^{r_1} \cdots w_n^{r_n}$ with $0 \leqslant r_i < p^d$. We show that the set W generates $H^*(Y;\mathbb{F}_p)$ as a module over $H^*(Y;\mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$. Consider any monomial $m = w_1^{s_1} \cdots w_n^{s_n}$ in $H^*(Y;\mathbb{F}_p)$. Writing the exponents $s_i = r_i + p^d t_i$ with $0 \leqslant r_i < p^d$, we express $m = x^{p^d} \cdot w$ for a monomial w in W and a homogeneous element x. By Lemma 2.2, x^{p^d} lifts to an element a in $H^*(Y;\mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$ and $m = \rho(a) \cdot w$. $\square$

Theorem 2.4. *Let Y be a connected space with finite fundamental group. The graded $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ is then Noetherian if and only if the graded $\mathbb{F}_p$-algebra $H^*(Y;\mathbb{F}_p)$ is Noetherian and the torsion in $H^*(Y;\mathbb{Z}_p^\wedge)$ is bounded.*

Proof. Assume first that $H^*(Y;\mathbb{Z}_p^\wedge)$ is a Noetherian $\mathbb{Z}_p^\wedge$-algebra. By Lemma 2.1, $H^*(Y;\mathbb{F}_p)$ is a finitely generated module over $H^*(Y;\mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$. Since $H^*(Y;\mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$ is a Noetherian $\mathbb{F}_p$-algebra, it follows from [**5**, Proposition 7.2] that $H^*(Y;\mathbb{F}_p)$ is also a Noetherian $\mathbb{F}_p$-algebra. The torsion part $\mathrm{T}_p H^*(Y;\mathbb{Z}_p^\wedge)$ is an ideal of the Noetherian algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ and hence is finitely generated. The order of the torsion is thus bounded by the order of its generators.

Suppose now that $H^*(Y;\mathbb{F}_p)$ is a Noetherian $\mathbb{F}_p$-algebra and that the torsion in $H^*(Y;\mathbb{Z}_p^\wedge)$ is bounded. Then, by Lemma 2.3, $H^*(Y;\mathbb{F}_p)$ is a finitely generated module over $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$. As a consequence of the graded version of the so-called Eakin–Nagata Theorem (see Proposition A 1), we infer that the graded subring $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$ of $H^*(Y;\mathbb{F}_p)$ is also Noetherian. Since $H^*(Y;\mathbb{F}_p)$ is finitely generated, Proposition 1.3 shows that $H^n(Y;\mathbb{Z}_p^\wedge)$ is a finitely generated $\mathbb{Z}_p^\wedge$-module, and is hence Hausdorff, in each degree. Thus $H^*(Y;\mathbb{Z}_p^\wedge)$ is a Noetherian $\mathbb{Z}_p^\wedge$-algebra by Corollary A 3. □

We end this section with an example showing that Theorem 2.4 does not hold without the condition on torsion.

Example 2.5. In [**2**] Aguadé, Broto and Notbohm constructed spaces $X_k(r)$ that, for any odd prime p with $r|p-1$ and $k\geqslant 0$, satisfy

$$H^*(X_k(r);\mathbb{F}_p)\cong\mathbb{F}_p[x_{2r}]\otimes E(\beta^{(k+1)}x_{2r}),$$

where $\beta^{(k+1)}$ denotes the Bockstein of order $k+1$. Observe that $H^*(X_k(r);\mathbb{F}_p)$ is a Noetherian $\mathbb{F}_p$-algebra. The torsion of $H^*(X_k(r);\mathbb{Z}_p^\wedge)$ is unbounded by [**2**, Remark 5.8]. Theorem 2.4 shows that the algebra $H^*(X_k(r);\mathbb{Z}_p^\wedge)$ is not Noetherian.

3. Cohomology with twisted coefficients

In this section we work over a ring R that is either $\mathbb{Z}_p^\wedge$ or $\mathbb{F}_p$. Let Y be a connected space whose fundamental group is a finite p-group. Let M be an $R[\pi_1 Y]$-module that is a finitely generated R-module. We aim to show that the cohomology with twisted coefficients $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;R)$ if $H^*(Y,R)$ is Noetherian. We shall deal separately with the field of p elements and with the ring of p-adic integers.

We start with a standard Noetherianity result.

Lemma 3.1. *Let $R=\mathbb{Z}_p^\wedge$ or $\mathbb{F}_p$. Let Y be a space and let $0\to N\to M\to Q\to 0$ be a short exact sequence of $R[\pi_1 Y]$-modules. If both $H^*(Y;N)$ and $H^*(Y;Q)$ are Noetherian modules over $H^*(Y;R)$, then so is $H^*(Y;M)$.*

Proof. The long exact sequence in cohomology induced by the short exact sequence of modules is one of $H^*(Y;R)$-modules. It exhibits $H^*(Y;M)$ as an extension of a submodule of $H^*(Y;Q)$ by a quotient of $H^*(Y;N)$. □

3.1. The case of $\mathbb{F}_p$-vector spaces

To prove the next result we follow Minh and Symonds's approach for profinite groups [**20**, Lemma 1].

Theorem 3.2. *Let Y be a connected space such that $\pi_1 Y$ is a finite p-group and let M be a finite $\mathbb{F}_p[\pi_1 Y]$-module. If the graded $\mathbb{F}_p$-algebra $H^*(Y;\mathbb{F}_p)$ is Noetherian, then $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;\mathbb{F}_p)$.*

Proof. We use induction on $\dim_{\mathbb{F}_p} M$. Since $G=\pi_1(Y)$ is a finite p-group, the invariant submodule M^G is not trivial when M is not trivial. The induction step follows by applying Lemma 3.1 to the short exact sequence $0\to M^G\to M\to M/M^G\to 0$. □

3.2. The case of $\mathbb{Z}_p^\wedge$-modules

In this section we consider the cohomology with twisted coefficients $H^*(Y;M)$ of a connected space Y, where M is a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module that is finitely generated over $\mathbb{Z}_p^\wedge$. In a first step, let M be a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module that is finite (meaning finite as a set).

Lemma 3.3. *Let Y be a connected space such that $\pi_1 Y$ is a finite p-group. Let M be a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module that is finite. If the graded $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ is Noetherian, then $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;\mathbb{Z}_p^\wedge)$.*

Proof. The module M being finite, M is a finite abelian p-group. We perform an induction on the exponent e of M. When $e = 1$, the module M has the structure of an $\mathbb{F}_p$-vector space. As $H^*(Y;\mathbb{F}_p)$ is a Noetherian $\mathbb{F}_p$-algebra by Theorem 2.4, we know from Theorem 3.2 that $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;\mathbb{F}_p)$. The Noetherian $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ acts on $H^*(Y;M)$ through $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$. By Lemma 2.1, $H^*(Y;\mathbb{F}_p)$ is finitely generated as an $H^*(Y;\mathbb{Z}_p^\wedge)$-module. Therefore, $H^*(Y;M)$ is a Noetherian module over $H^*(Y;\mathbb{Z}_p^\wedge)$.

Let us now assume that $e > 1$ and consider the short exact sequence

$$0 \to M_p \to M \to Q \to 0,$$

where M_p is the submodule of M consisting of elements of order 1 or p. The induction step follows from Lemma 3.1. □

Remark 3.4. In the case of trivial coefficient modules our main tool was the universal coefficient exact sequence, but this does not exist in general for twisted coefficients. One basic counter-example is given by the module $M = \mathbb{F}_p[G]$ for a finite group G whose order is divisible by p. Then $H^*(BG;M)$ is zero in positive degrees and the universal coefficient formula does not hold.

In a second step we consider cohomology with coefficients in a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module M, which is free of finite rank over $\mathbb{Z}_p^\wedge$.

Lemma 3.5. *Let Y be a connected space such that $\pi_1 Y$ is a finite p-group. Let M be a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module that is free of finite rank over $\mathbb{Z}_p^\wedge$. If the graded $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ is Noetherian, then $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;\mathbb{Z}_p^\wedge)$.*

Proof. The short exact sequence $0 \to M \xrightarrow{\cdot p} M \to M\otimes\mathbb{F}_p \to 0$ induces in cohomology a long exact sequence of $H^*(Y;\mathbb{Z}_p^\wedge)$-modules. We see that $H^*(Y;M)\otimes\mathbb{F}_p$ is a sub-$H^*(Y;\mathbb{Z}_p^\wedge)$-module of $H^*(Y;M\otimes\mathbb{F}_p)$. Since the action of $H^*(Y;\mathbb{Z}_p^\wedge)$ on both $H^*(Y;M\otimes\mathbb{F}_p)$ and $H^*(Y;M)\otimes\mathbb{F}_p$ factors through $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$, it follows that $H^*(Y;M)\otimes\mathbb{F}_p$ is a sub-$H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$-module of $H^*(Y;M\otimes\mathbb{F}_p)$.

This takes us back to the world of $\mathbb{F}_p$-vector spaces. We know from Theorem 3.2 that $H^*(Y;M\otimes\mathbb{F}_p)$ is a Noetherian module over $H^*(Y;\mathbb{F}_p)$, where $H^*(Y;\mathbb{F}_p)$ is a Noetherian algebra by Theorem 2.4. As this algebra is a finitely generated module over $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$ by Lemma 2.1, we infer that $H^*(Y;M\otimes\mathbb{F}_p)$ is a Noetherian module over $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$. Therefore, $H^*(Y;M)\otimes\mathbb{F}_p$ is a Noetherian module over $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$ as well, and since $H^*(Y;\mathbb{Z}_p^\wedge)$ acts on $H^*(Y;M)\otimes\mathbb{F}_p$ via $H^*(Y;\mathbb{Z}_p^\wedge)\otimes\mathbb{F}_p$, it is a Noetherian module over $H^*(Y;\mathbb{Z}_p^\wedge)$.

Set $A^* = H^*(Y;\mathbb{Z}_p^\wedge)$ and $N^* = H^*(Y;M)$. Both are finitely generated $\mathbb{Z}_p^\wedge$-modules in each degree by Proposition 1.3 and are thus also Hausdorff and complete. We then conclude by applying Proposition A 2. □

We now prove our main theorem.

Theorem 3.6. *Let Y be a connected space such that $\pi_1 Y$ is a finite p-group. Let M be a $\mathbb{Z}_p^\wedge[\pi_1 Y]$-module that is finitely generated over $\mathbb{Z}_p^\wedge$. If the graded $\mathbb{Z}_p^\wedge$-algebra $H^*(Y;\mathbb{Z}_p^\wedge)$ is Noetherian, then $H^*(Y;M)$ is Noetherian as a module over $H^*(Y;\mathbb{Z}_p^\wedge)$.*

Proof. Let TM be the torsion submodule of M and consider the short exact sequence of $\mathbb{Z}_p^\wedge[\pi_1 Y]$-modules $0\to TM\to M\to Q\to 0$. We know from Lemma 3.3 that $H^*(Y;TM)$ is a Noetherian $H^*(Y;\mathbb{Z}_p^\wedge)$-module and from Lemma 3.5 that $H^*(Y;Q)$ is as well. We conclude by using Lemma 3.1. □

Remark 3.7. Our main theorem assumes only that the fundamental group is a finite p-group. One could try to relax this assumption with transfer arguments, requiring a version of the transfer with twisted coefficients. However, recent work of Levi and Ragnarsson [**18**, Proposition 3.1], in the context of p-local finite group theory, provides an example showing that such a transfer might not have, in general, the properties we need when the fundamental group of the space is not a p-group.

4. The case of *p*-compact groups and *p*-local finite groups

We arrive at the promised application to p-compact groups and p-local finite groups. By definition, a p-compact group is a mod p finite loop space $X = \Omega BX$, where the 'classifying space' BX is p-complete [**13**].

Lemma 4.1. *Let X be a p-compact group. Then the p-torsion in $H^*(BX;\mathbb{Z}_p^\wedge)$ is bounded.*

Proof. By [**13**, Proposition 9.9], any p-compact group admits a maximal toral p-compact subgroup S such that $\iota\colon BS\to BX$ is a monomorphism and the Euler characteristic χ of the homotopy fibre is prime to p (see [**13**, proof of Theorem 2.4, p. 431]). The Euler characteristic is the alternating sum of the ranks of the $\mathbb{F}_p$-homology groups. Dwyer constructed a transfer map $\tau\colon \Sigma^\infty BX\to\Sigma^\infty BS$ in [**12**] such that $\iota\circ\tau$ induces multiplication by χ on mod p cohomology. This is an isomorphism, so that the homotopy cofibre C of $\iota\circ\tau$ has trivial mod p cohomology.

Moreover, both BX and BS have finite mod p cohomology in each degree and finite fundamental group [**13**, Lemma 2.1]. Proposition 1.3 applies, so, in any degree, the p-adic cohomology modules of BX and BS are finitely generated over $\mathbb{Z}_p^\wedge$. The long exact sequence in cohomology associated to a cofibration then shows that the $\mathbb{Z}_p^\wedge$-modules $H^n(C;\mathbb{Z}_p^\wedge)$ are finitely generated for all n. Since $H^*(C;\mathbb{F}_p)$ is trivial, it follows from the universal coefficient exact sequence (1.1) that $H^*(C; \mathbb{Z}_p^\wedge) \otimes \mathbb{F}_p$ is trivial as well. We conclude, by the Nakayama Lemma, that $H^*(C; \mathbb{Z}_p^\wedge)$ is trivial, i.e. $\iota \circ \tau$ also induces an isomorphism in cohomology with p-adic coefficients. Therefore, $\iota^* \colon H^*(BX; \mathbb{Z}_p^\wedge) \to H^*(BS; \mathbb{Z}_p^\wedge)$ is a monomorphism. It is thus sufficient to show that $H^*(BS; \mathbb{Z}_p^\wedge)$ has bounded torsion.

Now, a toral p-compact group S can be constructed, up to p-completion, as an extension of a finite p-group P and a discrete torus $H = \bigoplus \mathbb{Z}_{p^\infty}$. The fibration

$$BH_p^\wedge \simeq K\Big(\bigoplus \mathbb{Z}_p^\wedge, 2\Big) \to BS \to BP$$

yields a finite covering $BH_p^\wedge \to BS$, and a classical transfer argument then shows that multiplication by $|P|$ on $H^*(BS; \mathbb{Z}_p^\wedge)$ factors through the torsion-free module $H^*(BH_p^\wedge; \mathbb{Z}_p^\wedge)$. □

Theorem 4.2. *Let X be a p-compact group, let M be a finite $\mathbb{F}_p[\pi_1 BX]$-module, and let N be a $\mathbb{Z}_p^\wedge[\pi_1 BX]$-module that is finitely generated over $\mathbb{Z}_p^\wedge$. Then*

(1) *the $\mathbb{Z}_p^\wedge$-algebra $H^*(BX; \mathbb{Z}_p^\wedge)$ is Noetherian;*

(2) *the module $H^*(BX; M)$ is Noetherian over $H^*(BX; \mathbb{F}_p)$;*

(3) *the module $H^*(BX; N)$ is Noetherian over $H^*(BX; \mathbb{Z}_p^\wedge)$.*

Proof. The main theorem of Dwyer and Wilkerson [**13**, Theorem 2.4] asserts that $H^*(BX; \mathbb{F}_p)$ is Noetherian. Lemma 4.1 allows us to apply our Theorem 2.4 to prove the first claim. The second claim then follows from Theorem 3.2 because $\pi_1 BX$ is a finite p-group [**13**, Lemma 2.1]. Finally, Theorem 3.6 implies the third claim. □

Remark 4.3. Let us consider the case of $BO(n)$ at the prime 2 (the fundamental group is cyclic of order 2). Brown made an explicit computation of the integral cohomology in [**10**], proving that the square of any even Stiefel–Whitney class w_{2i}^2 belongs to the image of ρ, and the technique we use in Lemma 2.3 is somewhat inspired by his computations. Even though the relations in the mod p cohomology of an arbitrary p-compact group (one which is not p-torsion free) make it difficult to exhibit explicit generators for the p-adic cohomology, Theorem 4.2 (1) gains qualitative control over it.

As for twisted coefficients, let $\mathbb{Z}^\vee$ be a free abelian group of rank 1, endowed with the sign action of the fundamental group C_2. In [**11**, Theorem 1] Čadek exhibits an explicit *finite* set of generators of $H^*(BO(n); \mathbb{Z}^\vee)$ as a module over $H^*(BO(n); \mathbb{Z})$. This is one of the few available explicit computations illustrating our results.

In [**9**] Broto, Levi and Oliver defined the concept of a p-local finite group. It consists of a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a finite p-group and $\mathcal{F}$ and $\mathcal{L}$ are two categories whose objects are subgroups of S. The category $\mathcal{F}$ models abstract conjugacy relations among the subgroups of S, and $\mathcal{L}$ is an extension of $\mathcal{F}$ with enough information to define a classifying space $|\mathcal{L}|_p^\wedge$ that behaves like the p-completed classifying space of a finite group. In fact, to any finite group G corresponds a p-local finite group with $|\mathcal{L}|_p^\wedge \simeq (BG)_p^\wedge$, but there are also other 'exotic' p-local finite groups.

Lemma 4.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. The p-torsion in $H^*(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge)$ is then bounded.*

Proof. In [**9**, p. 815] Broto, Levi and Oliver show, following an idea due to Linckelmann and Webb (see also [**23**]), that the suspension spectrum $\Sigma^\infty(|\mathcal{L}|_p^\wedge)$ is a retract of $\Sigma^\infty BS$. Since the order of S annihilates all cohomology groups of BS, the same holds for $H^*(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge)$. □

Theorem 4.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group, let M be a finite $\mathbb{F}_p[\pi_1(|\mathcal{L}|_p^\wedge)]$-module, and let N be a $\mathbb{Z}_p^\wedge[\pi_1(|\mathcal{L}|_p^\wedge)]$-module that is finitely generated over $\mathbb{Z}_p^\wedge$. Then*

(1) *the $\mathbb{Z}_p^\wedge$-algebra $H^*(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge)$ is Noetherian;*

(2) *the module $H^*(|\mathcal{L}|_p^\wedge; M)$ is Noetherian over $H^*(|\mathcal{L}|_p^\wedge; \mathbb{F}_p)$;*

(3) *the module $H^*(|\mathcal{L}|_p^\wedge; N)$ is Noetherian over $H^*(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge)$.*

Proof. We follow the same steps we took for p-compact groups in Theorem 4.2. The first ingredient is the stable elements theorem [**9**, Theorem B], which also shows that $H^*(|\mathcal{L}|_p^\wedge; \mathbb{F}_p)$ is Noetherian. We just proved that the torsion in $H^*(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge)$ is bounded. Moreover, the fundamental group of $|\mathcal{L}|_p^\wedge$ is a finite p-group by [**9**, Proposition 1.12]. □

Appendix A

This short appendix deals with Noetherianity in the graded case over the p-adics. We start, however, with a more general result that is probably well known to the experts: the graded Eakin–Nagata Theorem. The non-graded version can be found, for example, in [**19**, Theorem 3.7 (i)].

Proposition A 1. *Let A^* be a graded subring of B^*. Assume that B^* is Noetherian as a ring and finitely generated as an A^*-module. Then A^* is also a Noetherian ring.*

Proof. By [**19**, Theorem 13.1], B^0 is Noetherian and B^* is a finitely generated B^0-algebra. Moreover, B^0 is a finitely generated A^0-module and therefore B^* is a finitely generated A^0-algebra. Also, A^0 is Noetherian by the classical Eakin–Nagata Theorem [**19**, Theorem 3.7 (i)]. Applying [**5**, Proposition 7.8] to the inclusions $A^0 \subset A^* \subset B^*$, we obtain that A^* is a finitely generated A^0-algebra. Again by [**19**, Theorem 13.1], A^* is a Noetherian ring. □

The following technical proposition allows us to deduce Noetherianity over the p-adics from the Noetherianity of the mod p reduction.

Proposition A 2. *Let A^* be a graded $\mathbb{Z}_p^\wedge$-algebra such that in each degree A^k is complete for the p-adic topology. Let N^* be a graded A^*-module such that for all k, N^k is Hausdorff for the p-adic topology. If $N^* \otimes \mathbb{F}_p$ is a Noetherian A^*-module, then so is N^*.*

Proof. Let us choose homogeneous elements $\nu_1, \ldots, \nu_t \in N^*$ such that $\nu_1 \otimes 1, \ldots, \nu_t \otimes 1$ generate $N^* \otimes \mathbb{F}_p$ as an A^*-module. We claim that $\nu_1, \ldots, \nu_t$ generate N^* as an A^*-module. Given $n \in N^*$ we may write $n \otimes 1 = \sum a_i^0 (\nu_i \otimes 1)$ for some $a_i^0 \in A^*$. Define $n_0 = \sum a_i^0 \nu_i$ and notice that $n - n_0 \in pN^*$. Thus, there exists an element $m_1 \in N^*$, homogeneous of degree at most $\deg n$, such that $n - n_0 = pm_1$. We iterate the procedure and find elements $a_i^1 \in A^*$ such that $m_1 \otimes 1 = \sum a_i^1 (\nu_i \otimes 1)$. We define

$$n_1 = n_0 + p \sum a_i^1 \nu_i = \sum (a_i^0 + p a_i^1) \nu_i.$$

In this way we construct, for any i, Cauchy sequences of coefficients $(a_i^0 + p a_i^1 + \cdots + p^k a_i^k)_k$ in A^*. By completeness this sequence converges to some $a_i \in A^*$. Since N^* is Hausdorff, the element $\sum a_i \nu_i$ is equal to n. □

In the following corollary, the assumption that A^* be connected, i.e. $A^0 = \mathbb{Z}_p^\wedge$, is important.

Corollary A 3. *Let A^* be a graded connected Hausdorff $\mathbb{Z}_p^\wedge$-algebra. If $A^* \otimes \mathbb{F}_p$ is a Noetherian $\mathbb{F}_p$-algebra, then A^* is a Noetherian $\mathbb{Z}_p^\wedge$-algebra.*

Proof. Since $\mathbb{Z}_p^\wedge$ is Noetherian and A^* is connected, A^* is a Noetherian $\mathbb{Z}_p^\wedge$-algebra if and only if A^* is a finitely generated $\mathbb{Z}_p^\wedge$-algebra [**19**, Theorem 13.1]. Note that $A^* \otimes \mathbb{F}_p$ is also a Noetherian $\mathbb{Z}_p^\wedge$-algebra via the mod p reduction $\mathbb{Z}_p^\wedge \to \mathbb{F}_p$. Let us choose homogeneous elements $\gamma_1, \ldots, \gamma_n \in A^*$ such that $\gamma_1 \otimes 1, \ldots, \gamma_n \otimes 1$ generate $A^* \otimes \mathbb{F}_p$ as a $\mathbb{Z}_p^\wedge$-algebra. For a fixed $k \geqslant 0$, $A^k \otimes \mathbb{F}_p$ is generated as a $\mathbb{Z}_p^\wedge$-module by the monomials $(\gamma_1 \otimes 1)^{e_1} \cdots (\gamma_n \otimes 1)^{e_n}$ with $\sum_{i=1}^n |\gamma_i| e_i = k$. Since A^k is a Hausdorff $\mathbb{Z}_p^\wedge$-module, the proof of Proposition A 2 shows that A^k is generated by the monomials $\gamma_1^{e_1} \cdots \gamma_n^{e_n}$ with $\sum_{i=1}^n |\gamma_i| e_i = k$. This shows that A^* is generated as a $\mathbb{Z}_p^\wedge$-algebra by the elements $\gamma_1, \ldots, \gamma_n \in A^*$, and therefore A^* is a Noetherian $\mathbb{Z}_p^\wedge$-algebra. □

Note added in proof

The first part of the statement in Proposition 1.3 is true without the finiteness assumption on $\pi_1 Y$. Indeed, if the mod p cohomology is degree-wise finite, then the integral cohomology $H^n(Y; \mathbb{Z}_p^\wedge)$ is equal to the limit $\lim_k H^n(Y; \mathbb{Z}/p^k)$, and hence is complete. This implies that the finiteness assumption on the fundamental group is not needed in Theorem 2.4.

Bill Dwyer pointed out that the second part of Proposition 1.3 is true under the alternative assumption that $\pi_1 Y$ is q-divisible for $q \neq p$, in particular, as soon as Y is p-complete. Indeed, this hypothesis ensures that $\pi_1 Y$ acts as a p-group on a finite vector space. As a consequence, Theorem 3.6 holds true with this weaker hypothesis.

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