First Steps towards Probabilistic Justification Logic

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Abstract

In this paper, we introduce the probabilistic justification logic PJ, a logic in which we can reason about the probability of justification statements. We present its syntax and semantics, and establish a strong completeness theorem. Moreover, we investigate the relationship between PJ and the logic of uncertain justifications.

Keywords: Justification logic, probability logic, uncertain justifications, probabilistic justifications, soundness, completeness

1 Introduction

The idea of probability logics was first proposed by Leibnitz and subsequently discussed by a number of his successors, such as Jacobus Bernoulli, Lambert, Boole, etc. The modern development of this topic, however, started only in the late 1970s and was initiated by H. Jerome Keisler in his seminal paper [10], where he introduced probability quantifiers of the form $Px > r$ (meaning that the probability of a set of objects is greater than r), thus providing a model-theoretic approach to the field. Another important effort came from Nils Nilsson, who tried to provide a logical framework for uncertain reasoning in [14]. For example, he was able to formulate a probabilistic generalization of modus ponens as *if α holds with probability s and β follows from α with probability t, then the probability of β is r.*

Following Nilsson, a number of logical systems appeared (see [17] for references) that extended the classical language with different probability operators. The standard semantics for this kind of probability logic is a special kind of Kripke models, where the accessibility relation between worlds is replaced with a finitely additive probability measure. As usual, the main logical problems in the proof-theoretical framework concern providing a sound and complete axiomatic system and decidability.

In fact, there are two kinds of completeness theorems: the simple completeness (every consistent formula is satisfiable) and the strong completeness theorem (every consistent set of formulas is satisfiable). In the first paper [7] along the lines of Nilsson's research, Fagin, Halpern and Meggido introduced a logic with arithmetical operations built into the syntax so that Boolean combinations of linear inequalities of probabilities of formulas can be expressed. A finite axiomatic system is given and proved to be simply complete. However, the corresponding strong completeness does not follow immediately (as in classical logic) because of the lack of compactness: there are unsatisfiable sets of formulas that are finitely satisfiable. An example is the set of probabilistic constraints saying that the probability of a formula is not zero, but that it is less than any positive rational number. Concerning this issue, the main contribution of [15, 18, 16] was the introduction of several infinitary inference rules (rules with countably many premises and one conclusion) that allowed proofs of strong completeness in the corresponding logics.

Traditional modal epistemic logic uses the formulas $\Box\alpha$ to express that an agent believes α. The language of justification logic [5, 19] 'unfolds' the $\Box$-modality into a family of so-called *justification terms*, which are used to represent evidence for the agent's belief. Hence, instead of $\Box\alpha$, justification logic includes formulas of the form $t : \alpha$ meaning

the agent believes α for reason t.

Artemov [1, 2] developed the first justification logic, the Logic of Proofs, to provide intuitionistic logic with a classical provability semantics. There, justification terms represent formal proofs in Peano Arithmetic. Later Fitting [8] introduced epistemic, that is Kripke, models for justification logic. In this semantics, justification terms represent evidence in a much more general sense [3, 6, 12]. For instance, our belief in α may be justified by direct observation of α or by learning that a friend of a friend has heard about α. Obviously these two situations are not equal: they provide different degrees of justification that α holds.

In this paper we introduce the system PJ, a combination of justification logic and probabilistic logic that makes it possible to adequately model different degrees of justification. We consider a language that features formulas of the form $P_{\geq r}\alpha$ to express that the probability of truthfulness of the justification logic formula α is equal to or greater than the rational number r. Hence we can study, for instance, the formula

$$P_{\geq r}(u : (\alpha \rightarrow \beta)) \rightarrow \big(P_{\geq s}(v : \alpha) \rightarrow P_{\geq r \cdot s}(u \cdot v : \beta)\big), \tag{1}$$

which states that the probability of the conclusion of an application axiom is greater than or equal to the product of the probabilities of its premises. We will see later that this, of course, only holds in models where the premises are independent.

Our semantics consists of a set of possible worlds, each a model of justification logic, and a probability measure $\mu(\cdot)$ on sets of possible worlds. We assign a probability to a formula α of justification logic as follows. We first determine the set $[\alpha]$ of possible worlds that satisfy α. Then we obtain the probability of α as $\mu([\alpha])$, i.e. by applying the measure function to the set $[\alpha]$. Hence our logic relies on the usual model of probability. This makes it possible, e.g., to explore the role of independence and to investigate formulas like (1) in full generality.

We study the basic properties of the probabilistic justification logic PJ, present an axiom system for PJ, and establish its soundness and completeness. In order to achieve strong completeness (i.e. every consistent set has a model), our axiom system includes an infinitary rule.

Related Work

So far, probabilistic justification logics have not been investigated. Closely related are Milnikel's proposal [13] for a system with uncertain justifications and Ghari's recent preprint [9] introducing fuzzy justification logics.

Milnikel introduces formulas of the form $t :_q \alpha$, which correspond to our $P_{\geq q}(t : \alpha)$. However, there are three important differences with our current work.

First, his semantics is completely different from the one we study. Instead of using a probability space, Milnikel uses a variation of Kripke-Fitting models. In his models, each triple (w, t, α) (of world, term and formula) is assigned an interval $E(w, t, \alpha)$ of the form $[0, r)$ or $[0, r]$ where r is a rational number from $[0, 1]$. Then the formula $t :_q \alpha$ is true at a world w iff $q \in E(w, t, \alpha)$ and also α is true in all worlds accessible from w. Because of this interval semantics, Milnikel can dispense with infinitary rules.

Second, Milnikel implicitly assumes that various pieces of evidence are independent. Hence the formula corresponding to (1) is an axiom in his system whereas (1) may or may not hold in a model of PJ depending on the independence of the premises of (1) in the given model.

Third, the logic of uncertain justification includes iterated statements of the form $s :_r t :_q \alpha$. In PJ we do not have this kind of iteration; that means $P_{\geq r}(u : P_{\geq s}(t : \alpha))$ is not a formula of PJ. However, we plan to study a system with formulas of this type in future work.

Ghari presents various justification logics where he replaces the classical base with well-known fuzzy logics. In particular, he studies a justification logic RPLJ that is defined over Pavelka logic, which includes constants for all rational numbers in the interval $[0, 1]$. This allows him to express statements of the form *t is a justification for believing α with certainty degree at least r*. Ghari shows that all principles of Milnikel's logic of uncertain justifications are valid in RPLJ.

Our probabilistic justification logic is inspired by the system LPP_2, which is a probability logic over classical propositional logic without iterations of probability operators [17]. The definitions of syntax and semantics of PJ follow the pattern of LPP_2 and our completeness proof is an adaptation of the completeness proof for LPP_2.

The possible worlds in the semantics of PJ are so-called basic modular models of justification logic. Artemov [4] originally proposed these models to provide an ontologically transparent semantics for justifications. Kuznets and Studer [11] further developed basic modular models so that they can be used as a semantics for many different justification logics.

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2 The Justification Logic J

In this section we present the basic justification logic J. We introduce its syntax and semantics and recall some fundamental properties of J.

2.1 Syntax

Justification terms are built from countably many constants and countably many variables according to the following grammar:

$$t ::= c \mid x \mid (t \cdot t) \mid (t + t) \mid !t$$

where c is a constant and x is a variable. Tm denotes the set of all terms. For any term t and non-negative integer n we define:

$$!^0 t := t \qquad \text{and} \qquad !^{n+1} t := !\ (!^n t)$$

We assume that ! has greater precedence than $\cdot$, which has greater precedence than $+$. The operators $\cdot$ and $+$ are assumed to be left-associative.

Let Prop denote a countable set of atomic propositions. Formulas of the language $\mathsf{L_J}$ (justification formulas) are built according to the following grammar:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid t : \alpha$$

where $t \in \mathsf{Tm}$ and $p \in \mathsf{Prop}$. We define the following abbreviations:

$$\begin{aligned} \alpha \vee \beta &\equiv \neg(\neg\alpha \wedge \neg\beta) \\ \alpha \to \beta &\equiv \neg\alpha \vee \beta \\ \alpha \leftrightarrow \beta &\equiv (\alpha \to \beta) \wedge (\beta \to \alpha) \\ \bot &\equiv \alpha \wedge \neg\alpha, \text{for some } \alpha \in \mathsf{L_J} \\ \top &\equiv \alpha \vee \neg\alpha, \text{for some } \alpha \in \mathsf{L_J} \end{aligned}$$

We assume that : and $\neg$ have higher precedence than $\wedge$ and $\vee$, which have higher precedence than $\to$ and $\leftrightarrow$.

Sometimes we will write $\alpha_1, \ \ldots \ , \alpha_n$ instead of $\{\alpha_1\} \cup \ \cdots \ \cup \{\alpha_n\}$ as well as T, α instead of $T \cup \{\alpha\}$ and X, Y instead of $X \cup Y$.

In Figure 1 we present the axioms of logic J. The axiom (J) is called the application axiom. It states that we can combine a justification (proof) for $\alpha \to \beta$ and a justification for α to obtain a justification for β. Axiom (+)

states that if u or v is a justification for α then the term $u + v$ is also a justification for α.

(P)	$\vdash \alpha$, where α is a propositional tautology
(J)	$\vdash u : (\alpha \to \beta) \to (v : \alpha \to u \cdot v : \beta)$
$(+)$	$\vdash u : \alpha \vee v : \alpha \to u + v : \alpha$

Figure 1: Axioms of J

We will call *constant specification* any set CS that satisfies the following condition:

$$\mathsf{CS} \subseteq \{(c, \alpha) \mid c \text{ is a constant and } \alpha \text{ is an instance of some axiom of } \mathsf{J}\}$$

So, CS determines some axiom instances for which the logic provides justifications (without any proof).

A constant specification CS is called *axiomatically appropriate* if for every axiom α of J, there exists some constant c such that $(c, \alpha) \in \mathsf{CS}$, i.e. every axiom of J is justified by at least one constant.

Let CS be any constant specification. The deductive system J_{CS} is the Hilbert system obtained by adding to the axioms of J the rules modus ponens, (MP), and axiom necessitation, $(\mathsf{AN!})$, as one can see in Figure 2. Rule $(\mathsf{AN!})$ makes the connection between the constant specification and the proofs in J_{CS}: if $(c, \alpha) \in \mathsf{CS}$ then we can prove that c is justification for α, that $!c$ is a justification for $c : \alpha$ and so on.

	axioms of J
	$+$
$(\mathsf{AN!})$	$\vdash\, !^n c : !^{n-1} c : \cdots : !c : c : \alpha$, where $(c, \alpha) \in \mathsf{CS}$ and $n \in \mathbb{N}$
(MP)	if $T \vdash \alpha$ and $T \vdash \alpha \to \beta$ then $T \vdash \beta$

Figure 2: System J_{CS}

Let L be a logic. As usual $T \vdash_{\mathsf{L}} A$ will mean that the formula A is deducible from the set of formulas T using the rules and axioms of L. When L is clear from the context, it will be omitted.

Let L be a logic and $\mathcal{L}$ be a language. A set T is said to be L-*deductively closed* for $\mathcal{L}$ iff for every $A \in \mathcal{L}$:

$$T \vdash_{\mathsf{L}} A \Longleftrightarrow A \in T.$$

2.2 Semantics

We use T to represent the truth value "true" and F to represent the truth value "false". Let $\mathcal{P}(W)$ denote the powerset of the set W.

Definition 1. Let $X, Y \subseteq \mathsf{L_J}$ and $t \in \mathsf{Tm}$. We define the following set:

$$X \cdot Y := \left\{ \alpha \in \mathsf{L_J} \;\middle|\; \beta \to \alpha \in X \text{ and } \beta \in Y \text{ for some formula } \beta \in \mathsf{L_J} \right\}$$

Definition 2 (Basic Evaluation)**.** Let CS be any constant specification. A *basic evaluation* for $\mathsf{J_{CS}}$, or a basic $\mathsf{J_{CS}}$-evaluation, is a function $*$ that maps atomic propositions to truth values and maps justification terms to sets of justification formulas, i.e. $* : \mathsf{Prop} \to \{\mathsf{T}, \mathsf{F}\}$ and $* : \mathsf{Tm} \to \mathcal{P}(\mathsf{L_J})$, such that for $u, v \in \mathsf{Tm}$, for a constant c and $\alpha \in \mathsf{L_J}$ we have:

(1) $u^* \cdot v^* \subseteq (u \cdot v)^*$

(2) $u^* \cup v^* \subseteq (u + v)^*$

(3) if $(c, \alpha) \in \mathsf{CS}$ then:

  (a) $\alpha \in c^*$

  (b) for all $n \in \mathbb{N}$ we have:

$$!^n c : !^{n-1} c : \cdots : !c : c : \alpha \in (!^{n+1} c)^*$$

We will usually write t^* and p^* instead of $*(t)$ and $*(p)$ respectively.

Now we will define the binary relation $\Vdash$.

Definition 3 (Truth under a Basic Evaluation)**.** Let $\alpha \in \mathsf{L_J}$. We define what it means for α to hold under a basic $\mathsf{J_{CS}}$-evaluation $*$ inductively as follows:

- If $\alpha = p \in \mathsf{Prop}$ then:
$$* \Vdash \alpha \iff p^* = \mathsf{T}$$
- If $\alpha = \neg\beta$ then:
$$* \Vdash \alpha \iff * \nVdash \beta$$
- If $\alpha = \beta \wedge \gamma$ then:
$$* \Vdash \alpha \iff \Big(* \Vdash \beta \text{ and } * \Vdash \gamma \Big)$$
- If $\alpha = t : \beta$ then:
$$* \Vdash \alpha \iff \beta \in t^*$$

Let $T \subseteq \mathsf{L_J}$, let $\alpha \in \mathsf{L_J}$ and let $*$ be a basic $\mathsf{J_{CS}}$-evaluation. $* \Vdash T$ means that $*$ satisfies all the members of the set T. $T \Vdash_{\mathsf{CS}} \alpha$ means that for every basic $\mathsf{J_{CS}}$-evaluation $*$, $* \Vdash T$ implies $* \Vdash \alpha$.

2.3 Fundamental Properties

Internalization states that justification logic internalizes its own notion of proof. The version without premises is an explicit form of the necessitation rule of modal logic. A proof of the following theorem can be found in [11].

Theorem 4 (Internalization)**.** *Let* CS *be an axiomatically appropriate constant specification. For any formulas* α, $\beta_1, \ldots, \beta_n \in \mathsf{L_J}$ *and terms* $t_1, \ldots, t_n$*, if:*

$$\beta_1,\ \ldots\ ,\ \beta_n \vdash_{\mathsf{J_{CS}}} \alpha$$

then there exists a term t *such that:*

$$t_1 : \beta_1,\ \ldots\ ,\ t_n : \beta_n \vdash_{\mathsf{J_{CS}}} t : \alpha$$

The deduction theorem is standard for justification logic [2]. Therefore, we omit its proof here.

Theorem 5 (Deduction Theorem for J)**.** *Let* $T \subseteq \mathsf{L_J}$ *and let* $\alpha, \beta \in \mathsf{L_J}$*. Then for any* $\mathsf{J_{CS}}$ *we have:*

$$T, \alpha \vdash_{\mathsf{J_{CS}}} \beta \Longleftrightarrow T \vdash_{\mathsf{J_{CS}}} \alpha \to \beta$$

Last but not least, we have soundness and completeness of J with respect to basic evaluations [4, 11].

Theorem 6 (Completeness of J)**.** *Let* CS *be any constant specification. Let* $\alpha \in \mathsf{L_J}$*. Then we have:*

$$\vdash_{\mathsf{J_{CS}}} \alpha \quad \Longleftrightarrow \quad \Vdash_{\mathsf{CS}} \alpha.$$

3 The Probabilistic Justification Logic PJ

The probabilistic justification logic PJ is a probabilistic logic over the justification logic J. We first introduce syntax and semantics of PJ and then establish some basic facts about it. Remarks 18, 21, and 23 make the relationship of PJ to the logic of uncertain justification formally precise.

3.1 Syntax

We will represent the set of all rational numbers with the symbol $\mathbb{Q}$. If X and Y are sets, we will sometimes write XY instead of $X \cap Y$. We define $\mathsf{S} := \mathbb{Q}[0,1]$, while $\mathsf{S}[0,t)$ will denote the set $[0,t) \cap \mathbb{Q}[0,1]$.

The formulas of the language $\mathsf{L_P}$ (the so called probabilistic formulas) are built according to the following grammar:

$$A ::= P_{\geq s}\alpha \mid \neg A \mid A \wedge A$$

where $s \in \mathsf{S}$, and $\alpha \in \mathsf{L_J}$.

We assume the same abbreviations and the same precedence for the propositional connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, as the ones we defined in subsection 2.1 for logic J. However, we need to define a bottom and a top element for the language $\mathsf{L_P}$. Hence we define:

$$\bot := A \wedge \neg A, \text{for some } A \in \mathsf{L_P}$$
$$\top := A \vee \neg A, \text{for some } A \in \mathsf{L_P}$$

It will always be clear by the context whether $\neg, \wedge, \top, \bot, \ldots$ refer to formulas of $\mathsf{L_J}$ or $\mathsf{L_P}$. The operator $P_{\geq s}$ is assumed to have greater precedence than all the propositional connectives. We will also use the following syntactical abbreviations:

$$\begin{aligned} P_{<s}\alpha &\equiv \neg P_{\geq s}\alpha \\ P_{\leq s}\alpha &\equiv P_{\geq 1-s}\neg\alpha \\ P_{>s}\alpha &\equiv \neg P_{\leq s}\alpha \\ P_{=s}\alpha &\equiv P_{\geq s}\alpha \wedge P_{\leq s}\alpha \end{aligned}$$

We will use capital Latin letters like $A,\ B,\ C,\ \ldots$ for members of $\mathsf{L_P}$ and the letters r, s for members of S, all of them possibly primed or with subscripts.

The axioms of PJ are presented in Figure 3. Axiom (PI) corresponds to the fact that the measure of the set of worlds satisfying a justification formula is at least 0. Observe that by substituting $\neg\alpha$ for α in (PI), we have $P_{\geq 0}\neg\alpha$, which by our syntactical abbreviations is $P_{\leq 1}\alpha$. Hence axiom (PI) also corresponds to the fact that the measure of the set of worlds satisfying a justification formula is at most 1. Axioms (WE) and (LE) state that our degree of confidence for the truth of a justification formula can be weakened. Axioms (DIS) and (UN) correspond to the additivity of measures. Axiom (DIS) states that if the set of worlds satisfying α and the set of worlds satisfying β are disjoint, then the measure of the set of worlds that satisfy $\alpha \vee \beta$ is at least the sum of the measures of the former two sets. Axiom (UN) states that the measure of the set of worlds that satisfy $\alpha \vee \beta$ cannot be greater than the sum of the measure of the worlds satisfying α and the measure of the worlds satisfying β.

(P)	$\vdash A$, where A is a propositional tautology
(PI)	$\vdash P_{\geq 0}\alpha$
(WE)	$\vdash P_{\leq r}\alpha \to P_{<s}\alpha$, where $s > r$
(LE)	$\vdash P_{<s}\alpha \to P_{\leq s}\alpha$
(DIS)	$\vdash P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}\neg(\alpha \wedge \beta) \to P_{\geq \min(1,r+s)}(\alpha \vee \beta)$
(UN)	$\vdash P_{\leq r}\alpha \wedge P_{<s}\beta \to P_{<r+s}(\alpha \vee \beta)$, where $r + s \leq 1$

Figure 3: Axioms of PJ

It is very important to note the different uses of axiom (P). As an axiom of J, (P) contains all the propositional tautologies that are members of L_{J}, e.g. $t : \alpha \to t : \alpha$. As an axiom of PJ, (P) contains all the propositional tautologies that are members of L_{P}, e.g. $P_{\geq s}(t : \alpha) \to P_{\geq s}(t : \alpha)$. Recall that any constant specification contains pairs of term constants and axioms of J. Thus $(c, t : \alpha \to t : \alpha)$ may belong to a constant specification whereas $(c, P_{\geq s}(t : \alpha) \to P_{\geq s}(t : \alpha))$ may not.

For any constant specification CS the deductive system $\mathsf{PJ}_{\mathsf{CS}}$ is the Hilbert system obtained by adding to the axioms of PJ the rules (MP), (CE) and (ST) (see Figure 4). Rule (CE) makes the connection between justification logic and probabilistic logic possible. It states that if a justification formula is a validity, then it has probability 1 because it holds in every possible world. Rule (CE) can also be considered as the analogue of the necessitation rule for modal logics. The rule (ST) intuitively states that if the probability of a justification formula is arbitrary close to s, then it is at least s. Observe that the rule (ST) is infinitary in the sense that it has an infinite number of premises. It corresponds to the Archimedean property for the real numbers (see Proposition 24). As a consequence, the depth of a proof is given by an (infinite) ordinal.

	axioms of PJ
	$+$
(MP)	if $T \vdash A$ and $T \vdash A \to B$ then $T \vdash B$
(CE)	if $\vdash_{\mathsf{J}_{\mathsf{CS}}} \alpha$ then $\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq 1}\alpha$
(ST)	if $T \vdash A \to P_{\geq s-\frac{1}{k}}\alpha$ for every integer $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \to P_{\geq s}\alpha$

Figure 4: System $\mathsf{PJ}_{\mathsf{CS}}$

When we present proofs in a logic we are going to use the following abbreviations:

P.R.: it stands for "propositional reasoning". E.g. when we have $\vdash A \to B$ we can claim that by **P.R.** we get $\vdash \neg B \to \neg A$. We can think of **P.R.** as an abbreviation of the phrase "by some applications of (P) and (MP)".

S.E.: it stands for "syntactical equivalence". E.g. according to our syntactical conventions the formulas $P_{\geq 1-s}(\alpha \vee \beta)$ and $P_{\leq s}(\neg\alpha \wedge \neg\beta)$ are syntactically equivalent. We will transform our formulas to syntactically equivalent ones (using the syntactical abbreviations defined in subsections 2.1 and 3.1), in order to increase readability of our proofs. We have to be very careful when we apply **S.E.**. For example the formulas $P_{\geq s}(\neg\alpha \vee \beta)$ and $P_{\geq s}(\alpha \to \beta)$ are syntactically equivalent, whereas the formulas $P_{\geq s}\alpha$ and $P_{\geq s}\neg\neg\alpha$ are not.

Theorem 7 (Deduction Theorem for PJ)**.** *Let $T \subseteq \mathsf{L}_{\mathsf{P}}$ and assume that $A, B \in \mathsf{L}_{\mathsf{P}}$. Then for any $\mathsf{PJ}_{\mathsf{CS}}$ we have:*

$$T, A \vdash_{\mathsf{PJ}_{\mathsf{CS}}} B \Longleftrightarrow T \vdash_{\mathsf{PJ}_{\mathsf{CS}}} A \to B$$

Proof. We only show the interesting cases of the direction $\Longrightarrow$, which is established as usual by induction on the depth of the proof $T, A \vdash_{\mathsf{PJ}_{\mathsf{CS}}} B$.

1. Assume that B is the result of an application of (CE). That means there exists $\alpha \in \mathsf{L}_{\mathsf{J}}$ such that $B = P_{\geq 1}\alpha$ and also $\vdash_{\mathsf{J}_{\mathsf{CS}}} \alpha$. Hence we

have:

$$\vdash_{\mathsf{J_{CS}}} \alpha \tag{2}$$

$$\vdash_{\mathsf{PJ_{CS}}} P_{\geq 1}\alpha \qquad [2, (\mathsf{CE})] \tag{3}$$

$$\vdash_{\mathsf{PJ_{CS}}} P_{\geq 1}\alpha \to (A \to P_{\geq 1}\alpha) \qquad [(\mathsf{P})] \tag{4}$$

$$\vdash_{\mathsf{PJ_{CS}}} A \to P_{\geq 1}\alpha \qquad [3, 4, (\mathsf{MP})] \tag{5}$$

$$T \vdash_{\mathsf{PJ_{CS}}} A \to B \tag{6}$$

2. Assume that B is the result of an application of (ST). That means that $B = C \to P_{\geq s}\alpha$ and also:

$$T, A \vdash_{\mathsf{PJ_{CS}}} C \to P_{\geq s-\frac{1}{k}}\alpha, \forall \text{ integer } k \geq \frac{1}{s}$$

Thus we have:

$$T \vdash_{\mathsf{PJ_{CS}}} A \to (C \to P_{\geq s-\frac{1}{k}}\alpha), \forall \text{ integer } k \geq \frac{1}{s} \qquad [\text{i.h.}] \tag{7}$$

$$T \vdash_{\mathsf{PJ_{CS}}} (A \wedge C) \to P_{\geq s-\frac{1}{k}}\alpha, \forall \text{ integer } k \geq \frac{1}{s} \qquad [7, \mathbf{P.R.}] \tag{8}$$

$$T \vdash_{\mathsf{PJ_{CS}}} (A \wedge C) \to P_{\geq s}\alpha \qquad [8, (\mathsf{ST})] \tag{9}$$

$$T \vdash_{\mathsf{PJ_{CS}}} A \to (C \to P_{\geq s}\alpha) \qquad [9, \mathbf{P.R.}] \tag{10}$$

$$T \vdash_{\mathsf{PJ_{CS}}} A \to B \qquad [10, \mathbf{S.E.}] \tag{11}$$

□

3.2 Semantics

Definition 8 (Algebra over a set)**.** Let W be a non-empty set and let H be a non-empty subset of $\mathcal{P}(W)$. H will be called an *algebra over* W iff the following hold:

- $W \in H$
- $U, V \in H \Longrightarrow U \cup V \in H$
- $U \in H \Longrightarrow W \setminus U \in H$

Definition 9 (Finitely Additive Measure)**.** Let H be an algebra over W and $\mu : H \to [0, 1]$. We call μ a *finitely additive measure* iff the following hold:

(1) $\mu(W) = 1$

(2) for all $U, V \in H$:

$$U \cap V = \emptyset \Longrightarrow \mu(U \cup V) = \mu(U) + \mu(V)$$

Definition 10 (Models)**.** Let CS be any constant specification. A $\mathsf{PJ}_{\mathsf{CS}}$-*model*, or simply a model, is a structure $M = \langle W, H, \mu, * \rangle$ where:

- W is a non-empty set of objects called worlds.
- H is an algebra over W.
- $\mu : H \to [0, 1]$ is a finitely additive measure.
- $*$ is a function from W to the set of all basic J_{CS}-evaluations, i.e. $*(w)$ is a basic J_{CS}-evaluation for each world $w \in W$. We will usually write $*_w$ instead of $*(w)$.

Definition 11 (Independent Sets in a Model)**.** Let $M = \langle W, H, \mu, * \rangle$ be a $\mathsf{PJ}_{\mathsf{CS}}$-model and let $U, V \in H$. U, V will be called *independent in* M iff the following holds:

$$\mu(U \cap V) = \mu(U) \cdot \mu(V)$$

Definition 12 (Measurable model)**.** Let $M = \langle W, H, \mu, * \rangle$ be a model and $\alpha \in \mathsf{L}_{\mathsf{J}}$. We define the following set:

$$[\alpha]_M = \{w \in W \mid *_w \Vdash \alpha\}$$

We will omit the subscript M, i.e. we will simply write $[\alpha]$, if M is clear from the context. A $\mathsf{PJ}_{\mathsf{CS}}$-model $M = \langle W, H, \mu, * \rangle$ is *measurable* iff $[\alpha]_M \in H$ for every $\alpha \in \mathsf{L}_{\mathsf{J}}$. The class of measurable $\mathsf{PJ}_{\mathsf{CS}}$-models will be denoted by $\mathsf{PJ}_{\mathsf{CS,Meas}}$.

We have the following standard properties of a finitely additive measure.

Lemma 13. *Let H be an algebra over some set W, $\mu : H \to [0, 1]$ be a finitely additive measure and $U, V \in H$. Then the following hold:*

(1) $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$

(2) $\mu(U) + \mu(W \setminus U) = 1$

(3) $U \supseteq V \Longrightarrow \mu(U) \geq \mu(V)$

Remark 14. Let $M = \langle W, H, \mu, *\rangle$ be a model and $\alpha, \beta \in \mathsf{L_J}$. It holds:

$$\begin{aligned}
[\alpha \vee \beta]_M =& \{w \in W \mid *_w \Vdash \alpha \vee \beta\} = \{w \in W \mid *_w \Vdash \alpha \text{ or } *_w \Vdash \beta\} = \\
& \{w \in W \mid *_w \Vdash \alpha\} \cup \{w \in W \mid *_w \Vdash \beta\} = [\alpha]_M \cup [\beta]_M \\
[\alpha \wedge \beta]_M =& \{w \in W \mid *_w \Vdash \alpha \wedge \beta\} = \{w \in W \mid *_w \models \alpha \text{ and } *_w \Vdash \beta\} = \\
& \{w \in W \mid *_w \Vdash \alpha\} \cap \{w \in W \mid *_w \Vdash \beta\} = [\alpha]_M \cap [\beta]_M \\
[\neg\alpha]_M =& \{w \in W \mid *_w \Vdash \neg\alpha\} = \{w \in W \mid *_w \nVdash \alpha\} = \\
& W \setminus \{w \in W \mid *_w \Vdash \alpha\} = W \setminus [\alpha]_M
\end{aligned}$$

Hence if $M \in \mathsf{PJ_{CS,Meas}}$ we get by Lemma 13:

$$\begin{aligned}
\mu([\alpha \vee \beta]_M) + \mu([\alpha \wedge \beta]_M) = \mu([\alpha]_M) + \mu([\beta]_M) \\
\mu([\alpha]_M) + \mu([\neg\alpha]_M) = 1
\end{aligned}$$

Definition 15 (Truth in a $\mathsf{PJ_{CS,Meas}}$-model)**.** Let CS be any constant specification. Let $M = \langle W, H, \mu, *\rangle$ be a $\mathsf{PJ_{CS,Meas}}$-model and $A \in \mathsf{L_P}$. We define what it means for A to hold in M inductively as follows[1]:

- If $A \equiv P_{\geq s}\alpha$ then:

$$M \models A \Longleftrightarrow \mu([\alpha]_M) \geq s$$

- If $A \equiv \neg B$ then:

$$M \models A \Longleftrightarrow M \not\models B$$

- If $A \equiv B \wedge C$ then:

$$M \models A \Longleftrightarrow \big(M \models B \text{ and } M \models C\big)$$

Let $T \subseteq \mathsf{L_P}$, $A \in \mathsf{L_P}$ and M be a $\mathsf{PJ_{CS,Meas}}$-model. Then $M \models T$ means that M satisfies all members of the set T. Further $T \models_{\mathsf{PJ_{CS,Meas}}} A$ means that for every $M \in \mathsf{PJ_{CS,Meas}}$, $M \models T$ implies $M \models A$.

Lemma 16 (Properties of the Class $\mathsf{PJ_{CS,Meas}}$)**.** *Let* CS *be any constant specification, let* $M = \langle W, H, \mu, *\rangle \in \mathsf{PJ_{CS,Meas}}$ *and let* $\alpha \in \mathsf{L_J}$*. Then the following hold:*

(1) $M \models P_{\leq s}\alpha \Longleftrightarrow \mu([\alpha]) \leq s$

(2) $M \models P_{< s}\alpha \Longleftrightarrow \mu([\alpha]) < s$

[1]observe that the satisfiability relation of a basic evaluation is represented with $\Vdash$ whereas the satisfiability relation of a model is represented with $\models$.

(3) $M \models P_{>s}\alpha \iff \mu([\alpha]) > s$

(4) $M \models P_{=s}\alpha \iff \mu([\alpha]) = s$

Proof. (1) We have:

$$M \models P_{\le s}\alpha \overset{\textbf{S.E.}}{\iff} M \models P_{\ge 1-s}\neg\alpha \overset{\text{Def. 15}}{\iff}$$
$$\mu([\neg\alpha]) \ge 1 - s \overset{\text{Remark 14}}{\iff}$$
$$1 - \mu([\alpha]) \ge 1 - s \iff \mu([\alpha]) \le s$$

(2) $M \models P_{<s}\alpha \overset{\textbf{S.E.}}{\iff} M \models \neg P_{\ge s}\alpha \iff M \not\models P_{\ge s}\alpha \overset{\text{Def. 15}}{\iff} \mu([\alpha]) < s$

(3) $M \models P_{>s}\alpha \overset{\textbf{S.E.}}{\iff} M \models \neg P_{\le s}\alpha \iff M \not\models P_{\le s}\alpha \overset{(1)}{\iff} \mu([\alpha]) > s$

(4) We have:

$$M \models P_{=s}\alpha \overset{\textbf{S.E.}}{\iff} \big(M \models P_{\ge s}\alpha \text{ and } M \models P_{\le s}\alpha\big) \overset{(1) \text{ and Def. 15}}{\iff}$$
$$\big(\mu([\alpha]) \ge s \text{ and } \mu([\alpha]) \le s\big) \iff \mu([\alpha]) = s$$

□

4 Relations with the Logic of Uncertain Justifications

The logic of uncertain justifications [13] consists of the following axioms (we use the notation and syntactical conventions from [13]):

*A*0. Propositional tautologies
*A*1. $s :_p (F \to G) \to (t :_q F \to (s \cdot t) :_{p \cdot q} G)$ (the application axiom scheme)
*A*2. $s :_r F \to (s + t) :_r F$ and
$t :_r F \to (s + t) :_r F$ (the monotonicity axiom schemes)
*A*3. $s :_p F \to s :_q F,$ where $p \ge q$ (the confidence weakening axiom scheme)

Further it includes modus ponens and a version of axiom necessitation that is slightly different from (AN!). Recall that Milnikel's formula $t :_q F$ should be read as the "term t justifies formula F with probability at least q" and therefore corresponds to our formula $P_{\ge q} t : F$.

The version of axiom necessitation that is studied in [13] cannot be expressed in the logic PJ since in this logic we cannot have iterations of the probability operators.

In this section we will show that all the axioms of the logic of uncertain justifications (without iterations of the form $s :_r t :_q \alpha$) can be expressed in our framework. Axioms $A0$, $A2$ and $A3$ are theorems of PJ and therefore hold in all $\mathsf{PJ}_{\mathsf{CS},\mathsf{Meas}}$-models, whereas axiom $A1$ holds in a $\mathsf{PJ}_{\mathsf{CS},\mathsf{Meas}}$-model only under the assumption that the premises of the the axiom are independent in the model.

Lemma 17. *Let* CS *be any constant specification. Then the following hold:*

(i) $\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)$

(ii) If $\vdash_{\mathsf{J}_{\mathsf{CS}}} \alpha \to \beta$ *then* $\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq s}\alpha \to P_{\geq s}\beta$

(iii) if $s > r$ *then* $\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq s}\alpha \to P_{>r}\alpha$

(iv) $\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{>r}\alpha \to P_{\geq r}\alpha$

(v) if $r \geq s$ *then* $\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq r}\alpha \to P_{\geq s}\alpha$

Proof. (i) We have:

$$
\begin{aligned}
&\vdash_{\mathsf{J}_{\mathsf{CS}}} \neg(\alpha \wedge \bot) && [(\mathsf{P})] \quad (12)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq 1}\neg(\alpha \wedge \bot) && [12, (\mathsf{CE})] \quad (13)\\
&\vdash_{\mathsf{J}_{\mathsf{CS}}} (\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha && [(\mathsf{P})] \quad (14)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq 1}\big((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha\big) && [14, (\mathsf{CE})] \quad (15)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} \Big(P_{\geq s}\alpha \wedge P_{\geq 0}\bot \wedge P_{\geq 1}\neg(\alpha \wedge \bot)\Big) \\
&\qquad \to P_{\geq s}(\alpha \vee \bot) && [(\mathsf{DIS})] \quad (16)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq 0}\bot && [(\mathsf{PI})] \quad (17)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq s}\alpha \to P_{\geq s}(\alpha \vee \bot) && [13, 16, 17, \mathbf{P.R.}] \quad (18)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} \big(P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \wedge P_{<s}\neg\neg\alpha\big) \\
&\qquad \to P_{<1}\big((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha\big) && (\mathsf{UN}) \quad (19)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} \neg\neg P_{\geq 1}\big((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha\big) && [15, \mathbf{P.R.}] \quad (20)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} \neg P_{<1}\big((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha\big) && [20, \mathbf{S.E.}] \quad (21)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} \neg\big(P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \wedge P_{<s}\neg\neg\alpha\big) && [19, 21, \mathbf{P.R.}] \quad (22)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \to \neg P_{<s}\neg\neg\alpha && [22, \mathbf{P.R.}] \quad (23)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \to \neg\neg P_{\geq s}\neg\neg\alpha && [23, \mathbf{S.E.}] \quad (24)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \to P_{\geq s}\neg\neg\alpha && [24, \mathbf{P.R.}] \quad (25)\\
&\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq s}(\alpha \vee \bot) \to P_{\geq s}\neg\neg\alpha && [25, \mathbf{S.E.}] \quad (26)
\end{aligned}
$$

$$\vdash_{\mathsf{PJ_{CS}}} P_{\geq s}\alpha \to P_{\geq s}\neg\neg\alpha \qquad [18, 26, \mathbf{P.R.}] \quad (27)$$

$$\vdash_{\mathsf{PJ_{CS}}} \neg\big(P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)\big) \to P_{\geq 1}(\alpha \to \beta) \wedge P_{\geq s}\alpha \wedge \neg P_{\geq s}\beta \qquad [(\mathsf{P})] \quad (28)$$

$$\vdash_{\mathsf{PJ_{CS}}} \neg\big(P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)\big) \to P_{\geq 1}(\alpha \to \beta) \wedge P_{\geq s}\neg\neg\alpha \wedge \neg P_{\geq s}\beta \qquad [27, 28, \mathbf{P.R.}] \quad (29)$$

$$\vdash_{\mathsf{PJ_{CS}}} \neg\big(P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)\big) \to P_{\geq 1}(\neg\alpha \vee \beta) \wedge P_{\leq 1-s}\neg\alpha \wedge P_{<s}\beta \qquad [29, \mathbf{S.E.}] \quad (30)$$

$$\vdash_{\mathsf{PJ_{CS}}} P_{\leq 1-s}\neg\alpha \wedge P_{<s}\beta \to P_{<1}(\neg\alpha \vee \beta) \qquad [(\mathsf{UN})] \quad (31)$$

$$\vdash_{\mathsf{PJ_{CS}}} \neg\big(P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)\big) \to \big(P_{\geq 1}(\neg\alpha \vee \beta) \wedge P_{<1}(\neg\alpha \vee \beta)\big) \qquad [30, 31, \mathbf{P.R.}] \quad (32)$$

$$\vdash_{\mathsf{PJ_{CS}}} \neg\big(P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)\big) \to \big(P_{\geq 1}(\neg\alpha \vee \beta) \wedge \neg P_{\geq 1}(\neg\alpha \vee \beta)\big) \qquad [32, \mathbf{S.E.}] \quad (33)$$

$$\vdash_{\mathsf{PJ_{CS}}} P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta) \qquad [33, \mathbf{P.R.}] \quad (34)$$

(ii) Follows by (CE), (i) and (MP).

(iii) It holds:

$$\vdash_{\mathsf{PJ_{CS}}} P_{\leq r}\alpha \to P_{<s}\alpha \qquad [(\mathsf{WE})] \quad (35)$$

$$\vdash_{\mathsf{PJ_{CS}}} \neg P_{<s}\alpha \to \neg P_{\leq r}\alpha \qquad [35, \mathbf{P.R.}] \quad (36)$$

$$\vdash_{\mathsf{PJ_{CS}}} \neg\neg P_{\geq s}\alpha \to P_{>r}\alpha \qquad [36, \mathbf{S.E.}] \quad (37)$$

$$\vdash_{\mathsf{PJ_{CS}}} P_{\geq s}\alpha \to P_{>r}\alpha \qquad [37, \mathbf{P.R.}] \quad (38)$$

(iv) Similar to case (iii).

(v) Follows by (iii) and (iv). $\square$

Remark 18. From statement (v) of Lemma 17 it follows that

$$\text{if } r \geq s, \text{ then } \vdash_{\mathsf{PJ_{CS}}} P_{\geq r}(u : \alpha) \to P_{\geq s}(u : \alpha)$$

and, indeed, this corresponds to Axiom A3 of the logic of uncertain justifications [13]. Moreover, from statement (ii) we get the following corollary, which corresponds to Axiom A2 of [13].

Corollary 19. *Let $\alpha \in \mathsf{L_J}$, $u, v \in \mathsf{Tm}$ and $r \in \mathsf{S}$. Then for any $\mathsf{PJ_{CS}}$ we have:*

(1) $\vdash_{\mathsf{PJ_{CS}}} P_{\geq r}(u : \alpha) \to P_{\geq r}(u + v : \alpha)$

(2) $\vdash_{\mathsf{PJ_{CS}}} P_{\geq r}(v : \alpha) \to P_{\geq r}(u + v : \alpha)$

Proof. (1) We have:

$$\vdash_{\mathsf{J_{CS}}} (u : \alpha \vee v : \alpha) \to u + v : \alpha \qquad [(+)] \tag{39}$$

$$\vdash_{\mathsf{J_{CS}}} u : \alpha \to u + v : \alpha \qquad [39, \textbf{P.R.}] \tag{40}$$

$$\vdash_{\mathsf{PJ_{CS}}} P_{\geq r}(u : \alpha) \to P_{\geq r}(u + v : \alpha) \qquad [\text{Lemma } 17(\text{ii}), 40] \tag{41}$$

(2) Similar to the previous case. □

Theorem 20 (Probabilistic Internalization)**.** *Let* CS *be an axiomatically appropriate constant specification. For any formulas* α*,* $\beta_1, \ \ldots \ , \beta_n \in \mathsf{L_J}$*, terms* $t_1, \ \ldots \ , t_n \in \mathsf{Tm}$ *and* $s \in \mathsf{S}$*, if:*

$$\beta_1, \ \ldots \ , \ \beta_n \vdash_{\mathsf{J_{CS}}} \alpha$$

then there exists a term t *such that:*

(1) $P_{\geq s}(t_1 : \beta_1 \wedge \ \ldots \ \wedge \ t_n : \beta_n) \vdash_{\mathsf{PJ_{CS}}} P_{\geq s}(t : \alpha)$

(2) for every $i \in \{1, \ldots, n\}$*:*

$$\left\{ \ P_{\geq 1}(t_j : \beta_j) \ \middle| \ j \neq i \ \right\}, P_{\geq s}(t_i : \beta_i) \vdash_{\mathsf{PJ_{CS}}} P_{\geq s}(t : \alpha)$$

Proof. By Theorem 4 we find that there exists a term t such that:

$$t_1 : \beta_1, \ \ldots \ , \ t_n : \beta_n \vdash_{\mathsf{J_{CS}}} t : \alpha$$

By repeatedly applying Theorem 5 we get:

$$\vdash_{\mathsf{J_{CS}}} t_1 : \beta_1 \to (\ \ldots \ \to (t_{n-1} : \beta_{n-1} \to (t_n : \beta_n \to t : \alpha)) \ldots) \tag{42}$$

So we have:

(1) By (42) and **P.R.** we get:

$$\vdash_{\mathsf{J_{CS}}} \big(t_1 : \beta_1 \wedge \ldots \wedge t_n : \beta_n\big) \to t : \alpha$$

By Lemma 17(ii):

$$\vdash_{\mathsf{PJ_{CS}}} P_{\geq s}\big(t_1 : \beta_1 \wedge \ldots \wedge t_n : \beta_n\big) \to P_{\geq s}\big(t : \alpha\big)$$

and by Theorem 7:

$$P_{\geq s}\big(t_1 : \beta_1 \wedge \ldots \wedge t_n : \beta_n\big) \vdash_{\mathsf{PJ_{CS}}} P_{\geq s}\big(t : \alpha\big)$$

(2) Let $i \in \{1, \dots, n\}$ and $\{j_1, \dots, j_{n-1}\} = \{1, \dots, n\} \setminus i$. By (42) and **P.R.** we get:

$$\vdash_{\mathsf{J}_{\mathsf{CS}}} t_{j_1} : \beta_{j_1} \to (\ \dots \ \to (t_{j_{n-1}} : \beta_{j_{n-1}} \to (t_i : \beta_i \to t : \alpha)) \dots)$$

By (CE) we get:

$$\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq 1}\Big(t_{j_1} : \beta_{j_1} \to (\ \dots \ \to (t_{j_{n-1}} : \beta_{j_{n-1}} \to (t_i : \beta_i \to t : \alpha)) \dots)\Big)$$

By repeatedly applying Lemma 17(i) and **P.R.** we get:

$$\vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq 1}(t_{j_1} : \beta_{j_1}) \to (\ \dots \ \to (P_{\geq 1}(t_{j_{n-1}} : \beta_{j_{n-1}}) \to (P_{\geq s}(t_i : \beta_i) \to P_{\geq s}(t : \alpha))) \dots)$$

And by repeatedly applying Theorem 7 we get:

$$P_{\geq 1}\big(t_{j_1} : \beta_{j_1}\big), \ \dots \ , P_{\geq 1}\big(t_{j_{n-1}} : \beta_{j_{n-1}}\big), P_{\geq s}\big(t_i : \beta_i\big) \vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq s}\big(t : \alpha\big)$$

i.e.

$$\big\{ \ P_{\geq 1}(t_j : \beta_j) \ \big| \ j \neq i \ \big\}, P_{\geq s}(t_i : \beta_i) \vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq s}(t : \alpha) \qquad \square$$

Remark 21. If we consider the formulation of probabilistic internalization without premises, then we obtain for an axiomatically appropriate CS that

$$\vdash_{\mathsf{J}_{\mathsf{CS}}} \alpha \qquad \text{implies} \qquad \vdash_{\mathsf{PJ}_{\mathsf{CS}}} P_{\geq 1}(t : \alpha) \quad \text{for some term } t.$$

This version corresponds to internalization for the logic of uncertain justifications, see Theorem 3 of [13].

Theorem 22. *Let* CS *be a constant specification. Let $u, v \in \mathsf{Tm}$, $\alpha, \beta \in \mathsf{L}_{\mathsf{J}}$ and M be a $\mathsf{PJ}_{\mathsf{CS},\mathsf{Meas}}$-model. Assume that $[u : (\alpha \to \beta)]_M$ and $[v : \alpha]_M$ are independent in M. Then for any $r, s \in \mathsf{S}$ we have:*

$$M \models P_{\geq r}(u : (\alpha \to \beta)) \to \big(P_{\geq s}(v : \alpha) \to P_{\geq r \cdot s}(u \cdot v : \beta)\big)$$

Proof. Assume that $M = \langle W, H, \mu, * \rangle$.

Let $w \in [u : (\alpha \to \beta)] \cap [v : \alpha]$. We have that $*_w \Vdash u : (\alpha \to \beta)$ and that $*_w \Vdash v : \alpha$. Since $*_w$ is a basic J_{CS}-evaluation, by Theorem 6 we get that $*_w$ satisfies all instances of axiom (J), i.e. $*_w \Vdash u : (\alpha \to \beta) \to (v : \alpha \to u \cdot v : \beta)$. Hence we have $*_w \Vdash u \cdot v : \beta$, i.e. $w \in [u \cdot v : \beta]$. So we proved that $[u : (\alpha \to \beta)] \cap [v : \alpha] \subseteq [u \cdot v : \beta]$. So by Lemma 13(3) we get:

$$\mu\big([u \cdot v : \beta]\big) \geq \mu\big([u : (\alpha \to \beta)] \cap [v : \alpha]\big)$$

And since $[u : (\alpha \to \beta)]$ and $[v : \alpha]$ are independent in M we have:

$$\mu\big([u \cdot v : \beta]\big) \geq \mu\big([u : (\alpha \to \beta)]\big) \cdot \mu\big([v : \alpha]\big) \tag{43}$$

Assume that:

$$M \models P_{\geq r}(u : (\alpha \to \beta)) \text{ and } M \models P_{\geq s}(v : \alpha), \text{ i.e.}$$
$$\mu\big([u : (\alpha \to \beta)]\big) \geq r \text{ and } \mu\big([v : \alpha]\big) \geq s$$

By (43) we have $\mu\big([u \cdot v : \beta]\big) \geq r \cdot s$, i.e. $M \models P_{\geq r \cdot s}(u \cdot v : \beta)$. Hence we proved that:

$$M \models P_{\geq r}(u : (\alpha \to \beta)) \to \big(P_{\geq s}(v : \alpha) \to P_{\geq r \cdot s}(u \cdot v : \beta)\big) \qquad \square$$

Remark 23. The previous theorem corresponds to Axiom A1 of [13]. However, we have to explicitly formulate the additional assumption that the premises are independent. In the logic of uncertain justifications, independence of evidential assertions is assumed implicitly.

5 Soundness and Completeness of PJ

In order to prove soundness for PJ we will need the Archimedean property for the real numbers.

Proposition 24 (Archimedean Property for the real numbers)**.** *For any real number $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.*

Theorem 25 (Soundness)**.** *Let* CS *be any constant specification. Then the system* $\mathsf{PJ}_{\mathsf{CS}}$ *is sound with respect to the class of* $\mathsf{PJ}_{\mathsf{CS,Meas}}$*-models. I.e. for any $T \subseteq \mathsf{L_P}$ and $A \in \mathsf{L_P}$ we have:*

$$T \vdash_{\mathsf{PJ}_{\mathsf{CS}}} A \Longrightarrow T \models_{\mathsf{PJ}_{\mathsf{CS,Meas}}} A$$

Proof. Let $T \subseteq \mathsf{L_P}$ and $A \in \mathsf{L_P}$. We prove the claim by transfinite induction on the depth of the derivation $T \vdash_{\mathsf{PJ}_{\mathsf{CS}}} A$. Let $M = \langle W, H, \mu, * \rangle \in \mathsf{PJ}_{\mathsf{CS,Meas}}$. We assume that $M \models T$. We distinguish the following cases:

(1) $A \in T$. Then M satisfies A by assumption.

(2) A is an instance of (P). Then obviously M satisfies A.

(3) A is an instance of (PI). This means:

$$A = P_{\geq 0}\alpha$$

Since $\mu : H \to [0, 1]$ and $[\alpha] \in H$ we have $\mu([\alpha]) \geq 0$, i.e. $M \models P_{\geq 0}\alpha$, i.e. $M \models A$.

(4) A is an instance of (WE). That means:

$$A = P_{\leq r}\alpha \to P_{<s}\alpha, \text{ with } s > r$$

We have:

$$\begin{array}{cr} M \models A & \iff \\ (M \models P_{\leq r}\alpha \Longrightarrow M \models P_{<s}\alpha) & \overset{\text{Lemma 16}}{\iff} \\ (\mu([\alpha]) \leq r \Longrightarrow \mu([\alpha]) < s) & \end{array}$$

The last statement is true since $r < s$. Thus $M \models A$.

(5) A is an instance of (LE). Similar to case (4).

(6) A is an instance of (DIS). Then we have:

$$A = \big(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}\neg(\alpha \wedge \beta)\big) \to P_{\geq \min(1,r+s)}(\alpha \vee \beta)$$

It holds:

$$\begin{array}{cr} M \models A & \iff \\ M \models \big(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}\neg(\alpha \wedge \beta)\big) \to P_{\geq \min(1,r+s)}(\alpha \vee \beta) & \overset{\textbf{S.E.}}{\iff} \\ M \models \big(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\leq 0}(\alpha \wedge \beta)\big) \to P_{\geq \min(1,r+s)}(\alpha \vee \beta) & \end{array}$$

By Lemma 16 the last statement is equivalent to:

$$\begin{array}{l} \big(\mu([\alpha]) \geq r \text{ and } \mu([\beta]) \geq s \text{ and } \mu([\alpha \wedge \beta]) \leq 0\big) \Longrightarrow \\ \mu([\alpha \vee \beta]) \geq \min(1, r+s) \end{array}$$

Let $\mu([\alpha]) \geq r$, $\mu([\beta]) \geq s$ and $\mu([\alpha \wedge \beta]) \leq 0$. By Remark 14 we have: $\mu([\alpha \vee \beta]) = \mu([\alpha]) + \mu([\beta]) - \mu([\alpha \wedge \beta]) \geq r + s$. Since $\mu([\alpha \vee \beta]) \leq 1$ we have $\mu([\alpha \vee \beta]) \geq \min(1, r+s)$. Thus, the last of the above statements is true, so $M \models A$.

(7) A is an instance of (UN). Then we have:

$$A = \big(P_{\leq r}\alpha \wedge P_{<s}\beta\big) \to P_{<r+s}(\alpha \vee \beta), \; r + s \leq 1$$

We have:

$$\begin{array}{cr} M \models A & \iff \\ \big(M \models \big(P_{\leq r}\alpha \wedge P_{<s}\beta\big) \to P_{<r+s}(\alpha \vee \beta)\big) & \overset{\text{Lemma 16}}{\iff} \\ \Big(\big(\mu([\alpha]) \leq r \text{ and } \mu([\beta]) < s\big) \Longrightarrow \mu([\alpha \vee \beta]) < r + s\Big) & \end{array}$$

Assume that $\mu([\alpha]) \leq r$ and $\mu([\beta]) < s$. By Remark 14 we have that $\mu([\alpha \vee \beta]) = \mu([\alpha]) + \mu([\beta]) - \mu([\alpha \wedge \beta]) < r + s - \mu([\alpha \wedge \beta])$. Since $\mu([\alpha \wedge \beta]) \geq 0$ we have $\mu([\alpha \vee \beta]) < r + s$. Thus, the last of the above statements is true, so $M \models A$.

(8) A is obtained by an application of the rule (MP). Thus there exists some $B \in \mathsf{L_P}$ such that $T \vdash_{\mathsf{PJ_{CS}}} B$ and $T \vdash_{\mathsf{PJ_{CS}}} B \to A$. By the inductive hypothesis we have that $M \models B$ and $M \models B \to A$. Thus $M \models A$.

(9) A is obtained by an application of (CE). That means $A = P_{\geq 1}\alpha$ and also $\vdash_{\mathsf{J_{CS}}} \alpha$ for some $\alpha \in \mathsf{L_J}$. By Theorem 6 we have $\Vdash_{\mathsf{CS}} \alpha$, which implies that $(\forall w \in W)[*_w \Vdash \alpha]$, i.e. $[\alpha] = W$. Thus $\mu([\alpha]) = 1$, i.e. $M \models P_{\geq 1}\alpha$.

(10) A is obtained by an application of (ST). That means $A = B \to P_{\geq s}\beta$ for $s > 0$ and also $T \vdash_{\mathsf{PJ_{CS}}} B \to P_{\geq s - \frac{1}{k}}\beta$ for every integer $k \geq \frac{1}{s}$. By the inductive hypothesis we have that $M \models B \to P_{\geq s-\frac{1}{k}}\beta$ for every integer $k \geq \frac{1}{s}$.
Assume that $M \models B$. This implies that for every integer $k \geq \frac{1}{s}$ we have $M \models P_{\geq s-\frac{1}{k}}\beta$, i.e.

$$\mu([\beta]) \geq s - \frac{1}{k} \qquad \text{for every integer } k \geq \frac{1}{s}. \tag{44}$$

Assume that $\mu([\beta]) < s$, i.e. $s - \mu([\beta]) > 0$. By the Archimedean property for the real numbers we know that there exists some integer n such that $\frac{1}{n} < s - \mu([\beta])$, which implies $n > \frac{1}{s-\mu([\beta])} \geq \frac{1}{s}$ since $s > \mu([\beta]) \geq 0$. Hence there exists some $n \geq \frac{1}{s}$ with $\mu([\beta]) < s - \frac{1}{n}$, which contradicts (44). Thus $\mu([\beta]) \geq s$, i.e. $M \models P_{\geq s}\beta$. Hence we proved that $M \models B$ implies $M \models P_{\geq s}\beta$. So we have that $M \models A$. $\square$

Now we define the notion of $\mathsf{PJ_{CS}}$-consistent sets.

Definition 26 ($\mathsf{PJ_{CS}}$-Consistent Sets)**.** Let CS be any constant specification and let T be a set of $\mathsf{L_P}$-formulas.

- T is said to be $\mathsf{PJ_{CS}}$-*consistent* iff $T \nvdash_{\mathsf{PJ_{CS}}} \bot$. Otherwise T is said to be $\mathsf{PJ_{CS}}$-*inconsistent*.
- T is said to be $\mathsf{L_P}$-maximal iff for every $A \in \mathsf{L_P}$ either $A \in T$ or $\neg A \in T$.
- T is said to be *maximal* $\mathsf{PJ_{CS}}$-*consistent* iff it is $\mathsf{L_P}$-maximal and $\mathsf{PJ_{CS}}$-consistent.

Alternatively we can say that T is $\mathsf{PJ}_{\mathsf{CS}}$-consistent iff there exists some $A \in \mathsf{L}_{\mathsf{P}}$ such that $T \nvdash_{\mathsf{PJ}_{\mathsf{CS}}} A$.

Before proving completeness for PJ we need to prove some auxiliary Lemmata and Theorems.

Lemma 27 (Properties of $\mathsf{PJ}_{\mathsf{CS}}$-Consistent Sets)**.** *Let CS be any constant specification and let T be a $\mathsf{PJ}_{\mathsf{CS}}$-consistent set of L_{P}-formulas.*

(1) For any formula $A \in \mathsf{L}_{\mathsf{P}}$ either T, A is $\mathsf{PJ}_{\mathsf{CS}}$-consistent or $T, \neg A$ is $\mathsf{PJ}_{\mathsf{CS}}$-consistent.

(2) If $\neg(A \to P_{\geq s}\beta) \in T$ for $s > 0$, then there is some integer $n \geq \frac{1}{s}$ such that $T, \neg(A \to P_{\geq s-\frac{1}{n}}\beta)$ is $\mathsf{PJ}_{\mathsf{CS}}$-consistent.

Proof. The proof of (1) is standard and therefore omitted. For (2) we have the following:

Assume that for every integer $n \geq \frac{1}{s}$ the set $T, \neg(A \to P_{\geq s-\frac{1}{n}}\beta)$ is $\mathsf{PJ}_{\mathsf{CS}}$-inconsistent. Then we have the following:

$$T, \neg(A \to P_{\geq s-\frac{1}{n}}\beta) \vdash_{\mathsf{PJ}_{\mathsf{CS}}} \bot, \forall \text{ integer } n \geq \frac{1}{s} \tag{45}$$

$$T \vdash_{\mathsf{PJ}_{\mathsf{CS}}} \neg(A \to P_{\geq s-\frac{1}{n}}\beta) \to \bot, \forall \text{ integer } n \geq \frac{1}{s} \qquad [\text{Thm.}7, 45] \tag{46}$$

$$T \vdash_{\mathsf{PJ}_{\mathsf{CS}}} A \to P_{\geq s-\frac{1}{n}}\beta, \forall \text{ integer } n \geq \frac{1}{s} \qquad [46, \textbf{P.R.}] \tag{47}$$

$$T \vdash_{\mathsf{PJ}_{\mathsf{CS}}} A \to P_{\geq s}\beta \qquad [47, (\mathsf{ST})] \tag{48}$$

$$T \vdash_{\mathsf{PJ}_{\mathsf{CS}}} \neg(A \to P_{\geq s}\beta) \tag{49}$$

$$T \vdash_{\mathsf{PJ}_{\mathsf{CS}}} \bot \qquad [48, 49, \textbf{P.R.}] \tag{50}$$

(50) contradicts the fact that T is $\mathsf{PJ}_{\mathsf{CS}}$-consistent. Thus there exists some $n \geq \frac{1}{s}$ such that $T, \neg(A \to P_{\geq s-\frac{1}{n}}\beta)$ is $\mathsf{PJ}_{\mathsf{CS}}$-consistent. □

Lemma 28 (Properties of Maximal $\mathsf{PJ}_{\mathsf{CS}}$-Consistent Sets)**.** *Let CS be any constant specification and let $\mathcal{T}$ be a maximal $\mathsf{PJ}_{\mathsf{CS}}$-consistent set. Then the following hold:*

(1) For any formula $A \in \mathsf{L}_{\mathsf{P}}$, exactly one member of $\{A, \neg A\}$ is in $\mathcal{T}$.

(2) For any formula $A \in \mathsf{L}_{\mathsf{P}}$:

$$\mathcal{T} \vdash_{\mathsf{PJ}_{\mathsf{CS}}} A \Longleftrightarrow A \in \mathcal{T}$$

(3) *For all formulas* $A, B \in \mathsf{L_P}$ *we have:*

$$A \wedge B \in \mathcal{T} \Longleftrightarrow \{A, B\} \subseteq \mathcal{T}$$

(4) *For all formulas* $A, B \in \mathsf{L_P}$ *we have:*

$$\{A, A \to B\} \subseteq \mathcal{T} \Longrightarrow B \in \mathcal{T}$$

(5) *Let* $\alpha \in \mathsf{L_J}$, $X = \{s \mid P_{\geq s}\alpha \in \mathcal{T}\}$ *and* $t = \sup(X)$. *Then:*

(i) *For all* $r \in \mathsf{S}[0, t)$ *we have that* $P_{>r}\alpha \in \mathcal{T}$

(ii) *For all* $r \in \mathsf{S}[0, t)$ *we have that* $P_{\geq r}\alpha \in \mathcal{T}$

(iii) *If* $t \in \mathsf{S}$ *then* $P_{\geq t}\alpha \in \mathcal{T}$.

Proof. The proofs of (1) to (4) are standard and therefore omitted. For (5) we have the following:

(i) Let $r \in \mathsf{S}[0, t)$. Assume that $P_{>r}\alpha \notin \mathcal{T}$. Then assume that for some $r' \in \mathsf{S}(r, 1]$ we have $P_{\geq r'}\alpha \in \mathcal{T}$. Since $r' > r$ by Lemma 17(iii) we have that $\mathcal{T} \vdash_{\mathsf{PJ_{CS}}} P_{\geq r'}\alpha \to P_{>r}\alpha$. By (2) we have $P_{\geq r'}\alpha \to P_{>r}\alpha \in \mathcal{T}$ and by (4) we have $P_{>r}\alpha \in \mathcal{T}$ which is absurd since we assumed that $P_{>r}\alpha \notin \mathcal{T}$. Thus for all $r' \in \mathsf{S}(r, 1]$ we have $P_{\geq r'}\alpha \notin \mathcal{T}$. Thus r is an upper bound of X, which is again absurd since $r < t$ and $t = \sup(X)$. Hence we conclude that $P_{>r}\alpha \in \mathcal{T}$.

(ii) Let $r \in \mathsf{S}[0, t)$. By (i) we have that $P_{>r}\alpha \in \mathcal{T}$. By Lemma 17(iv) we have $P_{>r}\alpha \to P_{\geq r}\alpha \in \mathcal{T}$ and by (4) we get $P_{\geq r}\alpha \in \mathcal{T}$.

(iii) If $t = 0$ then by (PI) we have that $\mathcal{T} \vdash_{\mathsf{PJ_{CS}}} P_{\geq 0}\alpha$. Thus by (2) we have that $P_{\geq t}\alpha \in \mathcal{T}$.

Let $t > 0$. By (ii) we have that for all $n \geq \frac{1}{t}$, $P_{\geq t - \frac{1}{n}}\alpha \in \mathcal{T}$. So by the rule ($\mathsf{ST}$) we get $P_{\geq t}\alpha \in \mathcal{T}$. □

Lemma 29 (Lindenbaum)**.** *Let* CS *be any constant specification. For every* $\mathsf{PJ_{CS}}$*-consistent set* T*, there exists a maximal* $\mathsf{PJ_{CS}}$*-consistent set* $\mathcal{T}$ *such that* $T \subseteq \mathcal{T}$.

Proof. Let T be a $\mathsf{PJ_{CS}}$-consistent set. Let $A_0,\ A_1,\ A_2,\ \ldots$ be an enumeration of all the formulas in $\mathsf{L_P}$. We define a sequence of sets $\{T_i\}_{i \in \mathbb{N}}$ such that:

(1) $T_0 := T$

(2) for every $i \geq 0$:

(a) if $T_i \cup \{A_i\}$ is $\mathsf{PJ_{CS}}$-consistent, then we set $T_{i+1} := T_i \cup \{A_i\}$, otherwise

(b) if A_i is of the form $B \to P_{\geq s}\gamma$ for $s > 0$ then we choose some integer $n \geq \frac{1}{s}$ such that $T_i \cup \{\neg A_i, \neg(B \to P_{\geq s-\frac{1}{n}}\gamma)\}$ is $\mathsf{PJ_{CS}}$-consistent[2] and we set $T_{i+1} := T_i \cup \{\neg A_i, \neg(B \to P_{\geq s-\frac{1}{n}}\gamma)\}$, otherwise

(c) we set $T_{i+1} := T_i \cup \{\neg A_i\}$

(3) $\mathcal{T} = \bigcup_{i=0}^{\infty} T_i$

By induction on i we will prove that T_i is $\mathsf{PJ_{CS}}$-consistent for every $i \in \mathbb{N}$.

(i) The consistency of T_0 follows from that of T.

(ii) Let $i \geq 0$. Assuming that T_i is $\mathsf{PJ_{CS}}$-consistent, we will prove that T_{i+1} is $\mathsf{PJ_{CS}}$-consistent. We have the following cases:

- If T_{i+1} is constructed using the case (2)(a) above, then it is obviously $\mathsf{PJ_{CS}}$-consistent.
- If T_{i+1} is constructed using the case (2)(b) above then we know that T_i, A_i is $\mathsf{PJ_{CS}}$-inconsistent, thus according to Lemma 27(1) we have that $T_i, \neg A_i$ is $\mathsf{PJ_{CS}}$-consistent. We also have that $A_i = B \to P_{\geq s}\gamma$ for $s > 0$. So according to Lemma 27(2) we know that there exists some $n \geq \frac{1}{s}$ such that $T_i, \neg A_i, \neg(B \to P_{\geq s-\frac{1}{n}}\gamma)$ is $\mathsf{PJ_{CS}}$-consistent, thus T_{i+1} is $\mathsf{PJ_{CS}}$-consistent.
- If T_{i+1} is constructed using the case (2)(c) above then we know that T_i, A_i is $\mathsf{PJ_{CS}}$-inconsistent, thus according to Lemma 27(1) we have that $T_i, \neg A_i$ is $\mathsf{PJ_{CS}}$-consistent, i.e. T_{i+1} is $\mathsf{PJ_{CS}}$-consistent.

Now we will show that $\mathcal{T}$ is a maximal $\mathsf{PJ_{CS}}$-consistent set.

We have that for every $A \in \mathsf{L_P}$ either $A \in \mathcal{T}$ or $\neg A \in \mathcal{T}$. Thus according to Definition 26, the set $\mathcal{T}$ is $\mathsf{L_P}$-maximal.

It remains to show that $\mathcal{T}$ is $\mathsf{PJ_{CS}}$-consistent. We will first show that $\mathcal{T}$ does not contain all $\mathsf{L_P}$-formulas (see (A) below) and then that $\mathcal{T}$ is $\mathsf{PJ_{CS}}$-deductively closed for $\mathsf{L_P}$ (see (B) below). The fact that $\mathcal{T}$ is $\mathsf{PJ_{CS}}$-consistent follows easily from (A) and (B).

[2] we will show in the case (ii) below that such an n always exists

(A) Assume that for some $A \in \mathsf{L_P}$ both A and $\neg A$ belong to $\mathcal{T}$. That means there are i, j such that $A \in T_i$ and $\neg A \in T_j$. Since $T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$, we have that $\{A_i, A_j\} \subseteq T_{\max(i,j)}$, which implies that $T_{\max(i,j)}$ is $\mathsf{PJ_{CS}}$-inconsistent, a contradiction. Thus $\mathcal{T}$ does not contain all members of $\mathsf{L_P}$.

(B) We show that $\mathcal{T}$ is $\mathsf{PJ_{CS}}$-deductively closed for $\mathsf{L_P}$-formulas.

Assume that for some $A \in \mathsf{L_P}$ we have that $\mathcal{T} \vdash_{\mathsf{PJ_{CS}}} A$. We will prove by transfinite induction on the depth of the derivation $\mathcal{T} \vdash_{\mathsf{PJ_{CS}}} A$ that $A \in \mathcal{T}$. We distinguish cases depending on the last rule or axiom used to obtain A from $\mathcal{T}$.

(1) If $A \in \mathcal{T}$ then we are done.

(2) Assume that A is an instance of some PJ-axiom. We know that there exists some k such that $A = A_k$. Assume that $\neg A_k \in T_{k+1}$. Then we have that $T_{k+1} \vdash_{\mathsf{PJ_{CS}}} \neg A_k$ and $T_{k+1} \vdash_{\mathsf{PJ_{CS}}} A_k$, which contradicts the fact that T_{k+1} is $\mathsf{PJ_{CS}}$-consistent. Hence $A_k \in T_{k+1}$, i.e. $A \in \mathcal{T}$.

(3) If A is obtained from $\mathcal{T}$ by an application of the rule (MP), then by the inductive hypothesis we have that all the premises of the rule are contained in $\mathcal{T}$. So there must exist some l such that T_l contains all the premises of the rule. So, $T_l \vdash_{\mathsf{PJ_{CS}}} A$. There exists also some k such that $A = A_k$. Assume that $\neg A \in T_{\max(k,l)+1}$. This implies that $T_{\max(k,l)+1} \vdash_{\mathsf{PJ_{CS}}} A$ and $T_{\max(k,l)+1} \vdash_{\mathsf{PJ_{CS}}} \neg A$, which contradicts the fact that $T_{\max(k,l)+1}$ is $\mathsf{PJ_{CS}}$-consistent. Thus we have that $A \in T_{\max(k,l)+1}$, i.e. $A \in \mathcal{T}$.

(4) Assume that A is obtained by $\mathcal{T}$ by an application of the rule (CE). This means that $A = P_{\geq 1}\alpha$ and that $\vdash_{\mathsf{J_{CS}}} \alpha$ for some $\alpha \in \mathsf{L_J}$. We know that there exists some k such that $A = A_k$. Using the same arguments with the case (2) we can prove that $A \in T_{k+1}$, i.e. $A \in \mathcal{T}$.

(5) Assume that A is obtained from $\mathcal{T}$ by the rule (ST). That means that $A = B \to P_{\geq s}\gamma$ for $s > 0$ and also that for every integer $k \geq \frac{1}{s}$ we have $\mathcal{T} \vdash_{\mathsf{PJ_{CS}}} B \to P_{\geq s-\frac{1}{k}}\gamma$. Assume that A does not belong to $\mathcal{T}$, thus $\neg A \in \mathcal{T}$, i.e. $\neg(B \to P_{\geq s}\gamma) \in \mathcal{T}$. Let m be such that $A_m = B \to P_{\geq s}\gamma$. We find that $\neg(B \to P_{\geq s}\gamma) \in T_{m+1}$ and by the construction of $\mathcal{T}$, there exists some $l \geq \frac{1}{s}$ such that $\neg(B \to P_{\geq s-\frac{1}{l}}\gamma) \in T_{m+1}$. However, we also find that the formula $B \to P_{\geq s-\frac{1}{l}}$ is a premise of (ST), thus by the inductive hypothesis $B \to P_{\geq s-\frac{1}{l}} \in \mathcal{T}$. So, there exists an m' such that

$B \to P_{\geq s-\frac{1}{l}} \in T_{m'}$. Thus

$$\{\neg(B \to P_{\geq s-\frac{1}{l}}), B \to P_{\geq s-\frac{1}{l}}\} \subseteq T_{\max(m+1,m')},$$

which contradicts the fact that $T_{\max(m+1,m')}$ is $\mathsf{PJ}_{\mathsf{CS}}$-consistent. Thus $A \in \mathcal{T}$.

So, we proved that $\mathcal{T}$ is a maximal $\mathsf{PJ}_{\mathsf{CS}}$-consistent set that contains the $\mathsf{PJ}_{\mathsf{CS}}$-consistent set T. □

Now we will define a canonical model for any maximal $\mathsf{PJ}_{\mathsf{CS}}$-consistent set of formulas.

Definition 30 (Canonical Model)**.** Let CS be any constant specification and let $\mathcal{T}$ be a maximal $\mathsf{PJ}_{\mathsf{CS}}$-consistent set of $\mathsf{L_P}$-formulas. The *canonical model* for $\mathcal{T}$ is the quadruple $M_{\mathcal{T}} = \langle W, H, \mu, * \rangle$, defined as follows:

- $W = \{w \mid w \text{ is a basic } \mathsf{J}_{\mathsf{CS}}\text{-evaluation}\}$
- $H = \{[\alpha]_{M_{\mathcal{T}}} \mid \alpha \in \mathsf{L_J}\}$
- for every $\alpha \in \mathsf{L_J}$, $\mu([\alpha]_{M_{\mathcal{T}}}) = \sup_s \{P_{\geq s}\alpha \in \mathcal{T}\}$
- for every $w \in W$, $*_w = w$

Remark 31. In Definition 30 the canonical model $M_{\mathcal{T}} = \langle W, H, \mu, * \rangle$ was defined. Observe that in definition of H we use the set $[\alpha]_{M_{\mathcal{T}}}$. This is not a problem since by Definition 12 we have that $[\alpha]_{M_{\mathcal{T}}}$ depends only on $*$, W, and the justification formula α, which do not depend on H. The same holds for μ. Thus, the canonical model is well-defined.

Lemma 32. *Let* CS *be any constant specification and let* $\mathcal{T}$ *be a maximal* $\mathsf{PJ}_{\mathsf{CS}}$*-consistent set. The canonical model for* $\mathcal{T}$*,* $M_{\mathcal{T}}$*, is a* $\mathsf{PJ}_{\mathsf{CS},\mathsf{Meas}}$*-model.*

Proof. Let $M_{\mathcal{T}} = \langle W, H, \mu, * \rangle$. Observe that according to Definition 30 for every $\alpha \in \mathsf{L_J}$ we have:

$$[\alpha]_{M_{\mathcal{T}}} = \{w \in W \mid *_w \Vdash \alpha\} = \{w \mid w \text{ is a basic } \mathsf{J}_{\mathsf{CS}}\text{-evaluation and } w \Vdash \alpha\}$$

In order for $M_{\mathcal{T}}$ to be a $\mathsf{PJ}_{\mathsf{CS},\mathsf{Meas}}$-model we have to prove the following:

(1) W **is a non-empty set:**

We know that there exists a basic J_{CS}-evaluation, thus $W \neq \emptyset$.

(2) H **is an algebra over** W**:**

It holds that $[\top] = W$. Thus $W \in H$. Hence $H \neq \emptyset$. Let $[\alpha] \in H$. It holds that $[\alpha] \subseteq W$. Thus $H \subseteq \mathcal{P}(W)$.

Let $\alpha, \beta \in \mathsf{L_J}$ and assume that $[\alpha], [\beta] \in H$. We have that $\neg\alpha, \alpha \vee \beta \in \mathsf{L_J}$ and by Remark 14 $[\alpha] \cup [\beta] = [\alpha \vee \beta] \in H$ and $W \setminus [\alpha] = [\neg\alpha] \in H$.

So, according to Definition 8, H is an algebra over W.

(3) μ **is a function from** H **to** $[0,1]$**:**

We have to prove the following:

(a) **the domain of** μ **is** H **and the codomain of** μ **is** $[0,1]$**:**
Let $[\alpha] \in H$ for some $\alpha \in \mathsf{L_J}$. We have that $P_{\geq 0}\alpha$ is an axiom of PJ, thus $P_{\geq 0}\alpha \in \mathcal{T}$. Hence the set $\{s \in S \mid P_{\geq s}\alpha \in \mathcal{T}\}$ is not empty which means that it has a supremum. We have that $\mu([\alpha]) = \sup_s\{P_{\geq s}\alpha \in \mathcal{T}\}$. Thus, μ is defined for all members of H, i.e. the domain of μ is H. In $\sup_s\{P_{\geq s}\alpha \in \mathcal{T}\}$ we have by definition that $s \in \mathsf{S}$, i.e. $s \leq 1$. By a previous argument it also holds that $\sup_s\{P_{\geq s}\alpha \in \mathcal{T}\} \geq 0$. Thus $0 \leq \sup_s\{P_{\geq s}\alpha \in \mathcal{T}\} \leq 1$, i.e. $0 \leq \mu([\alpha]) \leq 1$. So the codomain of μ is $[0,1]$.

(b) **for every** $U \in H$**,** $\mu(U)$ **is unique:**
Let $U \in H$ and assume that $U = [\alpha] = [\beta]$ for some $\alpha, \beta \in \mathsf{L_J}$. We will prove that $\mu([\alpha]) = \mu([\beta])$. Of course it suffices to prove that:

$$[\alpha] \subseteq [\beta] \Longrightarrow \mu([\alpha]) \leq \mu([\beta]) \tag{51}$$

We have:

$$
\begin{array}{rr}
[\alpha] \subseteq [\beta] & \text{implies} \\
(\forall w \in W)\big[w \in [\alpha] \Longrightarrow w \in [\beta]\big] & \text{implies} \\
(\forall w \in W)\big[w \Vdash \alpha \Longrightarrow w \Vdash \beta\big] & \text{implies} \\
(\forall w \in W)\big[w \Vdash \alpha \to \beta\big] & \text{implies} \\
\Vdash_{\mathsf{CS}} \alpha \to \beta & \text{implies by Theorem 6} \\
\vdash_{\mathsf{J_{CS}}} \alpha \to \beta & \text{implies by Lemma 17(ii)} \\
(\forall s \in \mathsf{S})\big[\vdash_{\mathsf{PJ_{CS}}} P_{\geq s}\alpha \to P_{\geq s}\beta\big] & \text{implies by Lemma 28(2)} \\
(\forall s \in \mathsf{S})\big[P_{\geq s}\alpha \to P_{\geq s}\beta \in \mathcal{T}\big] & \text{implies by Lemma 28(4)} \\
(\forall s \in \mathsf{S})\big[P_{\geq s}\alpha \in \mathcal{T} \Longrightarrow P_{\geq s}\beta \in \mathcal{T}\big] & \text{implies} \\
\{s \in \mathsf{S} \mid P_{\geq s}\alpha \in \mathcal{T}\} \subseteq \{s \in \mathsf{S} \mid P_{\geq s}\beta \in \mathcal{T}\} & \text{implies}
\end{array}
$$

$$\sup_s\{P_{\geq s}\alpha \in \mathcal{T}\} \leq \sup_s\{P_{\geq s}\beta \in \mathcal{T}\} \qquad \text{i.e.}$$

$$\mu([\alpha]) \leq \mu([\beta])$$

Hence (51) holds, which proves that $\mu(U)$ is unique.

(4) μ **is a finitely additive measure:**

Before proving that μ is a finitely additive measure we need to prove the following statement:

$$\mu([\alpha]) + \mu([\neg\alpha]) \leq 1 \tag{52}$$

Let:

$$\begin{aligned} X &= \{s \mid P_{\geq s}\alpha \in \mathcal{T}\} \\ Y &= \{s \mid P_{\geq s}\neg\alpha \in \mathcal{T}\} \\ r_1 &= \mu([\alpha]) = \sup(X) \\ r_2 &= \mu([\neg\alpha]) = \sup(Y) \end{aligned}$$

Let $s \in Y$. It holds that $P_{\geq s}\neg\alpha \in \mathcal{T}$. If $1 - s < r_1$ then by Lemma 28(5)(i) we would have $P_{>1-s}\alpha \in \mathcal{T}$. By **S.E.** we get $\neg P_{\leq 1-s}\alpha \in \mathcal{T}$ and by **S.E.** again we get $\neg P_{\geq s}\neg\alpha \in \mathcal{T}$ which contradicts the fact that $\mathcal{T}$ is $\mathsf{PJ_{CS}}$-consistent. Thus $1 - s \geq r_1$, i.e. $1 - r_1 \geq s$, i.e. $1 - r_1$ is an upper bound of Y, hence $1 - r_1 \geq r_2$, i.e. $r_1 + r_2 \leq 1$, i.e. (52) holds.

Now in order to prove that μ is a finitely additive measure we need to prove the following:

(i) $\mu(\mathbf{W}) = \mathbf{1}$

We have that $\vdash_{\mathsf{J_{CS}}} \top$. By the rule ($\mathsf{CE}$) we get $\vdash_{\mathsf{PJ_{CS}}} P_{\geq 1}\top$. By Lemma 28(2) we get $P_{\geq 1}\top \in \mathcal{T}$. It holds that $W = [\top]$. Thus $\mu(W) = \mu([\top]) = \sup_s\{P_{\geq s}\top \in \mathcal{T}\} \geq 1$, i.e. $\mu(W) = 1$.

(ii) $[\alpha] \cap [\beta] = \emptyset \Longrightarrow \mu([\alpha] \cup [\beta]) = \mu([\alpha]) + \mu([\beta])$

Let $\alpha, \beta \in \mathsf{L_J}$ such that:

$$\begin{aligned} &[\alpha] \cap [\beta] = \emptyset \\ &r = \mu([\alpha]) = \sup_s \left\{s \;\middle|\; P_{\geq s}\alpha \in \mathcal{T}\right\} \\ &s = \mu([\beta]) = \sup_r \left\{r \;\middle|\; P_{\geq r}\beta \in \mathcal{T}\right\} \end{aligned}$$

It holds $[\beta] \subseteq [\neg\alpha]$. By (51) we have $\mu([\beta]) \leq \mu([\neg\alpha])$ and by (52) we have:

$$\begin{aligned} \mu([\beta]) &\leq 1 - \mu([\alpha]) \\ &\text{i.e. } s \leq 1 - r \\ &\text{i.e. } r + s \leq 1 \end{aligned} \tag{53}$$

We also have that

$$\mu([\neg(\alpha \wedge \beta)]) = \mu(W \setminus ([\alpha] \cap [\beta])) = \mu(W) = 1.$$

Thus $1 = \sup_s\{P_{\geq s}\neg(\alpha \wedge \beta) \in \mathcal{T}\}$. So by Lemma 28(5)(iii) we find

$$P_{\geq 1}\neg(\alpha \wedge \beta) \in \mathcal{T} \tag{54}$$

We distinguish the following cases:

- Suppose that $r > 0$ and $s > 0$. By Lemma 28(5)(ii) we have that for every $r' \in \mathsf{S}[0, r)$ and every $s' \in \mathsf{S}[0, s)$, $P_{\geq r'}\alpha$, $P_{\geq s'}\beta \in \mathcal{T}$. It holds that $r' + s' < r + s$ and by (53) we get $r' + s' < 1$. Thus by (54) and by axiom (DIS) we get $P_{\geq r'+s'}(\alpha \vee \beta) \in \mathcal{T}$. Hence $t_0 = \sup_t\{P_{\geq t}(\alpha \vee \beta) \in \mathcal{T}\} \geq r + s$.

  If $r + s = 1$ then we have that $t_0 = 1$, i.e. $\mu([\alpha \vee \beta]) = \mu([\alpha]) + \mu([\beta])$.

  If $r + s < 1$ then since $r, s > 0$ we have that $r, s < 1$. Assume that $r + s < t_0$. By Lemma 28(5)(ii) for every $t' \in \mathsf{S}(r + s, t_0)$ we have $P_{\geq t'}(\alpha \vee \beta) \in \mathcal{T}$. We choose rational numbers r'' and s'' such that $t' = r'' + s''$ and $r'' > r$ and $s'' > s$. If we had $P_{\geq r''}\alpha, P_{\geq s''}\beta \in \mathcal{T}$ this would imply that

  $$\mu([\alpha]) = \sup_s\{s \mid P_{\geq s}\alpha \in \mathcal{T}\} = r \geq r''$$

  and

  $$\mu([\beta]) = \sup_r\{r \mid P_{\geq r}\beta \in \mathcal{T}\} = s \geq s''$$

  which is absurd since $r'' > r$ and $s'' > s$. Thus we have:

  $$\neg P_{\geq r''}\alpha \in \mathcal{T}, \neg P_{\geq s''}\beta \in \mathcal{T}$$

  by **S.E.** we get:

  $$P_{<r''}\alpha \in \mathcal{T}, P_{<s''}\beta \in \mathcal{T}$$

  By Axiom (LE) we get:

  $$P_{\leq r''}\alpha \in \mathcal{T}, P_{<s''}\beta \in \mathcal{T}$$

It holds that $r'' + s'' = t' < t_0 \leq 1$. Thus by Axiom ($\mathsf{UN}$) we get:

$P_{<r''+s''}(\alpha \vee \beta) \in \mathcal{T}$ and by **S.E.** $\neg P_{\geq r''+s''}(\alpha \vee \beta) \in \mathcal{T}$, i.e. $\neg P_{\geq t'}(\alpha \vee \beta) \in \mathcal{T}$

which is a contradiction since $P_{\geq t'}(\alpha \vee \beta) \in \mathcal{T}$ and $\mathcal{T}$ is $\mathsf{PJ}_{\mathsf{CS}}$-consistent. Thus $r + s = t_0$, i.e. $\mu([\alpha] \cup [\beta]) = \mu([\alpha]) + \mu([\beta])$.

- Assume that at least one of r, s is equal to 0. Then we can reason as in the above case with the only exception that $r' = 0$ or $s' = 0$ (depending on whether $r = 0$ or $s = 0$ respectively).

(5) **for all $w \in W$, $*_w$ is a basic J_{CS}-evaluation:**

It holds by the construction of $M_{\mathcal{T}}$.

(6) **for all $\alpha \in \mathsf{L}_{\mathsf{J}}$, $[\alpha]_{M_{\mathcal{T}}} \in H$**

It holds by the construction of $M_{\mathcal{T}}$. □

Lemma 33 (Truth Lemma)**.** *Let CS be a constant specification. Let $\mathcal{T}$ be a maximal $\mathsf{PJ}_{\mathsf{CS}}$-consistent set of L_{P}-formulas and let $M_{\mathcal{T}}$ be the canonical model for $\mathcal{T}$. We have:*

$$(\forall A \in \mathsf{L}_{\mathsf{P}})[A \in \mathcal{T} \Longleftrightarrow M_{\mathcal{T}} \models A]$$

Proof. We prove the claim by induction on the structure of $A \in \mathsf{L}_{\mathsf{P}}$. We distinguish the following cases:

$A \equiv P_{\geq s}\alpha$**:** ($\Longrightarrow$) Assume that $P_{\geq s}\alpha \in \mathcal{T}$. By definition of the canonical model we have:

$$\mu([\alpha]) = sup_r\{P_{\geq r}\alpha \in \mathcal{T}\}$$

Thus $\mu([\alpha]) \geq s$. We conclude that $M_{\mathcal{T}} \models P_{\geq s}\alpha$.

($\Longleftarrow$) Assume that $M_{\mathcal{T}} \models P_{\geq s}\alpha$. That means:

$$s \leq \mu([\alpha]) = \sup_r\{P_{\geq r}\alpha \in \mathcal{T}\}$$

By Lemma 28(5)(ii)-(iii) we have that $P_{\geq s}\alpha \in \mathcal{T}$.

$A \equiv \neg B$ **or** $A \equiv B \wedge C$**:** These cases are standard and therefore omitted. □

Theorem 34 (Strong Completeness for PJ)**.** *Let CS be any constant specification, let $T \subseteq \mathsf{L}_{\mathsf{P}}$ and let $A \in \mathsf{L}_{\mathsf{P}}$. Then we have:*

$$T \models_{\mathsf{PJ}_{\mathsf{CS},\mathsf{Meas}}} A \Longrightarrow T \vdash_{\mathsf{PJ}_{\mathsf{CS}}} A$$

Proof. We prove the claim by contraposition. Assume that $T \nvdash_{\mathsf{PJ_{CS}}} A$. This means that $T \nvdash_{\mathsf{PJ_{CS}}} (\neg A) \to \bot$. By Theorem 7 we get $T, \neg A \nvdash_{\mathsf{PJ_{CS}}} \bot$, i.e. the set $T, \neg A$ is $\mathsf{PJ_{CS}}$-cosistent. By Lemma 29 there exists a maximal $\mathsf{PJ_{CS}}$-consistent set $\mathcal{T}$ such that $\mathcal{T} \supseteq T \cup \{\neg A\}$. By Lemma 33 we have that $M_{\mathcal{T}} \models T$ and $M_{\mathcal{T}} \models \neg A$. By Lemma 32 we have that $M_{\mathcal{T}} \in \mathsf{PJ_{CS,Meas}}$. Hence $T \not\models_{\mathsf{PJ_{CS,Meas}}} A$. □

6 Conclusion

In this paper we introduced the probabilistic justification logic PJ, a logic in which we can reason about the probability of justification statements. To our knowledge, we are the first to study probabilistic justification logic using the standard model for probability.

Some natural questions about the logic PJ, which we plan to work on, are the decidability and the complexity of PJ. However, the main direction for further research is how probability and justification may interact. We will study a system where interleaving and iteration of probabilistic and justification operators is possible, i.e. a system that includes formulas like $P_{\geq s}P_{\geq r}\beta$ and $t : (P_{\geq s}(u : P_{\geq r}\alpha))$.

Also it will be interesting to investigate a system where statistical evidence can serve as justification. For instance, if we know that the conditional probability of β given α is 0.6, it seems natural to use α (or, better, a term representing α) as a justification for β with probability 0.6.

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