Theories of proof-theoretic strength $\psi(\Gamma_{\Omega+1})$

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Abstract

The purpose of this article is to present a range of theories with proof-theoretic ordinal $\psi(\Gamma_{\Omega+1})$. This ordinal parallels the ordinal of predicative analysis, Γ_0, and our theories are parallel to classical theories of strength Γ_0 such as $\widehat{\mathsf{ID}}_{<\omega}$, FP_0, ATR_0, Σ^1_1-$\mathsf{DC}_0 + (\mathsf{SUB})$, and Σ^1_1-$\mathsf{AC}_0 + (\mathsf{SUB})$. We also relate these theories to the unfolding of ID_1 which was already presented in the PhD thesis of the first author as a system of strength $\psi(\Gamma_{\Omega+1})$.

Keywords: Subsystems of second order arithmetic and inductive definitions, proof-theoretic ordinals, unfolding

1 Introduction

The ordinal $\psi(\Gamma_{\Omega+1})$ appeared first in Bachmann [2], there denoted by $\varphi_{F_{\omega_2+1}(1)+1}(1)$.[1] This was the paper where Bachmann introduced the idea of using assigned fundamental sequences to ordinals of the third number class in order to define large countable ordinals, and this is what Howard [17] uses in his original ordinal analysis of ID_1. ID_1 is the theory of one generalized positive inductive definition, and its proof-theoretic ordinal is now known as the Bachmann-Howard ordinal.

Miller [22] proposed that $\psi(\Gamma_{\Omega+1})$ should be the proof-theoretic ordinal of a theory that relates to ID_1 as predicative analysis relates to first order arithmetic. Feferman's unfolding program [13] provides a way to identity such a system because the unfolding of first order arithmetic is proof-theoretically equivalent to predicative analysis with proof-theoretic ordinal Γ_0 (cf. Feferman and Strahm [15]).

For a history of the Bachmann method of describing constructive ordinals, and how it gave way to the more modern approach, we refer to Crossley and Bridge Kister [11] and the preface of Buchholz, Feferman, Pohlers and Sieg [7].

[1] See section 2 for details on the notation system.

Buchholtz [5] recently proved that the unfolding of ID_1 has proof-theoretic ordinal $\psi(\Gamma_{\Omega+1})$, which indeed relates to $\psi(\varepsilon_{\Omega+1})$ (the ordinal of ID_1) as Γ_0 relates to ε_0 (the ordinal of first order Peano arithmetic).

In this paper we survey a range of further systems which also have proof-theoretic strength $\psi(\Gamma_{\Omega+1})$, for example Σ^1_1-$\mathsf{DC}^\bullet_0 + (\mathsf{SUB}^\bullet)$, $\mathsf{ATR}^\bullet_0 + (\mathsf{SUB}^\bullet)$, $\mathsf{FP}^\bullet_0$ and $\widehat{\mathsf{ID}}^\bullet_{<\omega}$.

Hancock [16] separately conjectured that $\psi(\Gamma_{\Omega+1})$ is the ordinal of a certain kind of Martin-Löf type theory. This is made precise and verified in a companion article, which also identifies a system of explicit mathematics of strength $\psi(\Gamma_{\Omega+1})$.

The remainder of this paper is organized as follows: In the next section we set up some ordinal-theoretic preliminaries, including the definition of $\psi(\Gamma_{\Omega+1})$ and a review of derivation operators in the sense of Buchholz. Section 3 is centered around subsystems of second order arithmetic. Namely, we introduce systems Σ^1_1-$\mathsf{DC}^\bullet_0 + (\mathsf{SUB}^\bullet)$, $\mathsf{ATR}^\bullet_0$, and $\mathsf{FP}^\bullet_0$ resulting from their well-known relatives Σ^1_1-$\mathsf{DC}_0 + (\mathsf{SUB})$, ATR_0, and FP_0 by admitting least fixed points of arithmetical operators in the base language of second order arithmetic. In Section 4 we review the unfolding of ID_1; it is employed in Section 5 to establish the lower bound $\psi(\Gamma_{\Omega+1})$ of the above-mentioned systems via a formalized inductive model construction. Section 6 is devoted to the definition of finitely iterated fixed point theories $\widehat{\mathsf{ID}}^\bullet_n$ for $n < \omega$ and a reduction of $\mathsf{FP}^\bullet_0$ to the union of these theories, $\widehat{\mathsf{ID}}^\bullet_{<\omega}$. In Section 7, we sketch the main lines of the ordinal analysis of $\widehat{\mathsf{ID}}^\bullet_{<\omega}$, determining its proof-theoretic upper bound $\psi(\Gamma_{\Omega+1})$. The paper concludes with a final discussion on related systems of strength $\psi(\Gamma_{\Omega+1})$ in the setting of Martin-Löf type theory as well as Feferman's explicit mathematics.

2 Ordinal notations

In this section we try to give an account of the ordinal-theoretic environment and the ordinal-theoretic tools needed for putting the results of this article into perspective. We assume that the reader is familiar with the basic ordinal theory, the Veblen hierarchy of normal functions and collapsing functions à la Buchholz. A full exposition can be found in Buchholz and Schütte [8], Pohlers [23, 24], and Schütte [25].

Let On be the collection of all ordinals, Ω the least uncountable ordinal, and AP the collection of all *additive principal numbers*, meaning that $\alpha \in AP$ iff $(\forall \eta, \xi < \alpha)(\eta + \xi < \alpha)$. By $\alpha =_{NF} \alpha_1 + \cdots + \alpha_n$ we express that

$$\alpha = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \alpha_1, \ldots, \alpha_n \in AP \quad \text{and} \quad \alpha_n \leq \cdots \leq \alpha_1 < \alpha.$$

Then the following result about the existence of the *Cantor normal form* is standard.

Lemma 1. *For every ordinal α with $0 < \alpha$ and $\alpha \notin AP$ there exist uniquely determined ordinals $\alpha_1, \dots, \alpha_n$ such that $\alpha =_{NF} \alpha_1 + \dots + \alpha_n$.*

The binary *Veblen function* φ is inductively defined by $\varphi 0 \xi := \omega^\xi$ for all ordinals ξ and by choosing $\varphi\alpha$ to be the enumeration function of the closed and unbounded collection $\{\xi \in On : \forall \beta < \alpha(\varphi\beta\xi = \xi)\}$ if $\alpha > 0$. An ordinal α is called *strongly critical* iff $\alpha = \varphi\alpha 0$, and we let SC be the collection of all strongly critical ordinals. Now we set

$$\alpha =_{NF} \varphi\beta\gamma \quad :\Longleftrightarrow \quad \alpha = \varphi\beta\gamma \quad \text{and} \quad \beta, \gamma < \alpha$$

and obtain the following normal form property. For a proof see, for example, Pohlers [23] or Schütte [25].

Lemma 2. *For every ordinal $\alpha \in AP \setminus SC$ there exist uniquely determined ordinals β and γ such that $\alpha =_{NF} \varphi\beta\gamma$.*

It is common to write Γ_α for the α-th strongly critical ordinal, hence Γ is the normal function enumerating SC. Since $\Gamma_\Omega = \Omega$, it follows that $\Gamma_{\Omega+1}$ is the least strongly critical ordinal greater than Ω.

We now introduce for all ordinals α and β sets of ordinals $\mathbf{C}(\alpha, \beta)$ and ordinals $\psi\alpha$ following Buchholz [6].

Definition 3. The sets of ordinals $\mathbf{C}(\alpha, \beta)$ and the ordinals $\psi\alpha$ are defined for all ordinals α and β by induction on α.

1. $\{0, \Omega\} \cup \beta \subseteq \mathbf{C}(\alpha, \beta)$.
2. If $\eta, \xi \in \mathbf{C}(\alpha, \beta)$, then $\eta + \xi \in \mathbf{C}(\alpha, \beta)$ and $\varphi\eta\xi \in \mathbf{C}(\alpha, \beta)$.
3. If $\xi < \alpha$ and $\xi \in \mathbf{C}(\alpha, \beta)$, then $\psi\xi \in \mathbf{C}(\alpha, \beta)$.
4. $\psi\alpha := \min\{\beta \in On : \mathbf{C}(\alpha, \beta) \cap \Omega \subseteq \beta\}$.

The sets $\mathbf{C}(\alpha, \beta)$ and the ordinals $\psi\alpha$ have a series of important properties whose proofs are not difficult and can be found in the references mentioned above.

Lemma 4. *For all ordinals $\alpha, \alpha_1, \alpha_2, \beta, \gamma_1, \ldots, \gamma_n$, we have:*

1. *If β is a limit ordinal, then $\mathbf{C}(\alpha, \beta) = \bigcup\{\mathbf{C}(\alpha, \xi) : \xi < \beta\}$.*
2. $\mathbf{C}(\alpha, \psi\alpha) \cap \Omega = \psi\alpha$.
3. $\psi\alpha \in SC$.
4. *If $\gamma =_{NF} \gamma_1 + \cdots + \gamma_n$ and $\gamma \in \mathbf{C}(\alpha, \beta)$, then $\gamma_1, \ldots, \gamma_n \in \mathbf{C}(\alpha, \beta)$.*
5. *If $\gamma =_{NF} \varphi\gamma_1\gamma_2$ and $\gamma \in \mathbf{C}(\alpha, \beta)$, then $\gamma_1, \gamma_2 \in \mathbf{C}(\alpha, \beta)$.*
6. *If $\alpha_1 < \alpha_2$ and $\alpha_1 \in \mathbf{C}(\alpha_2, \psi\alpha_2)$, then $\psi\alpha_1 < \psi\alpha_2$.*
7. *If $\alpha_1 \leq \alpha_2$, then $\psi\alpha_1 \leq \psi\alpha_2$ and $\mathbf{C}(\alpha_1, \psi\alpha_1) \subseteq \mathbf{C}(\alpha_2, \psi\alpha_2)$.*
8. $\mathbf{C}(\alpha, 0) = \mathbf{C}(\alpha, \psi\alpha)$.

So ψ is a weakly monotone function from On to the strongly critical ordinals less than or equal to $\psi(\Gamma_{\Omega+1})$. It also follows that $\psi(\Gamma_{\Omega+1})$ is the largest segment of ordinals in $\mathbf{C}(\Gamma_{\Omega+1}, \psi\Gamma_{\Omega+1})$, i.e., the least ordinal that cannot be generated by closing $\{0, \Omega\}$ under addition, ω-exponentiation, the binary Veblen function φ, and the function ψ.

The function ψ is weakly monotone but not strictly monotone: for example, if $\alpha = \min\{\xi \in On : \Gamma_\xi = \xi\}$, then $\psi\beta = \Gamma_\beta$ for all $\beta \leq \alpha$ and $\psi\gamma = \alpha$ for all γ such that $\alpha \leq \gamma \leq \Omega$. In order to obtain unique representations of the ordinals in $\mathbf{C}(\Gamma_{\Omega+1}, \psi\Gamma_{\Omega+1})$ we introduce a further normal form. Given ordinals α and β we define

$$\alpha =_{NF} \psi\beta \quad :\Longleftrightarrow \quad \alpha = \psi\beta \quad \text{and} \quad \beta \in \mathbf{C}(\beta, \psi\beta).$$

A detailed proof of the following normal form theorem for the function ψ can be found in Pohlers [23].

Lemma 5. *For every strongly critical ordinal $\alpha \in \mathbf{C}(\Gamma_{\Omega+1}, \psi\Gamma_{\Omega+1})$ there exists a uniquely determined ordinal β such that $\alpha =_{NF} \psi\beta$.*

We end this section with some remarks about ordinal operators that will used to define *operator controlled derivations* in the sense of Buchholz [6].

Definition 6. Let $Pow(On)$ denote the collection of all sets of ordinals.

1. A class function
$$\mathcal{H} : Pow(On) \to Pow(On)$$
is called a *derivation operator* iff it is monotone, inclusive plus idempotent and satisfies the following properties for all $X \in Pow(On)$ and all ordinals $\alpha, \alpha_1, \ldots, \alpha_n$:

(i) $\{0, \Omega\} \subseteq \mathcal{H}(X)$.

(ii) If $\alpha =_{NF} \alpha_1 + \ldots \alpha_n$, then

$$\alpha \in \mathcal{H}(X) \iff \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{H}(X).$$

(iii) If $\alpha =_{NF} \varphi\alpha_1\alpha_2$, then

$$\alpha \in \mathcal{H}(X) \iff \{\alpha_1, \alpha_2\} \subseteq \mathcal{H}(X).$$

2. If $\mathcal{H}$ is a derivation operator, we define for all finite sets of ordinals $\mathfrak{m}$ and all ordinals σ operators

$$\mathcal{H}[\mathfrak{m}],\ \mathcal{H}[\sigma],\ \mathcal{H}_\sigma : Pow(On) \to Pow(On)$$

by setting for all $X \in Pow(On)$:

$$\begin{aligned}
\mathcal{H}[\mathfrak{m}](X) &\;:=\; \mathcal{H}(X \cup \mathfrak{m}), \\
\mathcal{H}[\sigma](X) &\;:=\; \mathcal{H}(X \cup \{\sigma\}), \\
\mathcal{H}_\sigma(X) &\;:=\; \bigcap\{\mathbf{C}(\alpha, \beta) : X \subseteq \mathbf{C}(\alpha, \beta) \text{ and } \sigma < \alpha\}.
\end{aligned}$$

Buchholz [6] provides a detailed analysis of derivation operators from which, in particular, we get all the properties summarized in the next lemma.

Lemma 7. *If $\mathcal{H}$ is a derivation operator, then we have for all finite sets of ordinals $\mathfrak{m}$, all ordinals σ, and all $X \in Pow(On)$:*

1. *$\mathcal{H}[\mathfrak{m}]$, $\mathcal{H}[\sigma]$, and $\mathcal{H}_\sigma$ are derivation operators.*
2. *$\mathfrak{m} \subseteq \mathcal{H}(\emptyset) \quad \Longrightarrow \quad \mathcal{H}[\mathfrak{m}] = \mathcal{H}$.*
3. *$\sigma \in \mathcal{H}(\emptyset) \quad \Longrightarrow \quad \mathcal{H}[\sigma] = \mathcal{H}$.*

3 Subsystems of second order arithmetic

Our language $\mathcal{L}_2$ of second order arithmetic contains number variables $a, b, c, u, v, w, x, y, z, \ldots$ and set variables $U, V, W, X, Y, Z, \ldots$ (both possibly with subscripts), function symbols for all primitive recursive functions, relation symbols for all primitive recursive relations, The relation symbol $\in$ for the element relation between natural numbers and sets of natural numbers as well as the standard logical connectives and auxiliary symbols. In addition, we have a distinguished anonymous unary relation symbol R that we use to define proof-theoretic ordinals and also plays a special role in the unfolding systems (see below). The *number terms* and *formulas* of $\mathcal{L}_2$ are defined as usual, and the *arithmetic* formulas of $\mathcal{L}_2$ are those without bounded set quantifiers; number and set parameters are permitted in arithmetic formulas.

Moreover, we frequently make use of the vector notation $\vec{\mathfrak{e}}$ as shorthand for a finite string $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ of expressions whose length is not important or is evident from the context. Suppose now that $\vec{a}$ is the string of variables $a_1, \ldots, a_n$ and $\vec{r}$ the string of number terms $r_1, \ldots, r_n$. Then $A[\vec{r}/\vec{a}]$ is the formula that is obtained from A by simultaneously replacing all free occurrences of the variables $\vec{a}$ by the number terms $\vec{r}$; in order to avoid collision of variables, a renaming of bounded variables may be necessary. If the formula A is written as $B[\vec{a}]$, then we often simply write $B[\vec{r}]$ instead of $A[\vec{r}/\vec{a}]$; further variants of this notation below will be obvious.

The binary (infix) relation symbol $=$ stands for the primitive recursive equality relation, the binary (infix) relation symbol $<$ for the primitive recursive less-than relation, and t' for the successor of t. Very often we also write the same expression for a primitive recursive function (relation) as for the associated function (relation) symbol. Equality is only taken as basic symbol between numbers; equality between sets of numbers and functions is defined as

$$(U = V) \quad := \quad \forall a(a \in U \leftrightarrow a \in V).$$

In the following we make use of the standard primitive recursive coding machinery in $\mathcal{L}_2$: $\langle r_1, \ldots, r_n \rangle$ stands for the primitive recursively formed n-tuple of the number terms $r_1, \ldots, r_n$; *Seq* is the primitive recursive set of sequence numbers; $lh(r)$ denotes the length of (the sequence number coded by) r; if $i < lh(r)$, then $(r)_i$ is the i-th component of (the sequence coded by) r, i.e., $r = \langle (r)_0, \ldots, (r)_{lh(r) \dot{-} 1} \rangle$ provided that r is a sequence number.

The first order language $\mathcal{L}_1$ is the sub-language of $\mathcal{L}_2$ in which only formulas of $\mathcal{L}_2$ without set variables are permitted. Now we pick a fresh unary relation symbol P and write $\mathcal{L}_1(P)$ for the extension of $\mathcal{L}_1$ by P, i.e., expressions of the form $P(t)$ are permitted as atomic formulas of $\mathcal{L}_1(P)$. An $\mathcal{L}_1(P)$ formula is called *P-positive* if each occurrence of P in this formula is positive. We call P-positive formulas that contain at most u free *inductive operator forms* and let $\mathfrak{A}[P, u]$ range over such forms. If $\mathfrak{A}[P, u]$ does not contain the anonymous relation symbol R, it is called a *pure* inductive operator form.

Now we extend the language $\mathcal{L}_2$ to a new second order language $\mathcal{L}_2^\bullet$ by adding a fresh unary relation symbol $P_{\mathfrak{A}}$ for every pure inductive operator form $\mathfrak{A}[P, u]$; the number terms of $\mathcal{L}_2^\bullet$ are, of course, the number terms of $\mathcal{L}_2$. An $\mathcal{L}_2^\bullet$ formula is called *elementary* in case it does not contain bounded set variables. As syntactic variables we use $r, s, t, r_0, s_0, t_0, \ldots$ for number terms and $A, B, C, A_0, B_0, C_0, \dots$ for formulas of $\mathcal{L}_2^\bullet$.

In the following we introduce a series of theories of second order arithmetic. The weakest of those, the theory ACA_0 is formulated in $\mathcal{L}_2$ and has the usual axioms and rules of inference of two-sorted logic with equality for the first sort, the axioms of primitive recursive arithmetic PRA for the primitive recursive functions and relations plus the axiom schema of *arithmetic comprehension*, i.e.,

$$\exists X \forall a (a \in X \leftrightarrow A[a])$$

for all arithmetic formulas $A[u]$ of $\mathcal{L}_2$, and the *induction axiom*

$$\forall X (0 \in X \ \land \ \forall a (a \in X \ \rightarrow \ a' \in X) \ \rightarrow \ \forall a (a \in Y)).$$

Well-known extensions of ACA_0 are obtained by adding axioms about comprehension and choice, for example

$$(\Delta^1_1\text{-}\mathsf{CA}) \quad \forall a (\exists X A[X,a] \leftrightarrow \forall X B[X,a]) \ \rightarrow \ \exists Y \forall a (a \in Y \ \leftrightarrow \ \exists X A[X,a]),$$

$$(\Sigma^1_1\text{-}\mathsf{AC}) \qquad \forall a \exists X C[a,X] \ \rightarrow \ \exists Y \forall a C[a,(Y)_a],$$

$$(\Sigma^1_1\text{-}\mathsf{DC}) \qquad \forall a \forall X \exists Y D[a,X,Y] \ \rightarrow \ \exists Z \forall a D[a,(Z)^a,(Z)_a],$$

where $A[U,v]$, $B[U,v]$, $C[u,V]$, and $D[u,V,W]$ are arithmetic formulas of $\mathcal{L}_2$. In these formulations we are using the abbreviations

$$\begin{aligned} r \in (U)_s \quad &:= \quad \langle r, s \rangle \in U, \\ r \in (U)^s \quad &:= \quad r \in U \ \land \ r = \langle (r)_0, (r)_1 \rangle \ \land \ (r)_1 < s. \end{aligned}$$

We write Δ^1_1-CA_0, Σ^1_1-AC_0, and Σ^1_1-DC_0 for the theories $\mathsf{ACA}_0 + (\Delta^1_1\text{-}\mathsf{CA})$, $\mathsf{ACA}_0 + (\Sigma^1_1\text{-}\mathsf{AC})$, and $\mathsf{ACA}_0 + (\Sigma^1_1\text{-}\mathsf{DC})$, respectively, and recall that Δ^1_1-CA_0 and Σ^1_1-AC_0 are conservative extensions of Peano arithmetic PA, whereas Σ^1_1-DC_0 has proof-theoretic ordinal $\varphi\omega 0$. For details see Barwise and Schlipf [3], Buchholz, Feferman, Pohlers, and Sieg [7], and Cantini [10].

Before turning to the next principle, we introduce some notation: If A and $B[v]$ are $\mathcal{L}_2$ formulas, then $A_U[\{a : B[a]\}]$ indicates the result of substituting $B[r]$ for each occurrence of $(r \in U)$ in A. The *substitution rule* is the rule of inference

$$(\mathsf{SUB}) \qquad \frac{\forall X A}{A_X[\{a : B[a]\}]}$$

for arithmetic $\mathcal{L}_2$ formulas $A[U]$ and arbitrary $\mathcal{L}_2$ formulas $B[v]$. Obviously, the *bar rule*

$$\frac{\forall X\, TI[\lhd, X]}{TI[\lhd, \{a : B[a]\}]}$$

for binary primitive recursive relations $\triangleleft$ is a special case of (SUB). Here $TI[\triangleleft, U]$ stands for the formula

$$\forall x(\forall y(y \triangleleft x \ \rightarrow \ y \in U) \rightarrow x \in U) \ \rightarrow \ \forall x(x \in U).$$

From Feferman and Jäger [14] we know that Δ^1_1-$\mathsf{CA}_0 + (\mathsf{SUB})$, Σ^1_1-$\mathsf{AC}_0 + (\mathsf{SUB})$, and Σ^1_1-$\mathsf{DC}_0 + (\mathsf{SUB})$ are proof-theoretically equivalent and of proof-theoretic strength Γ_0.

In the later considerations two further theories in $\mathcal{L}_2$ will play an important role: The first is the theory ATR_0 – the fourth system of Friedman's program of reverse mathematics – that extends ACA_0 by the schema of *arithmetic transfinite recursion*; a standard reference is Simpson [26]. The second is the fixed point theory FP_0, resulting from ACA_0 by adding the fixed point axioms

$$(\mathsf{FP}) \qquad \exists X \forall a(a \in X \ \leftrightarrow \ A[X, a])$$

for all U-positive arithmetic formulas $A[U, v]$. As shown in Avigad [1], ATR_0 and FP_0 are equivalent.

Theorem 8. *An $\mathcal{L}_2$ formula is provable in ATR_0 if and only if it is provable in FP_0.*

After these preliminary remarks we now turn to the theories that interest us most in this article. They are all formulated in the language $\mathcal{L}_2^\bullet$ and comprise the following *least fixed point axioms*

$$(\mathsf{ID.1}) \qquad \forall a(\mathfrak{A}[P_\mathfrak{A}, a] \rightarrow P_\mathfrak{A}(a)),$$

$$(\mathsf{ID.2}) \qquad \forall X(\forall a(\mathfrak{A}[X, a] \rightarrow a \in X) \rightarrow \forall a(P_\mathfrak{A}(a) \rightarrow a \in X)).$$

for all inductive operator forms $\mathfrak{A}[P, u]$. Please observe that $(\mathsf{ID.2})$ only claims minimality with respect to sets, not with respect to $\mathcal{L}_2^\bullet$ definable classes.

The theory $\mathsf{ACA}_0^\bullet$ is the $\mathcal{L}_2^\bullet$ system that contains the axioms of ACA_0 (formulated for $\mathcal{L}_2^\bullet$), all least fixed point axioms $(\mathsf{ID.1})$ and $(\mathsf{ID.2})$ plus the comprehension schema

$$(\mathsf{E\text{-}CA}) \qquad \exists X \forall a(a \in X \leftrightarrow A[a]),$$

for elementary $\mathcal{L}_2^\bullet$ formulas $A[u]$. As a consequence, any $P_\mathfrak{A}$ defines a set in $\mathsf{ACA}_0^\bullet$.

It is an easy exercise to show that $\mathsf{ACA}_0^\bullet$ is a conservative extension of the famous theory ID_1 of non-iterated positive inductive definitions; for details about ID_1 cf., for example, Buchholz, Feferman, Pohlers, and Sieg [7] or Pohlers [24].

The schemas (Δ^1_1-$\mathsf{CA}^\bullet$), (Σ^1_1-$\mathsf{AC}^\bullet$), and (Σ^1_1-$\mathsf{DC}^\bullet$) are the analogues of (Δ^1_1-CA), (Σ^1_1-AC), and (Σ^1_1-DC) with the arithmetic $\mathcal{L}_2$ formulas replaced by elementary $\mathcal{L}_2^\bullet$ formulas. Accordingly, the theories Δ^1_1-$\mathsf{CA}_0^\bullet$, Σ^1_1-$\mathsf{AC}_0^\bullet$, and Σ^1_1-$\mathsf{DC}_0^\bullet$ are defined to be the theories $\mathsf{ACA}_0^\bullet + (\Delta^1_1\text{-}\mathsf{CA}^\bullet)$, $\mathsf{ACA}_0^\bullet + (\Sigma^1_1\text{-}\mathsf{AC}^\bullet)$, and $\mathsf{ACA}_0^\bullet + (\Sigma^1_1\text{-}\mathsf{DC}^\bullet)$.

Of course, there is also an analogue of the substitution rule for the language $\mathcal{L}_2^\bullet$. Simply consider

$$(\mathsf{SUB}^\bullet) \qquad \frac{\forall X A}{A_X[\{a : B[a]\}]}$$

for all elementary $\mathcal{L}_2^\bullet$ formulas $A[U]$ and arbitrary $\mathcal{L}_2^\bullet$ formulas $B[v]$. Thus the bar rule for $\mathcal{L}_2^\bullet$ reads as

$$\frac{\forall X\, TI[\lhd, X]}{TI[\lhd, \{a : B[a]\}]}$$

for binary relations $\lhd$ that are primitive recursive in the least fixed points $P_{\mathfrak{A}}$ and arbitrary $\mathcal{L}_2^\bullet$ formulas $B[v]$, and is a special case of ($\mathsf{SUB}^\bullet$).[2]

In the following we shall prove that Δ^1_1-$\mathsf{CA}_0^\bullet + (\mathsf{SUB}^\bullet)$, Σ^1_1-$\mathsf{AC}_0^\bullet + (\mathsf{SUB}^\bullet)$, and Σ^1_1-$\mathsf{DC}_0^\bullet + (\mathsf{SUB}^\bullet)$ are theories with proof-theoretic ordinal $\psi(\Gamma_{\Omega+1})$. Two other interesting systems of the same strength are $\mathsf{ATR}_0^\bullet$ and $\mathsf{FP}_0^\bullet$, obtained from ATR_0 and FP_0, respectively, by relativizing them to the language $\mathcal{L}_2^\bullet$.

More precisely, let the schema of *elementary transfinite recursion* be as the schema of arithmetic transfinite recursion but with elementary $\mathcal{L}_2^\bullet$ formulas instead of arithmetic $\mathcal{L}_2$ formulas. Then $\mathsf{ATR}_0^\bullet$ is the extension of $\mathsf{ACA}_0^\bullet$ by elementary transfinite recursion. Similarly, the fixed point axioms of $\mathcal{L}_2$ are lifted to

$$(\mathsf{FP}^\bullet) \qquad \exists X \forall a (a \in X \leftrightarrow A[X, a])$$

for arbitrary U-positive elementary formulas $\mathfrak{A}[U, v]$ of $\mathcal{L}_2^\bullet$, and $\mathsf{FP}_0^\bullet$ is the $\mathcal{L}_2^\bullet$ theory $\mathsf{ACA}_0^\bullet + (\mathsf{FP}^\bullet)$.

There exists a close relationship between our theories formulated in $\mathcal{L}_2$ and their counterparts in $\mathcal{L}_2^\bullet$. Consider an inductive operator form $\mathfrak{A}[U, v]$ together with the axiom schema

$$(\mathsf{LFP}) \quad \exists X \Big(\forall a(\mathfrak{A}[X, a] \to a \in X) \land \forall Y \big(\forall a(\mathfrak{A}[Y, a] \to a \in Y) \to X \subseteq Y\big)\Big)$$

for all inductive operator forms $\mathfrak{A}[U, v]$. Added to ACA_0 it implies that every inductive operator form has a least fixed point, where "least" means least with respect to all sets that are fixed points.

[2] It is because of ($\mathsf{SUB}^\bullet$) that we restrict ourselves to *pure* operator forms in $\mathcal{L}_2^\bullet$.

Theorem 9. *Let T be one of the theories* ACA_0*,* Δ^1_1-CA_0*,* Σ^1_1-AC_0*,* Σ^1_1-DC_0*,* ATR_0*, or* FP_0*, which are all formulated in $\mathcal{L}_2$. Then we have:*

1. *$T^\bullet$, which is formulated in $\mathcal{L}_2^\bullet$, is a conservative extension of $T+(\mathsf{LFP})$ with respect to all $\mathcal{L}_2$ sentences.*

2. $\mathsf{ATR}_0^\bullet$ *and* $\mathsf{FP}_0^\bullet$ *prove the same $\mathcal{L}_2^\bullet$ formulas.*

Proof. Since every relation constant $P_{\mathfrak{A}}$ for an inductive operator form $\mathfrak{A}[U,v]$ defines a set, all instances of (LFP) are provable in the theories $T^\bullet$. Therefore, $T+(\mathsf{LFP}) \subseteq T^\bullet$. To establish conservativity, we simply fix for each inductive operator form $\mathfrak{A}[U,v]$ the uniquely determined least fixed point U, which exists according to (LFP), and interpret $P_{\mathfrak{A}}(t)$ as $(t \in U)$. These considerations together with Theorem 8 also yield the second assertion. □

We write $\mathfrak{M}[U]$ to express that U is (the range of the sets of) a *countable coded ω-model of* Σ^1_1-DC_0 in the sense of Simpson [26]. Elementhood in such an U is then abbreviated by

$$V \mathrel{\dot{\in}} U \quad := \quad \exists a(V = (U)_a),$$

and $\vec{V} \mathrel{\dot{\in}} U$ means that all components of $\vec{V}$ belong to U. Moreover, given an arbitrary $\mathcal{L}_2$ formula A, its relativization A^U to U is obtained from A by replacing all quantifiers $\exists X(\ldots X \ldots)$ and $\forall X(\ldots X \ldots)$ by $\exists x(\ldots (U)_x \ldots)$ and $\forall x(\ldots (U)_x \ldots)$, respectively. Note that A^U is always arithmetic.

The theory ATR_0 has the following important property; cf. Simpson [26] for a detailed proof and a discussion of the general context.

Theorem 10. *The theory* ATR_0 *proves that*

$$\forall X \exists Y (X \mathrel{\dot{\in}} Y \land \mathfrak{M}[Y]).$$

For the subsequent considerations we let $\mathfrak{A}_0[U,v], \mathfrak{A}_1[U,v], \mathfrak{A}_2[U,v], \ldots$ be an arbitrary (but fixed) enumeration of all inductive operator forms and write

$$\mathfrak{F}_i[X] := \forall a(\mathfrak{A}_i[U,a] \to a \in U) \land \forall Y\big(\forall a(\mathfrak{A}_i[Y,a] \to a \in Y) \to X \subseteq Y\big),$$

expressing that set X is the least fixed point of $\mathfrak{A}_i[U,v]$. For any natural number n we write $For(n)$ for the collection of all $\mathcal{L}_2^\bullet$ formulas that do not contain relation symbols $P_{\mathfrak{A}_i}$ with $n < i$.

Lemma 11. *Let n be an arbitrary natural number. Under the assumptions*

(*i*) *$A[\vec{U}]$ is an $\mathcal{L}_2$ formula whose free set variables are from the list $\vec{U}$,* *(ii)* Σ^1_1-$\mathsf{DC}_0^\bullet$ + $(\mathsf{SUB}^\bullet)$ *proves* $A[\vec{U}]$ *by a proof* $\mathcal{P}$ *such that all formulas occurring in* $\mathcal{P}$ *belong to* $For(n)$*,*

(iii) *the set variables* $W_0, \dots, W_n$ *do not occur in* $\mathcal{P}$*, and* $B[\vec{U}]$ *is the formula obtained from* $A[\vec{U}]$ *by substituting* $(t \in W_i)$ *for each subformula* $P_{\mathfrak{A}_i}(t)$*,* $i = 0, \dots, n$*,*

the theory ATR_0 *proves*

$$\forall Z\Big(\big(\bigwedge_{i=0}^{n}(\mathfrak{F}_i[W_i] \wedge W_i \mathrel{\dot\in} Z) \wedge \vec{U} \mathrel{\dot\in} Z \ \wedge\ \mathfrak{M}[Z]\big) \to B^Z[\vec{U}]\Big).$$

Proof. We proceed by induction on $\mathcal{P}$. If $A[\vec{U}]$ is an axiom of Σ^1_1-$\mathsf{DC}_0^\bullet$, then our assertion is obvious since we relativize with respect to countable coded ω-models of Σ^1_1-DC_0; if $A[\vec{U}]$ is the conclusion of a rule of inference different from $(\mathsf{SUB}^\bullet)$, then our assertion follows directly from the induction hypothesis. So it only remains to discuss the case that $A[\vec{U}]$ is the conclusion of $(\mathsf{SUB}^\bullet)$. Then this inference has the form

$$\frac{\forall X C[\vec{U}, X]}{C[\vec{U}, \{x : D[\vec{U}, x]\}]},$$

where $A[\vec{U}]$ is the formula $C[\vec{U}, \{x : D[\vec{U}, x]\}]$, $C[\vec{U}, V]$ is an elementary $\mathcal{L}_2^\bullet$ formula, and $D[\vec{U}, v]$ an arbitrary $\mathcal{L}_2^\bullet$ formula. Let $E[\vec{U}, V]$and $F[\vec{U}, v]$ be the $\mathcal{L}_2$ formulas obtained from $C[\vec{U}, V]$ and $D[\vec{U}, v]$, respectively, by substituting $(t \in W_i)$ for each subformula $P_{\mathfrak{A}_i}(t)$, $i = 0, \dots, n$. In view of the induction hypothesis ATR_0 proves

$$(*) \quad \forall Z\Big(\big(\bigwedge_{i=0}^{n}(\mathfrak{F}_i[W_i] \wedge W_i \mathrel{\dot\in} Z) \wedge \vec{U} \mathrel{\dot\in} Z \wedge \mathfrak{M}[Z]\big) \to (\forall X \mathrel{\dot\in} Z) E[\vec{U}, X]\Big),$$

and we have to show in ATR_0 that

$$\forall Z\Big(\big(\bigwedge_{i=0}^{n}(\mathfrak{F}_i[W_i] \wedge W_i \mathrel{\dot\in} Z) \wedge \vec{U} \mathrel{\dot\in} Z \wedge \mathfrak{M}[Z]\big) \to E[\vec{U}, \{x : D^Z[\vec{U}, x]\}]\Big).$$

Working within ATR_0, pick a Z such that

$$\bigwedge_{i=0}^{n}(\mathfrak{F}_i[W_i] \ \wedge\ W_i \mathrel{\dot\in} Z) \ \wedge\ \vec{U} \mathrel{\dot\in} Z \ \wedge\ \mathfrak{M}[Z].$$

By Theorem 10 there exists an Y_0 for which

$$\bigwedge_{i=0}^{n}(\mathfrak{F}_i[W_i] \ \wedge \ W_i \mathrel{\dot{\in}} Y_0) \ \wedge \ \vec{U} \mathrel{\dot{\in}} Y_0 \ \wedge \ Z \mathrel{\dot{\in}} Y_0 \ \wedge \ \mathfrak{M}[Y_0].$$

Hence $\{x : D^Z[\vec{U}, x]\}$ is a set in Y_0, i.e., there exists an X_0 with

$$X_0 \mathrel{\dot{\in}} Y_0 \ \wedge \ \forall a(a \in X_0 \ \leftrightarrow \ D^Z[\vec{U}, a]).$$

Now we go back to (*), and obtain (by inserting Y_0 for Z and X_0 for X) that $E[\vec{U}, X_0]$, hence $E[\vec{U}, \{x : D^Z[\vec{U}, x]\}]$. This is what we had to show. $\square$

Theorem 12. *Let $A[U]$ be an arithmetic formula of $\mathcal{L}_2$ with no set variables besides U. Then we have that*

$$\Sigma^1_1\text{-}\mathsf{DC}^\bullet_0 + (\mathsf{SUB}^\bullet) \ \vdash \ \forall X A[X] \quad \Longrightarrow \quad \mathsf{ATR}_0 + (\mathsf{LFP}) \ \vdash \ \forall X A[X].$$

Proof. From our assumption we obtain that there exists a natural number n and a proof of $A[\vec{U}]$ in $\Sigma^1_1\text{-}\mathsf{DC}^\bullet_0 + (\mathsf{SUB}^\bullet)$ such that all formulas in this proof belong to $For(n)$. We choose fresh set variables $W_0, \ldots, W_n$ not occurring in this proof and see by the previous lemma that ATR_0 proves

$$\forall Z\Big(\Big(\bigwedge_{i=0}^{n}(\mathfrak{F}_i[W_i] \wedge W_i \mathrel{\dot{\in}} Z) \wedge U \mathrel{\dot{\in}} Z \wedge \mathfrak{M}[Z]\Big) \to A[U]\Big),$$

hence also

$$\exists X_0 \ldots \exists X_n \exists Z\Big(\bigwedge_{i=0}^{n}(\mathfrak{F}_i[X_i] \wedge X_i \mathrel{\dot{\in}} Z) \wedge U \mathrel{\dot{\in}} Z \wedge \mathfrak{M}[Z]\Big) \to A[U].$$

In view of the schema (LFP) and Theorem 10, this means that $A[U]$ is provable in $\mathsf{ATR}_0 + (\mathsf{LFP})$. Consequently, $\mathsf{ATR}_0 + (\mathsf{LFP})$ proves $\forall X A[X]$. $\square$

Corollary 13. *Let $A[U]$ be an arithmetic formula of $\mathcal{L}_2$ with no set variables besides U. Then we have that*

$$\Sigma^1_1\text{-}\mathsf{DC}^\bullet_0 + (\mathsf{SUB}^\bullet) \ \vdash \ \forall X A[X] \quad \Longrightarrow \quad \mathsf{ATR}^\bullet_0 \ \vdash \ \forall X A[X].$$

Let us conclude this section with some remarks on the anonymous relation symbol R. Clearly, in the context of $\mathcal{L}_2$ and $\mathcal{L}^\bullet_2$ it plays the same role as any unspecified free set variable and would have been superfluous. More specifically: if T is one of the $\mathcal{L}_2$ or $\mathcal{L}^\bullet_2$ theories considered so far, then T proves A if and only if it proves $\forall X A_{\mathsf{R}}[X]$, where $A_{\mathsf{R}}[U]$ is obtained from A by replacing all occurrences of $\mathsf{R}(t)$ by $(t \in U)$.

In the next sections we shall turn to several first order theories, and then it is convenient to work within a syntax that provides for a place holder for arbitrary non-specified properties. A typical example is the first order definition of the proof-theoretic ordinal of a theory. For any primitive recursive relation $\lhd$ we set

$$TI[\lhd, \mathsf{R}] \quad := \quad \forall x(\forall y(y \lhd x \to \mathsf{R}(y)) \to \mathsf{R}(x)) \to \forall x \mathsf{R}(x).$$

Then if T is a theory formulated in a language containing the first order part of $\mathcal{L}_2$, the ordinal α is called *provable in T* if and only if there exists a primitive recursive well ordering $\lhd$ of order type α such that T proves $TI[\lhd, \mathsf{R}]$. The *proof-theoretic ordinal* $|T|$ of T is defined to be the least ordinal not provable in T.

Furthermore, it is more or less obvious that in the presence of $(\mathsf{SUB}^\bullet)$ the following variant of the substitution rule is available. It will be needed for interpreting the substitution rule of the unfolding of ID_1, described in the next section.

Lemma 14. *Let A be an elementary $\mathcal{L}_2^\bullet$ formula and $B[u]$ an arbitrary $\mathcal{L}_2^\bullet$ formula. If Σ^1_1-$\mathsf{DC}_0^\bullet + (\mathsf{SUB}^\bullet)$ proves A, then Σ^1_1-$\mathsf{DC}_0^\bullet + (\mathsf{SUB}^\bullet)$ also proves $A_\mathsf{R}[\{a : B[a]\}]$, where $A_\mathsf{R}[\{a : B[a]\}]$ here indicates the result of substituting $B[t]$ for each occurrence of $\mathsf{R}(t)$ in A.*

4 Unfolding of ID_1

In this section we define the unfolding of ID_1 as in Buchholtz [5]. This is an instance of Feferman's unfolding program [13], for a general exposition, see also Buchholtz [5]. This is defined in two steps; first the operational unfolding $\mathcal{U}_0(\mathsf{ID}_1)$ is introduced, and then the full unfolding $\mathcal{U}(\mathsf{ID}_1)$ is defined as an extension.

Let $\mathcal{L}_1^\bullet$ denote the fragment of $\mathcal{L}_2^\bullet$ without free or bound set variables (and thus also without the $\in$ relation). We use here a version of the unfolding in which the operational structure is given by a partial combinatory algebra. The language of $\mathcal{U}_0(\mathsf{ID}_1)$ is the language $\mathcal{L}_1^\bullet$ extended with new constants k and s (combinators), p, p_0 and p_1 (pairing and projection), d (definition by cases), tt (true) and ff (false), e (equality), and the binary function symbol $\cdot$ (application). Terms are built in the usual way using variables and constants and closing under application and the function symbols of $\mathcal{L}_2^\bullet$. Further, we add a unary relation symbol N (natural numbers), and to account for partiality of application we also add a unary relation symbol $\downarrow$ (defined; expressing that a term has a value). When writing terms we drop the symbol for application and use the convention that application is left-associative to leave out parentheses. We often write $f(a_1, \ldots, a_n)$ for $f\, a_1 \ldots a_n$ and $\langle\!\langle a, b \rangle\!\rangle$ for $\mathsf{p}\, a\, b$.

For $\mathcal{U}_0(\mathsf{ID}_1)$ we use Beeson's Logic of Partial Terms with strictness and equality, see Beeson [4]. The non-logical axioms of $\mathcal{U}_0(\mathsf{ID}_1)$ are:

1. The usual axioms of arithmetic, relativized to N, with the schematic form of complete induction on the natural numbers,

$$\mathsf{R}(0) \wedge \forall x(\mathsf{N}(x) \wedge \mathsf{R}(x) \rightarrow \mathsf{R}(x')) \rightarrow \forall x(\mathsf{N}(x) \rightarrow \mathsf{R}(x)).$$

2. The least fixed point axioms for each $P_{\mathfrak{A}}$ in schematic form relativized to N,[3]

$$\forall a(\mathsf{N}(a) \wedge \mathfrak{A}^{\mathsf{N}}[P_{\mathfrak{A}}, a] \rightarrow P_{\mathfrak{A}}(a)),$$
$$(\forall a(\mathsf{N}(a) \wedge \mathfrak{A}^{\mathsf{N}}[\mathsf{R}, a] \rightarrow \mathsf{R}(a)) \rightarrow \forall a(\mathsf{N}(a) \wedge P_{\mathfrak{A}}(a) \rightarrow \mathsf{R}(a))).$$

3. Partial combinatory algebra (PCA) axioms with pairing and definition by cases:

   (a) $\mathsf{k}\, a\, b = a$.

   (b) $\mathsf{s}\, a\, b\!\downarrow \wedge\, \mathsf{s}\, a\, b\, c \simeq a\, c\, (b\, c)$.

   (c) $\mathsf{p}_0 \langle\!\langle a, b \rangle\!\rangle = a \wedge \mathsf{p}_1 \langle\!\langle a, b \rangle\!\rangle = b$.

   (d) $\mathsf{d}\, a\, b\, \mathsf{tt} = a \wedge \mathsf{d}\, a\, b\, \mathsf{ff} = b$.

4. Decidable equality on natural numbers:

   (a) $\forall x, y(\mathsf{N}(x) \wedge \mathsf{N}(y) \rightarrow \mathsf{e}\, x\, y = \mathsf{tt} \vee \mathsf{e}\, x\, y = \mathsf{ff})$.

   (b) $\forall x, y(\mathsf{N}(x) \wedge \mathsf{N}(y) \rightarrow (\mathsf{e}\, x\, y = \mathsf{tt} \leftrightarrow x = y))$.

In addition, $\mathcal{U}_0(\mathsf{ID}_1)$ includes the unrestricted substitution rule,

$$(\mathsf{SUB}) \qquad \frac{A}{A_{\mathsf{R}}[\{a : B[a]\}]},$$

where, because of partiality, $A_{\mathsf{R}}[\{a : B[a]\}]$ indicates the result of substituting $(r\!\downarrow \wedge\, B[r])$ for each occurrence of $\mathsf{R}(r)$ in A.

Abstraction terms $\lambda x.t$ can be defined as usual, and from the PCA axioms we can show in $\mathcal{U}_0(\mathsf{ID}_1)$:

1. $(\lambda x.t)\!\downarrow \wedge\, (\lambda x.t)\, x \simeq t$

[3] As usual, $\mathfrak{A}^{\mathsf{N}}$ denotes the formula $\mathfrak{A}$ with all quantifiers relativized to N.

2. $s\downarrow \to (\lambda x.t)\, s \simeq t[s/x]$

Here, $t \simeq s$ is an abbreviation for $t\downarrow \vee s\downarrow \to t = s$. Note that we use the notation of the λ-calculus even though the conversion relation is not exactly the same (in particular, it does not validate the (ξ)-rule of the λ-calculus).

The PCA axioms allow us to introduce a fixed point operator, but we cannot prove that it produces *least* fixed points.

Theorem 15 (Fixed point). *There is a closed term* fix *of* $\mathcal{U}_0(\mathsf{ID}_1)$ *such that*

$$\mathcal{U}_0(\mathsf{ID}_1) \vdash \mathsf{fix}\, f\downarrow \wedge\, \mathsf{fix}\, f\, x \simeq f\, (\mathsf{fix}\, f)\, x.$$

The language of the full unfolding $\mathcal{U}(\mathsf{ID}_1)$ extends the language by additional constants to reflect the predicates of $\mathcal{U}_0(\mathsf{ID}_1)$: nat (natural number), $\mathsf{i}_{\mathfrak{A}}$ (inductive set), eq (equality), pr_{R} (anonymous relation symbol R), inv (inverse image), conj (conjunction), neg (negation), un (universal quantification over the natural numbers), join (join, that is, disjoint union). In addition, we add the unary relation symbol Π (predicates) and the binary relation symbol $\in$ (predication). The axioms of $\mathcal{U}(\mathsf{ID}_1)$ extend the ones of $\mathcal{U}_0(\mathsf{ID}_1)$ by

4. Basic axioms about predicates:
   (a) $\Pi(\mathsf{nat}) \wedge \forall x(x \in \mathsf{nat} \leftrightarrow \mathsf{N}(x))$.
   (b) $\Pi(\mathsf{i}_{\mathfrak{A}}) \wedge \forall x(x \in \mathsf{i}_{\mathfrak{A}} \leftrightarrow P_{\mathfrak{A}}(x))$.
   (c) $\Pi(\mathsf{eq}) \wedge \forall x(x \in \mathsf{eq} \leftrightarrow \exists y(x = \langle\!\langle y, y\rangle\!\rangle))$.
   (d) $\Pi(\mathsf{pr}_{\mathsf{R}}) \wedge \forall x(x \in \mathsf{pr}_{\mathsf{R}} \leftrightarrow \mathsf{R}(x))$.
   (e) $\Pi(a) \to \Pi(\mathsf{inv}(a, f)) \wedge \forall x(x \in \mathsf{inv}(a, f) \leftrightarrow f\, x \in a)$.
   (f) $\Pi(a) \wedge \Pi(b) \to \Pi(\mathsf{conj}(a, b)) \wedge \forall x(x \in \mathsf{conj}(a, b) \leftrightarrow x \in a \wedge x \in b)$.
   (g) $\Pi(a) \to \Pi(\mathsf{neg}\, a) \wedge \forall x(x \in \mathsf{neg}(a) \leftrightarrow \neg(x \in a))$.
   (h) $\Pi(a) \to \Pi(\mathsf{un}\, a) \wedge \forall x(x \in \mathsf{un}(a) \leftrightarrow \forall y(\mathsf{N}(y) \to \langle\!\langle x, y\rangle\!\rangle \in a))$.

5. The dependent join axiom:

$$\begin{aligned}&\Pi(a) \wedge (\forall y \in a)\Pi(f\, y) \to \Pi(\mathsf{join}(f, a))\\ &\qquad \wedge \forall x(x \in \mathsf{join}(f, a) \leftrightarrow \exists y, z(x = \langle\!\langle y, z\rangle\!\rangle \wedge y \in a \wedge z \in f(y))).\end{aligned}$$

Finally, $\mathcal{U}(\mathsf{ID}_1)$ contains the restricted substitution rule

$$(\mathsf{SUB}) \qquad \frac{A}{A_{\mathsf{R}}[\{a : B[a]\}]},$$

where A is any formula in the language of $\mathcal{U}_0(\mathsf{ID}_1)$ and B is any formula in the language of $\mathcal{U}(\mathsf{ID}_1)$ (with the same convention as for the substitution rule for $\mathcal{U}_0(\mathsf{ID}_1)$).

5 Lower proof-theoretic bound

We define an interpretation of $\mathcal{U}(\mathsf{ID}_1)$ into $\Sigma^1_1\text{-}\mathsf{AC}^\bullet_0 + (\mathsf{SUB}^\bullet)$ in which we interpret the operational constants using indices of partial recursive functions. The predicates are then interpreted via a fixed point of an elementary positive operator form $\mathcal{A}[Q^+, x, y, z]$ where Q is a new ternary relation symbol. For the interpretation we need only consider a particular proof in $\mathcal{U}(\mathsf{ID}_1)$, and since such a proof refers only to finitely many of the least fixed points $P_{\mathfrak{A}}$ we can for simplicity (and because there is a universal such case), assume that $\mathcal{A}$ refers to a single inductive predicate $P_{\mathfrak{A}}$ (which we fix throughout this section).

The fixed point $Q_{\mathcal{A}}[x, y, z]$ of $\mathcal{A}[Q^+, x, y, z]$ is obtained in $\Sigma^1_1\text{-}\mathsf{AC}^\bullet_0$ using, as usual, Aczel's trick (cf. Feferman [12]). In particular, we can consider the class $\mathcal{C}$ of formulas $\exists X A[X, \vec{x}]$, where A is an elementary formula with the same restrictions as for the operator $\mathcal{A}$. There is a quinary $\mathcal{C}$ formula $E[z, x_1, x_2, x_3, x_4]$ that enumerates the quaternary $\mathcal{C}$ formulas. Using diagonalization we can then obtain a ternary $\mathcal{C}$ formula $P_{\mathcal{A}}$ that is our desired fixed point of the operator $\mathcal{A}$. We record this as a lemma.

Lemma 16. *There is a $\mathcal{C}$ formula $Q_{\mathcal{A}}[x, y, z]$ such that $\Sigma^1_1\text{-}\mathsf{AC}^\bullet_0$ proves*

$$\forall x, y, z(\mathcal{A}_Q[Q_{\mathcal{A}}, x, y, z] \leftrightarrow Q_{\mathcal{A}}[x, y, z])$$

where $\mathcal{A}_Q[Q_{\mathcal{A}}, x, y, z]$ denotes the formula obtained from $\mathcal{A}[Q^+, x, y, z]$ by replacing each occurrence of $Q(s, t, u)$ with $Q_{\mathcal{A}}[s, t, u]$.

The operational unfolding $\mathcal{U}_0(\mathsf{ID}_1)$ is interpreted in the usual way using its model in the partial recursive functions. See for example Feferman and Strahm [15]. In particular, $a\,b$ is interpreted as $\{a\}(b)$ in the sense of ordinary recursion theory.

In order to interpret predicates we need codes of the following forms:

Predicate	Code
nat	$\langle 0, 0\rangle$
eq	$\langle 1, 0\rangle$
$\mathsf{i}_{\mathfrak{A}}$	$\langle 2, 0\rangle$
pr_{R}	$\langle 3, 0\rangle$
$\mathsf{neg}(a)$	$\langle 4, a\rangle$
$\mathsf{un}(a)$	$\langle 5, a\rangle$
$\mathsf{conj}(a, b)$	$\langle 6, a, b\rangle$
$\mathsf{inv}(a, f)$	$\langle 7, a, f\rangle$
$\mathsf{join}(f, a)$	$\langle 8, f, a\rangle$

Thus, for example, the constant neg is interpreted as the index $\widehat{\mathsf{neg}}$ for a partial recursive function such that $\{\widehat{\mathsf{neg}}\}(a) = \langle 4, a\rangle$.

We use the ternary fixed point $Q_{\mathcal{A}}[x, y, z]$ with following informal interpretation:

$$\begin{array}{ll} Q_{\mathcal{A}}[x,0,0] & x \text{ is a predicate} \\ Q_{\mathcal{A}}[x,y,1] & y \in x \\ Q_{\mathcal{A}}[x,y,2] & y \notin x \end{array}$$

The operator form $\mathcal{A}[Q, a, b, c]$ is now defined to be the disjunction of the following 26 clauses:

1. $a = \langle 0, 0\rangle \wedge b = 0 \wedge c = 0$,
2. $a = \langle 0, 0\rangle \wedge c = 1$,
3. $a = \langle 1, 0\rangle \wedge b = 0 \wedge c = 0$,
4. $a = \langle 1, 0\rangle \wedge \exists x(b = \langle x, x\rangle) \wedge c = 1$,
5. $a = \langle 1, 0\rangle \wedge \forall x(b \neq \langle x, x\rangle) \wedge c = 2$,
6. $a = \langle 2, 0\rangle \wedge b = 0 \wedge c = 0$,
7. $a = \langle 2, 0\rangle \wedge P_{\mathfrak{A}}(b) \wedge c = 1$,
8. $a = \langle 2, 0\rangle \wedge \neg P_{\mathfrak{A}}(b) \wedge c = 2$,
9. $a = \langle 3, 0\rangle \wedge b = 0 \wedge c = 0$,
10. $a = \langle 3, 0\rangle \wedge \mathsf{R}(b) \wedge c = 1$,
11. $a = \langle 3, 0\rangle \wedge \neg\mathsf{R}(b) \wedge c = 2$,
12. $\exists x(a = \langle 4, x\rangle \wedge Q(x, 0, 0)) \wedge b = 0 \wedge c = 0$,
13. $\exists x(a = \langle 4, x\rangle \wedge Q(x, 0, 0) \wedge Q(x, b, 2)) \wedge c = 1$,
14. $\exists x(a = \langle 4, x\rangle \wedge Q(x, 0, 0) \wedge Q(x, b, 1)) \wedge c = 2$,
15. $\exists x(a = \langle 5, x\rangle \wedge Q(x, 0, 0)) \wedge b = 0 \wedge c = 0$,
16. $\exists x(a = \langle 5, x\rangle \wedge Q(x, 0, 0) \wedge \forall y Q(x, \langle b, y\rangle, 1) \wedge c = 1$,
17. $\exists x(a = \langle 5, x\rangle \wedge Q(x, 0, 0) \wedge \exists y Q(x, \langle b, y\rangle, 2) \wedge c = 2$,
18. $\exists x, y(a = \langle 6, x, y\rangle \wedge Q(x, 0, 0) \wedge Q(y, 0, 0)) \wedge b = 0 \wedge c = 0$,

19. $\exists x, y(a = \langle 6, x, y\rangle \wedge Q(x,0,0) \wedge Q(y,0,0) \wedge Q(x,b,1) \wedge Q(y,b,1)) \wedge c = 1.$

20. $\exists x, y(a = \langle 6, x, y\rangle \wedge Q(x,0,0) \wedge Q(y,0,0) \wedge (Q(x,b,2) \vee Q(y,b,2))) \wedge c = 2.$

21. $\exists x, f(a = \langle 7, x, f\rangle \wedge Q(x,0,0)) \wedge b = 0 \wedge c = 0,$

22. $\exists x, f(a = \langle 7, x, f\rangle \wedge Q(x,0,0) \wedge Q(x, \{f\}(b), 1)) \wedge c = 1,$

23. $\exists x, f(a = \langle 7, x, f\rangle \wedge Q(x,0,0) \wedge (\{f\}(b)\uparrow \vee Q(x, \{f\}(b), 2)) \wedge c = 2,$

24. $\exists f, x(a = \langle 8, f, x\rangle \wedge Q(x,0,0) \wedge \forall y(Q(x,y,2) \vee Q(\{f\}(y),0,0)))$
$\wedge\, b = 0 \wedge c = 0,$

25. $\exists f, x(a = \langle 8, f, x\rangle \wedge Q(x,0,0) \wedge \forall y(Q(x,y,2) \vee Q(\{f\}(y),0,0))$
$\wedge\, \exists u, v(b = \langle u, v\rangle \wedge Q(x,u,1) \wedge Q(\{f\}(u),v,1)) \wedge c = 1,$

26. $\exists f, x(a = \langle 8, f, x\rangle \wedge Q(x,0,0) \wedge \forall y(Q(x,y,2) \vee Q(\{f\}(y),0,0))$
$\wedge\, \forall u, v(b \neq \langle u, v\rangle \vee Q(x,u,2) \vee Q(\{f\}(u),v,2)) \wedge c = 2.$

We can now define

$$\begin{aligned} \Pi(x) &:= Q_{\mathcal{A}}[x,0,0] \wedge \forall y(Q_{\mathcal{A}}[x,y,2] \leftrightarrow \neg Q_{\mathcal{A}}[x,y,1]), \\ y \in x &:= \Pi(x) \wedge Q_{\mathcal{A}}[x,y,1]. \end{aligned}$$

A similar trick was used by Feferman [12] in order to model universes in type theory and explicit mathematics. It is now a matter of routine to verify that this defines an interpretation * of $\mathcal{U}(\mathsf{ID}_1)$ into Σ^1_1-$\mathsf{AC}^\bullet_0 + (\mathsf{SUB}^\bullet)$. Note that according to this interpretation, the premise of the substitution rule translates into an elementary formula of $\mathcal{L}^\bullet_2$.

Theorem 17. *The system $\mathcal{U}(\mathsf{ID}_1)$ is contained in Σ^1_1-$\mathsf{AC}^\bullet_0 + (\mathsf{SUB}^\bullet)$ via the translation *.*

Hence, using Buchholtz [5], we get following:

Corollary 18. $\psi(\Gamma_{\Omega+1}) \leq |\mathcal{U}(\mathsf{ID}_1)| \leq |\Sigma^1_1\text{-}\mathsf{AC}^\bullet_0 + (\mathsf{SUB}^\bullet)|$.

In fact, the lower bound proof in [5] can also be carried through in Δ^1_1-$\mathsf{CA}^\bullet_0 + (\mathsf{SUB}^\bullet)$. As usual, jump hierarchies of elementary operators can be built using Δ^1_1 comprehension, see for example Schütte [25].

Theorem 19. $\psi(\Gamma_{\Omega+1}) \leq |\Delta^1_1\text{-}\mathsf{CA}^\bullet_0 + (\mathsf{SUB}^\bullet)|$.

6 Finitely iterated fixed point theories

The aim of this section is to introduce first order theories $\widehat{\mathsf{ID}}^\bullet_n$, for all natural numbers $n \geq 1$, and to reduce $\mathsf{FP}^\bullet_0$ to the union of those. In the next section we shall then carry through the ordinal analysis of the theories $\widehat{\mathsf{ID}}^\bullet_n$ and thus determine the upper proof-theoretic bound of $\mathsf{FP}^\bullet_0$ and the systems equivalent to $\mathsf{FP}^\bullet_0$.

The theories $\widehat{\mathsf{ID}}^\bullet_n$ are the analogues of the well-known fixed point theories $\widehat{\mathsf{ID}}_n$, see Feferman [12] or Jäger, Kahle, Setzer, and Strahm [18], but with ID_1 rather than PA as the base theory. The languages $\mathcal{L}^\bullet(n)$ are defined by induction on n as follows:

(i) $\mathcal{L}^\bullet(0)$ is the first order part $\mathcal{L}^\bullet_1$ of the language $\mathcal{L}^\bullet_2$.

(ii) Given $\mathcal{L}^\bullet(n-1)$, we first determine the collection $\mathcal{C}(n-1)$ of all (not necessarily pure) inductive operator forms $\mathfrak{A}[P,u]$ formulated in $\mathcal{L}^\bullet(n-1)$, then select a fresh unary relation symbol $P^{(n)}_{\mathfrak{A}}$ for each $\mathfrak{A}[P,u]$ from $\mathcal{C}(n-1)$, and let $\mathcal{L}^\bullet(n)$ be the extension of $\mathcal{L}^\bullet(n-1)$ by these new relation symbols, i.e.,

$$\mathcal{L}^\bullet(n) := \mathcal{L}^\bullet(n-1) \cup \{P^{(n)}_{\mathfrak{A}} : \mathfrak{A}[P,u] \in \mathcal{C}(n-1)\}.$$

For any natural number $n \geq 1$, the the theory $\widehat{\mathsf{ID}}^\bullet_n$ is formulated in the language $\mathcal{L}^\bullet(n)$, its logic is the usual first order predicate logic with equality. The non-logical axioms of $\widehat{\mathsf{ID}}^\bullet_n$ are:

(A1) All axioms of primitive recursive arithmetic PRA plus the schema of complete induction on the natural numbers for all formulas of $\mathcal{L}^\bullet(n)$.

(A2) The *least fixed point axioms*

$$\forall a(\mathfrak{A}[P_{\mathfrak{A}}, a] \rightarrow P_{\mathfrak{A}}(a)), \tag{1}$$

$$\forall a(\mathfrak{A}[\{x : B[x]\}, a] \rightarrow B[a]) \rightarrow \forall a(P_{\mathfrak{A}}(a) \rightarrow B[a]) \tag{2}$$

for all inductive operator forms $\mathfrak{A}[P,u]$ of $\mathcal{L}_1$ and all formulas $B[u]$ of $\mathcal{L}^\bullet(n)$.

(A3) The *fixed point axioms*

$$\forall a(\mathfrak{A}[P^{(m)}_{\mathfrak{A}}, a] \;\leftrightarrow\; P^{(m)}_{\mathfrak{A}}(a))$$

for all natural numbers m with $1 \leq m \leq n$ and all inductive operator forms $\mathfrak{A}[P,u]$ from $\mathcal{C}(m-1)$.

$\mathcal{L}^\bullet(<\omega)$ is the union of the languages $\mathcal{L}^\bullet(n)$ and $\widehat{\mathsf{ID}}^\bullet_{<\omega}$ is the union of the theories $\widehat{\mathsf{ID}}^\bullet_n$,

$$\widehat{\mathsf{ID}}^\bullet_{<\omega} := \bigcup\{\widehat{\mathsf{ID}}^\bullet_n : 1 \leq n < \omega\}.$$

The following theorem is the analogue of Avigad's reduction of FP_0 to the theory $\widehat{\mathsf{ID}}_{<\omega}$ of finitely iterated fixed points above PA; for details see [1].

Theorem 20. $\mathsf{FP}^\bullet_0$ *is a conservative extension of* $\widehat{\mathsf{ID}}^\bullet_{<\omega}$ *with respect to all formulas of* $\mathcal{L}^\bullet(0)$. *In other words, if* $\mathsf{FP}^\bullet_0$ *proves the* $\mathcal{L}^\bullet(0)$ *formula* A, *then* $\widehat{\mathsf{ID}}^\bullet_{<\omega}$ *proves* A *as well.*

Proof. Taking up the strategy of [1], we can establish this theorem by a model-theoretic argument. All we have to show is that any (first order) model of $\widehat{\mathsf{ID}}^\bullet_{<\omega}$ can be extended to a (second order) model of $\mathsf{FP}^\bullet_0$ that validates the same formulas of $\mathcal{L}^\bullet(0)$.

So let $\mathbb{M}$ be a model of $\widehat{\mathsf{ID}}^\bullet_{<\omega}$, write $|\mathbb{M}|$ for the universe of $\mathbb{M}$, and denote the $\mathbb{M}$-interpretations of the relation symbols $P_\mathfrak{A}$ for least fixed points and $P^{(n)}_\mathfrak{A}$ for arbitrary fixed points by $\mathbb{M}(P_\mathfrak{A})$ and $\mathbb{M}(P^{(n)}_\mathfrak{A})$, respectively. Then we define $\mathbb{S}_\mathbb{M}$ to be the collection of all these sets $\mathbb{M}(P_\mathfrak{A})$ and $\mathbb{M}(P^{(n)}_\mathfrak{A})$ plus their projections. Finally, $(\mathbb{M}, \mathbb{S}_\mathbb{M})$ is the second order extension of $\mathbb{M}$ where $\mathbb{S}_\mathbb{M}$ takes care of the second order part. Clearly, we have

$$\mathbb{M} \models A \iff (\mathbb{M}, \mathbb{S}_\mathbb{M}) \models A$$

for all formulas A of $\mathcal{L}^\bullet(0)$. It remains to show that $(\mathbb{M}, \mathbb{S}_\mathbb{M})$ is a model of $\mathsf{FP}^\bullet_0$. For dealing with the fixed point axioms, consider an U-positive elementary formula $A[U, V_1, \ldots, V_m, x, y_1, \ldots, y_n]$ with at most the indicated free set and number variables. To simplify notation we assume $m = n = 1$. We have to show that

$$(*) \qquad (\mathbb{M}, \mathbb{S}_\mathbb{M}) \models \forall X \forall a \exists Y \forall b (b \in Y \leftrightarrow A[Y, X, b, a]).$$

To do so, choose an element $p \in |\mathbb{M}|$ and a set $M \in \mathbb{S}_\mathbb{M}$, given, for example, as

$$M = \left\{ i \in |\mathbb{M}| : \langle i, q \rangle \in \mathbb{M}(P^{(k)}_\mathfrak{B}) \right\}$$

for some fixed point relation symbol $P^{(k)}_\mathfrak{B}$ and some $q \in |\mathbb{M}|$. Now we define the formula $\mathfrak{C}[P, u]$ to be

$$u = \langle (u)_0, (u)_1, (u)_2 \rangle \ \wedge \\ A[\{x : P(\langle x, (u)_1, (u)_2 \rangle)\}, \{x : P^{(k)}_\mathfrak{B}(\langle x, (u)_2 \rangle)\}, (u)_0, (u)_1].$$

and observe that $\mathfrak{C}[P,u]$ is an inductive operator form with respect to the language $\mathcal{L}^\bullet(k)$. For the set

$$N := \{i \in |\mathbb{M}| : \langle i,p,q\rangle \in \mathbb{M}(P_{\mathfrak{C}}^{(k+1)})\}$$

and all $i \in |\mathbb{M}|$ we thus have

$$\begin{aligned} i \in N &\iff \langle i,p,q\rangle \in \mathbb{M}(P_{\mathfrak{C}}^{(k+1)}), \\ &\iff (\mathbb{M},\mathbb{S}_{\mathbb{M}}) \models \mathfrak{C}[P_{\mathfrak{C}}^{(k+1)}, \langle i,p,q\rangle], \\ &\iff (\mathbb{M},\mathbb{S}_{\mathbb{M}}) \models A[N,M,i,p]. \end{aligned}$$

Hence N is the required fixed point, and (*) has been validated. All other cases are straightforward or treated similarly. $\square$

By methods similar to those in Avigad [1], the previous theorem can also be proved in a purely syntactic and proof-theoretic manner. We also conjecture that the speed-up result of [1] carries over to $\mathsf{FP}_0^\bullet$ and $\widehat{\mathsf{ID}}_{<\omega}^\bullet$.

7 Upper proof-theoretic bound

To establish the upper proof-theoretic bounds on the theories $\widehat{\mathsf{ID}}_n^\bullet$ we shall combine methods of predicative and impredicative cut-elimination. To this end we first extend the languages $\mathcal{L}^\bullet(n)$ to languages $\mathcal{L}^{\bullet\infty}(n)$ by adding for inductive operator form $\mathfrak{A}[P,u]$ and each ordinal $\alpha < \Omega$ a new unary relation symbol $P_{\mathfrak{A}}^{<\alpha}$. These relation symbols are used to represent the stages of the least fixed points $P_{\mathfrak{A}}$. Then we restrict ourselves to the fragment of closed $\mathcal{L}^{\bullet\infty}(n)$ formulas A in negation-normal form and define $\neg A$ by de Morgan's rules and the law of double negation.

We now turn to infinite calculi, and in order to measure and control the complexities of infinite derivations we need control over the ranks and ordinal parameters of formulas occurring in infinite derivations.

Definition 21 (Rank and parameter set)**.**

1. The *rank*, $rk(A)$, of a closed $\mathcal{L}^{\bullet\infty}(n)$ formula A in negation-normal form is defined inductively as follows:

   (1) $rk(A) := rk(\neg A) := 0$ for closed atomic $\mathcal{L}_1$ formulas A,

   (2) $rk(P_{\mathfrak{A}}^{<\alpha}(t)) := rk(\neg P_{\mathfrak{A}}^{<\alpha}(t)) := \omega\alpha$ for $\alpha < \Omega$,

   (3) $rk(P_{\mathfrak{A}}(t)) := rk(\neg P_{\mathfrak{A}}(t)) := \Omega$,

   (4) $rk(P_{\mathfrak{A}}^{(m)}(t)) := rk(\neg P_{\mathfrak{A}}^{(m)}(t)) := \Omega$,

(5) $rk(A \land B) := rk(A \lor B) := \max\{rk(A), rk(B)\} + 1$,

(6) $rk(\exists x A[x]) := rk(\forall x A[x]) := rk(A[0]) + 1$,

2. The *parameter set*, $|A|$, of a closed $\mathcal{L}^{\bullet\infty}(n)$ formula A in negation-normal form is defined to be the set of the ordinals α occurring in subformulas $P_{\mathfrak{A}}^{<\alpha}(t)$ in A.

Note that the definition of rank ensures that $rk(\mathfrak{A}[P^{<\beta}, s]) < rk(P_{\mathfrak{A}}^{<\alpha}(s))$ for $\beta < \alpha$.

Given any natural number n, we now introduce an infinitary system $\widehat{\mathsf{ID}}_n^{\bullet\infty}$ in Tait-style, and use the capital Greek letters Γ, Θ, Λ, possibly with subscripts, for finite sets of closed $\mathcal{L}^{\bullet\infty}(n)$ formulas in negation-normal form. Also, we write (for example) Γ, Θ, A, B for $\Gamma \cup \Theta \cup \{A, B\}$. If Γ is the set $\{A_1, \ldots, A_n\}$ of closed $\mathcal{L}^{\bullet\infty}(n)$ formulas in negation-normal form, then $|\Gamma| := |A_1| \cup \cdots \cup |A_n|$ is the parameter set of Γ.

Axioms of $\widehat{\mathsf{ID}}_n^{\bullet\infty}$:

(A1) Γ, A whenever A is a true atomic $\mathcal{L}_1$ formula.

(A2) $\Gamma, \neg B$ whenever B is a false atomic $\mathcal{L}_1$ formula.

(A3) $\Gamma, \neg \mathsf{R}(s), \mathsf{R}(t)$ for numerically equivalent closed terms s and t.

(A4) $\Gamma, \neg P_{\mathfrak{A}}^{(m)}(s), P_{\mathfrak{A}}^{(m)}(t)$ for numerically equivalent closed terms s and t.

Basic rules of inference of $\widehat{\mathsf{ID}}_n^{\bullet\infty}$:

$$(\lor)\ \frac{\Gamma, A, B}{\Gamma, A \lor B} \qquad\qquad (\land)\ \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \land B}$$

$$(\exists)\ \frac{\Gamma, A[s]}{\Gamma, \exists x A[x]} \qquad\qquad (\forall)\ \frac{\Gamma, A[s] \text{ for all closed } s}{\Gamma, \forall x A[x]}$$

$$(P_{\mathfrak{A}}^{(m)})\ \frac{\Gamma, \mathfrak{A}[P_{\mathfrak{A}}^{(m)}, s]}{\Gamma, P_{\mathfrak{A}}^{(m)}(s)} \text{ if } m \leq n \qquad (\neg P_{\mathfrak{A}}^{(m)})\ \frac{\Gamma, \neg\mathfrak{A}[P_{\mathfrak{A}}^{(m)}, s]}{\Gamma, \neg P_{\mathfrak{A}}^{(m)}(s)} \text{ if } m \leq n$$

Closure rules of $\widehat{\mathsf{ID}}_n^{\bullet\infty}$:

$$(\text{Cl-}P_{\mathfrak{A}})\ \frac{\Gamma, \mathfrak{A}[P_{\mathfrak{A}}, s]}{\Gamma, P_{\mathfrak{A}}(s)}$$

Ordinal rules of inference of $\widehat{\mathsf{ID}}_n^{\bullet\infty}$:

$$(P_{\mathfrak{A}}^{<\alpha})\ \frac{\Gamma, \mathfrak{A}[P_{\mathfrak{A}}^{<\beta}, s]}{\Gamma, P_{\mathfrak{A}}^{<\alpha}(s)} \text{ if } \beta < \alpha \qquad (\neg P_{\mathfrak{A}}^{<\alpha})\ \frac{\Gamma, \neg\mathfrak{A}[P_{\mathfrak{A}}^{<\xi}, s] \text{ for all } \xi < \alpha}{\Gamma, \neg P_{\mathfrak{A}}^{<\alpha}(s)}$$

$$(P_{\mathfrak{A}})\ \frac{\Gamma, \mathfrak{A}[P_{\mathfrak{A}}^{<\beta}, s]}{\Gamma, P_{\mathfrak{A}}(s)} \text{ if } \beta < \Omega \qquad (\neg P_{\mathfrak{A}})\ \frac{\Gamma, \neg\mathfrak{A}[P_{\mathfrak{A}}^{<\xi}, s] \text{ for all } \xi < \Omega}{\Gamma, \neg P_{\mathfrak{A}}(s)}$$

Cuts of $\widehat{\mathsf{ID}}_n^{\bullet\infty}$:

$$(\text{cut})\ \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

The formulas A and $\neg A$ in the premises of (cut) are called the *cut formulas* of this cut. The *rank* of a cut is the rank of its cut formulas.

For the ordinal assignment to proofs and the subsequent cut elimination and collapsing we follow Buchholz [6] and make use of his approach to operator controlled derivations.

Definition 22. Let $\mathcal{H}$ be a derivation operator and let Γ be a finite set of closed $\mathcal{L}^{\bullet\infty}(n)$ formulas in negation-normal form. Then $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$ is defined for all ordinals α and ρ by induction on α.

1. If Γ is an axiom of $\widehat{\mathsf{ID}}_n^{\bullet\infty}$ and $|\Gamma| \cup \{\alpha\} \subseteq \mathcal{H}(\emptyset)$, then $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$ for all ordinals ρ.

2. If $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma_i$ and $\alpha_0 < \alpha$ for every premise of a basic inference of $\widehat{\mathsf{ID}}_n^{\bullet\infty}$ or a cut of rank less than ρ and if $|\Gamma| \cup \{\alpha\} \subseteq \mathcal{H}(\emptyset)$ for the conclusion Γ of this rule, then $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$.

3. If $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, \mathfrak{A}[P_{\mathfrak{A}}^{<\sigma}, s]$ for some $\sigma < \tau$ and $\sigma, \alpha_0 < \alpha$ and if $|\Gamma, P_{\mathfrak{A}}^{<\tau}(s)| \cup \{\alpha\} \subseteq \mathcal{H}(\emptyset)$, then $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, P_{\mathfrak{A}}^{<\tau}(s)$.

4. If $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H}[\sigma] \overset{\alpha_\sigma}{\underset{\rho}{\vdash}} \Gamma, \neg\mathfrak{A}[P_{\mathfrak{A}}^{<\sigma}, s]$ and $\alpha_\sigma < \alpha$ for all $\sigma < \tau$ and if $|\Gamma, \neg P_{\mathfrak{A}}^{<\tau}(s)| \cup \{\alpha\} \subseteq \mathcal{H}(\emptyset)$, then $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \neg P_{\mathfrak{A}}^{<\tau}(s)$.

5. If $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, \mathfrak{A}[P_{\mathfrak{A}}^{<\sigma}, s]$ for some $\sigma < \Omega$ and $\sigma, \alpha_0 < \alpha$ and if $|\Gamma, P_{\mathfrak{A}}(s)| \cup \{\alpha\} \subseteq \mathcal{H}(\emptyset)$, then $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, P_{\mathfrak{A}}(s)$.

6. If $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H}[\sigma] \overset{\alpha_\sigma}{\underset{\rho}{\vdash}} \Gamma, \neg\mathfrak{A}[P_{\mathfrak{A}}^{<\sigma}, s]$ and $\alpha_\sigma < \alpha$ for all $\sigma < \Omega$ and if $|\Gamma, \neg P_{\mathfrak{A}}(s)| \cup \{\alpha\} \subseteq \mathcal{H}(\emptyset)$, then $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \neg P_{\mathfrak{A}}(s)$.

7. If $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H}[\sigma] \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, \mathfrak{A}[P_{\mathfrak{A}}, s]$ and $\alpha_0 + 1 < \alpha$ and if $|\Gamma, P_{\mathfrak{A}}(s)| \cup \{\alpha\} \subseteq \mathcal{H}(\emptyset)$, then $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, P_{\mathfrak{A}}(s)$.

We write $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \mathrel{|}\!\frac{<\alpha}{\rho}\, \Gamma$ to mean there exists an $\alpha_0 < \alpha$ with $\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \mathrel{|}\!\frac{\alpha_0}{\rho}\, \Gamma$.

In the remaining part of this article we describe (or sketch) how to employ the infinitary systems for establishing the upper proof-theoretic bound of the theories $\widehat{\mathsf{ID}}_n$. Our main reference is again Buchholz [6] where an analogous analysis has been carried out in full details for a theory that is similar to (or even more complicated than) our systems as far as impredicative methods of proof theory are concerned. We state the main results and believe that it should not be too complicated for the reader to fill in the missing details.

First observe that the infinitary systems $\widehat{\mathsf{ID}}_n^{\bullet\infty}$ have the property that all instances of complete induction on the natural numbers and all instances of least fixed point induction are provable, in particular by making use of the infinitary rules $(\forall)$ and $(\neg P_{\mathfrak{A}})$. However, the price is that we have to deal with complex derivations of infinitary depths. As a consequence we obtain a canonical embedding theorem.

Theorem 23 (Embedding). *Let A be a closed $\mathcal{L}^{\bullet}(n)$ formula A in negation-normal form. If $\widehat{\mathsf{ID}}_n^{\bullet} \vdash A$, then there exists a natural number k such that for all derivation operators $\mathcal{H}$,*

$$\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \mathrel{|}\!\frac{<\varepsilon_{\Omega+1}}{\Omega+k}\, A.$$

Now we move on to cut elimination. It is easy to convince oneself that the axioms and rules of inference of the infinitary systems $\widehat{\mathsf{ID}}_n^{\bullet\infty}$ and the definitions of the ranks of closed $\mathcal{L}^{\bullet\infty}(n)$ formulas in negation-normal form are so that cuts of ranks greater than Ω can be eliminated without any problems. As usual, $\omega_0(\alpha) := \alpha$ and $\omega_{k+1}(\alpha) := \omega^{\omega_k(\alpha)}$ for all ordinals α and natural numbers k.

Lemma 24 (Partial cut elimination). *If Γ is a finite set of closed $\mathcal{L}^{\bullet\infty}(n)$ formulas in negation-normal form, then we have for all derivation operators $\mathcal{H}$, all ordinals α, and all natural numbers k:*

$$\widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \mathrel{|}\!\frac{\alpha}{\Omega+1+k}\, \Gamma \quad\Longrightarrow\quad \widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \mathrel{|}\!\frac{\omega_k(\alpha)}{\Omega+1}\, \Gamma.$$

The next step is to eliminate the fixed points $P_{\mathfrak{A}}^{(m)}$. To achieve this, we can make use of standard elimination procedures for finitely many fixed points by asymmetric interpretations as, for example, in Cantini [9], Jäger and Strahm [19], or Marzetta and Strahm [21].

Lemma 25 (Elimination of fixed points). *If Γ is a finite set of closed $\mathcal{L}^{\bullet\infty}(n)$ formulas in negation-normal form, then we have for all derivation operators $\mathcal{H}$ and all ordinals α:*

$$\widehat{\mathsf{ID}}_{n+1}^{\bullet\infty}, \mathcal{H} \mathrel{|}\!\frac{<\alpha}{\Omega+1}\, \Gamma \quad\Longrightarrow\quad \widehat{\mathsf{ID}}_n^{\bullet\infty}, \mathcal{H} \mathrel{|}\!\frac{<\varphi\alpha 0}{\Omega+1}\, \Gamma.$$

Cut formulas of rank less than Ω are eliminated by methods of predicative cut elimination as presented in Schütte [25]; for all details concerning predicative cut elimination in the presence of derivation functions see Buchholz [6].

Lemma 26 (Predicative cut elimination)**.** *If Γ is a finite set of closed $\mathcal{L}^{\bullet\!\infty}(0)$ formulas in negation-normal form, then we have for all derivation operators $\mathcal{H}$, all ordinals α, and all ordinals β and ρ with $\beta, \rho < \Omega$ and $\rho \in \mathcal{H}(\emptyset)$:*

$$\widehat{\mathsf{ID}}_n^{\bullet\!\infty}, \mathcal{H} \,\Big|\frac{\alpha}{\beta+\omega^\rho}\, \Gamma \quad\Longrightarrow\quad \widehat{\mathsf{ID}}_n^{\bullet\!\infty}, \mathcal{H} \,\Big|\frac{\varphi\rho\alpha}{\beta}\, \Gamma.$$

So it only remains to deal with cut formulas of the form $P_{\mathfrak{A}}(t)$ and $\neg P_{\mathfrak{A}}(t)$, and here the boundedness and collapsing techniques enter the picture. Let POS be the collection of all closed $\mathcal{L}^{\bullet\!\infty}(0)$ formulas in negation-normal form that do not contain subformulas of the form $\neg P_{\mathfrak{A}}(t)$; i.e., POS is the collection of all closed $\mathcal{L}^{\bullet\!\infty}(0)$ formulas in negation-normal form that are positive in the least fixed point relations $P_{\mathfrak{A}}$. In addition, if A belongs to POS and α is an ordinal less than Ω, then $A^{(<\alpha)}$ is the formula obtained from A if all occurrences of $P_{\mathfrak{A}}(t)$ are replaced by $P_{\mathfrak{A}}^{<\alpha}(t)$. For the proof of this boundedness and collapsing lemma consult again Buchholz [6].

Lemma 27 (Boundedness and Collapsing)**.**

1. *For all finite sets Γ of closed $\mathcal{L}^{\bullet\!\infty}(n)$ formulas in negation-normal form, all and elements A of POS, all derivation operators $\mathcal{H}$, and all ordinals α, β, ρ such that $\alpha \le \beta < \Omega$ and $\beta \in \mathcal{H}(\emptyset)$ we have:*

$$\widehat{\mathsf{ID}}_0^{\bullet\!\infty}, \mathcal{H} \,\Big|\frac{\alpha}{\rho}\, \Gamma, A \quad\Longrightarrow\quad \widehat{\mathsf{ID}}_0^{\bullet\!\infty}, \mathcal{H} \,\Big|\frac{\alpha}{\rho}\, \Gamma, A^{(<\beta)}.$$

2. *Suppose that Γ is a finite subset of POS and σ an ordinal such that $|\Gamma| \subseteq \mathbf{C}(\sigma+1, \psi(\sigma+1))$ and $\sigma \in \mathcal{H}_\sigma[|\Gamma|](\emptyset)$. Then we have for all ordinals α and $\beta := \sigma + \omega^{\Omega+\alpha}$:*

$$\widehat{\mathsf{ID}}_0^{\bullet\!\infty}, \mathcal{H}_\sigma[|\Gamma|] \,\Big|\frac{\alpha}{\Omega+1}\, \Gamma \quad\Longrightarrow\quad \widehat{\mathsf{ID}}_0^{\bullet\!\infty}, \mathcal{H}_\beta[|\Gamma|] \,\Big|\frac{\psi\beta}{\psi\beta}\, \Gamma.$$

Combining Theorem 23 with the series of Lemmas 24 to 27 and carrying through some ordinal calculations, we obtain complete cut elimination for the closed $\mathcal{L}_1$ formulas provable in one of the theories $\widehat{\mathsf{ID}}_n$.

Theorem 28 (Complete cut elimination)**.** *Let A be a closed formula of the language $\mathcal{L}_1$ and suppose that A is provable in $\widehat{\mathsf{ID}}_n$ for some natural number $n \ge 1$. Then there exist a derivation operator $\mathcal{H}$ and an ordinal $\alpha < \psi(\Gamma_{\Omega+1})$ such that $\widehat{\mathsf{ID}}_0^{\bullet\!\infty}, \mathcal{H} \,\Big|\frac{\alpha}{0}\, A$.*

By standard proof theory this result immediately gives us the upper bound result for the theory $\widehat{\mathsf{ID}}^\bullet_{<\omega}$: If $\widehat{\mathsf{ID}}^\bullet_{<\omega}$ proves $TI[\lhd, \mathsf{R}]$ for some primitive recursive well ordering $\lhd$, then there exists a natural number n large enough such that $\widehat{\mathsf{ID}}_n$ proves $TI[\lhd, \mathsf{R}]$. In view of the previous theorem this implies

$$\widehat{\mathsf{ID}}^{\bullet\infty}_0, \mathcal{H} \,\Big|\!\frac{\alpha}{0}\; TI[\lhd, \mathsf{R}]$$

for some derivation operator $\mathcal{H}$ and some ordinal $\alpha < \psi(\Gamma_{\Omega+1})$. Finally, as shown in Schütte [25], we can conclude that the depth of a cut free derivation of $TI[\lhd, \mathsf{R}]$ essentially bounds the order type of this well ordering.

Corollary 29. $|\widehat{\mathsf{ID}}^\bullet_{<\omega}| \leq \psi(\Gamma_{\Omega+1})$.

It only remains to see this upper bound result in the context of Theorem 9, Corollary 13, Theorem 17, Corollary 18, and Theorem 20 in order to conclude the ordinal analysis of the main theories of this article.

Corollary 30 (Proof-theoretic ordinal)**.**

$$\begin{aligned}\psi(\Gamma_{\Omega+1}) &= |\widehat{\mathsf{ID}}^\bullet_{<\omega}| = |\mathsf{FP}^\bullet_0| = |\mathsf{ATR}^\bullet_0| \\ &= |\Sigma^1_1\text{-}\mathsf{DC}^\bullet_0 + (\mathsf{SUB}^\bullet)| = |\Sigma^1_1\text{-}\mathsf{AC}^\bullet_0 + (\mathsf{SUB}^\bullet)| = |\mathcal{U}(\mathsf{ID}_1)|.\end{aligned}$$

8 Discussion

We have identified several systems in classical logic of strength $\psi(\Gamma_{\Omega+1})$ (cf. Corollary 30). Our results are a parallel to those characterizing classical systems of strength Γ_0,

$$\begin{aligned}\Gamma_0 &= |\widehat{\mathsf{ID}}_{<\omega}| = |\mathsf{FP}_0| = |\mathsf{ATR}_0| \\ &= |\Sigma^1_1\text{-}\mathsf{DC}_0 + (\mathsf{SUB})| = |\Sigma^1_1\text{-}\mathsf{AC}_0 + (\mathsf{SUB})| = |\mathcal{U}(\mathsf{NFA})|.\end{aligned}$$

(See Feferman [12] for $\widehat{\mathsf{ID}}_{<\omega}$, Avigad [1] for FP_0, Simpson [26] for the subsystems of second order arithmetic, and Feferman and Strahm [15] for $\mathcal{U}(\mathsf{NFA})$.)

A companion article in preparation shall establish similar results for constructive systems. In particular, we shall verify the conjecture of Hancock [16] by studying a predicative type theory in the style of Martin-Löf [20] (i.e., a dependent type theory with an externally indexed hierarchy of predicative universes $(U_n)_{n<\omega}$) extended with a single well-ordering type (belonging to all universes) corresponding to the constructive tree ordinals (a type Ord with constructors zero of type Ord, $\mathsf{successor}$ of type $Ord \to Ord$ and limit of type $(Nat \to Ord) \to Ord$). This will parallel the result of Feferman [12] that the strength of the predicative type theory itself is Γ_0. Also of interest is an analogous system of explicit mathematics, similarly containing a hierarchy of universes (as in Feferman [12]) and a type of constructive tree ordinals.

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