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On supergravity theories, after ~ 40 years

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Abstract. An introduction to and a partial review of supergravity theories is given, insisting on concepts and on some important technical aspects. Topics covered include elements of global supersymmetry, a derivation of the simplest $\mathcal{N} = 1$ supergravity theory, a discussion of $\mathcal{N} = 1$ matter–supergravity couplings, of the scalar sector and of some simple models. Space-time is four-dimensional.

1. Foreword

Almost forty years ago, in 1974 and soon after, most of the important properties of quantum field theories with linear supersymmetry¹ were displayed in a brilliant series of papers. These fundamental results include what is now called the Wess-Zumino model [1], super-Yang-Mills theory [2]², exceptional renormalization properties [4], spontaneous supersymmetry breaking [5, 6], the current structure [7] and the development of superspace and superfield techniques [8, 9].

Two attractive complementary aspects of supersymmetry were recognized. Firstly, it is an extension compatible with quantum mechanics of the Poincaré algebra, the symmetry of relativistic field theory [10, 11, 4]. Secondly, it can be implemented in the Standard Model of particle interactions and be experimentally tested [12], leading to numerous models and analysis and, now, to LHC (preliminary and future) results.

Almost forty years ago, in 1976, the first supergravity theory, the gauge theory of supersymmetry, was invented by Ferrara, Freedman and van Nieuwenhuizen [13] and by Deser and Zumino [14], opening decades of developments which installed supergravity at the meeting point of two independent approaches to the unification program.

¹ In four dimensions.

² And later on [3].



On the low energy side, the success of the Standard Model emphasizes the enigma of the large hierarchical ratio M_P/M_W . Global supersymmetry helps with his capacity to forbid destabilizing quantum corrections [4, 15]. But it does not spontaneously break, as observations obviously require, and this is where supergravity helps, by proposing a source for supersymmetry breaking effects at low energies [16, 17, 18] and also in proposing a scheme to radiatively induce a small scale (M_W) from supersymmetry breaking at a much higher scale [19, 20, 21, 22, 23].

On the high energy scale, at present, the only concrete proposal for a coherent description of all particle interactions including gravitation involves models based on strings with scale close or related to the Planck scale. Coherence and stability of these models require some amount of supersymmetry. Supergravity appears then as the natural tool to describe the light sector of superstring theories, at energy scales where string excitations decouple and where supersymmetry breaking should be able to generate the so-called soft breaking terms needed in the supersymmetric Standard Model [16, 17, 18].

More pragmatically, the possible relevance of supersymmetry to particle physics, in relation with the weak interaction scale, is concrete and testable at LHC (and some other) experiments. This is maybe a strong enough motivation.

Over the years, supergravity theories have of course found many other more theoretical roles, in gauge/gravity dualities, to understand the structure of gravitational scattering amplitudes, in the description of superstring solutions and compactifications, in studies of black holes, solutions of gravitational theories with tensor fields, attractors, in sigma models living on particular geometries ...

This contribution tries to provide an introduction to (four-dimensional) supergravity and a detailed discussion of some aspects of simple theories and of their applications. It begins with basic aspects of global supersymmetry (section 2), with a detailed discussion of the role of auxiliary fields. The simplest $\mathcal{N} = 1$ supergravity and its Anti-de Sitter deformation are then derived (section 3). After a brief discussion of theories with extended supersymmetry, general properties of the $\mathcal{N} = 1$ supergravity–matter couplings are described (section 4), with particular attention to the scalar potential and the gravitino sector. The examples of simple no-scale or dilaton supergravities are the subject of section 5.

2. Elements of global supersymmetry

Relativistic quantum field theories and strong or electroweak interactions are invariant under global transformations of the Poincaré group, *i.e.* under Lorentz (proper and orthochronous) transformations and translations. In other words, there are Lorentz generators $M^{\mu\nu} = -M^{\nu\mu}$ and translation generators P^μ acting on coordinates and fields and leaving the dynamical equations unchanged. Translation generators are universal, $P_\mu = -i\partial_\mu$. Lorentz generators

act on coordinates according to

$$\delta x^\mu = \frac{i}{2} \omega_{\rho\sigma} M^{\rho\sigma} x^\mu = -\omega^{\mu\nu} x_\nu, \quad M^{\rho\sigma} = -i(x^\rho \partial^\sigma - x^\sigma \partial^\rho). \quad (2.1)$$

They act on fields with generators in representations depending on the spins of the fields. For a set of fields $\Phi(x)$,

$$\delta \Phi(x) = \frac{i}{2} \omega_{\rho\sigma} \Sigma^{\rho\sigma} \Phi(x) - \delta x^\mu \partial_\mu \Phi(x) = \frac{i}{2} \omega_{\rho\sigma} M^{\rho\sigma} \Phi(x) \quad (2.2)$$

and the information on spins is in the choice of linear operators $\Sigma^{\mu\nu}$. These variations represent the Poincaré Lie algebra

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= -i(\eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma}), \\ [P^\mu, M^{\nu\rho}] &= i(\eta^{\mu\nu} P^\rho - \eta^{\mu\rho} P^\nu), \\ [P^\mu, P^\nu] &= 0. \end{aligned} \quad (2.3)$$

2.1. The Poincaré superalgebra

Global supersymmetry is an extension compatible with quantum field theory of Poincaré symmetry adding spin 1/2 generators, the *supercharges*, with an algebra of anticommutators. The relevant algebra includes commutators

$$[M^{\mu\nu}, Q_\alpha^i] = -\frac{i}{4}([\sigma^\mu, \bar{\sigma}^\nu] Q^i)_\alpha, \quad [P_\mu, Q_\alpha^i] = 0. \quad (2.4)$$

The index $i = 1, \dots, \mathcal{N}$ labels the supercharges and the first relation indicates that the Q_α^i 's have spin 1/2. The superalgebra is completed by the anticommutators³

$$\{Q_\alpha^i, \bar{Q}_{\dot{\alpha}}^j\} = -2i \delta^{ij} (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu = 2 \delta^{ij} (\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu. \quad (2.5)$$

The representations of \mathcal{N} -extended supersymmetry share several important properties:

- Firstly, the particle states in a supermultiplet have helicities extending from a maximal λ to $\lambda - \mathcal{N}/2$ (and the opposite helicities if $\lambda - \mathcal{N}/2 \neq -\lambda$). Since quantum field theory admits helicities $|\lambda| \leq 1$, it admits at most $\mathcal{N} = 4$ global supersymmetries. For supergravity, $|\lambda| \leq 2$ and then $\mathcal{N} \leq 8$.
- Secondly, all particle states have the same mass (which can be zero) and the numbers of fermionic and bosonic particle states are equal, $n_B = n_F$. Similarly, for representations in terms of fields (unconstrained by a dynamical field equation) bosonic and fermionic component fields come in equal numbers. The number of helicity zero states is then always even.

³ We disregard the possibility of central charges for $\mathcal{N} > 1$.

- Thirdly, for a supermultiplet of fields, the allowed interactions are strongly constrained and related.
- Finally, the divergences of supersymmetric quantum field theories are much softer than in a generic case: quadratic divergences in the scalar sector are absent and, in particular, scalar and Yukawa (scalar–fermion) interactions are not renormalized.

Of course, these properties immediately indicate that supersymmetry cannot be an exact symmetry of Nature. Realistic theories with supersymmetry must include a mechanism of supersymmetry breaking, a condition which turns out to be a challenge to model builders.

It should also be mentioned that Poincaré supersymmetry is a limiting case of the supersymmetric extension of anti-de Sitter space-time symmetry $SO(2,3)$ (a contraction of the AdS superalgebra). But it is not compatible with the de Sitter $SO(1,4)$ algebra. This algebraic fact has important dynamical implications for supergravity theories, which are our main subject of interest here.

2.2. The simplest supermultiplet and auxiliary fields

The simplest representation of $\mathcal{N} = 1$ supersymmetry is the chiral multiplet, which describes particle states with helicities $\pm 1/2, 0, 0$. It then includes a Weyl (or Majorana) spinor ψ and a complex scalar z . But it also introduces the concept of auxiliary fields of central importance in field representations of the supersymmetry algebra. The relevance of auxiliary fields, with algebraic, non-propagating field equations, is related to the requirement $n_B = n_F$, and the fact that counting degrees of freedom on-shell and off-shell gives different numbers.

To illustrate the use of auxiliary fields, consider the sum of the free massless Klein-Gordon and Dirac lagrangians for a complex scalar and a Weyl (or Majorana) spinor:

$$\mathcal{L}_0 = (\partial_\mu \bar{z})(\partial^\mu z) + \frac{i}{2} \psi \sigma^\mu \partial_\mu \bar{\psi} - \frac{i}{2} \partial_\mu \psi \sigma^\mu \bar{\psi}. \quad (2.6)$$

It is a supersymmetric theory: under the variations

$$\delta z = \sqrt{2} \epsilon \psi, \quad \delta \psi_\alpha = -\sqrt{2} i \partial_\mu z (\sigma^\mu \bar{\epsilon})_\alpha, \quad (2.7)$$

the lagrangian changes by a derivative and the action is then invariant. There is however trouble in the algebra. Firstly,

$$[\delta_1, \delta_2] z = -2i(\epsilon_2 \sigma^\mu \bar{\epsilon}_1 - \epsilon_1 \sigma^\mu \bar{\epsilon}_2) \partial_\mu z, \quad (2.8)$$

which is a translation $\delta x^\mu = 2(\epsilon_2 \sigma^\mu \bar{\epsilon}_1 - \epsilon_1 \sigma^\mu \bar{\epsilon}_2) = 2\bar{\epsilon}_2 \gamma^\mu \epsilon_1$ as required by the supersymmetry algebra (2.5). But

$$\begin{aligned} [\delta_1, \delta_2] \psi_\alpha &= -2i(\epsilon_2 \sigma^\mu \bar{\epsilon}_1 - \epsilon_1 \sigma^\mu \bar{\epsilon}_2) \partial_\mu \psi_\alpha \\ &\quad + 2i(\partial_\mu \psi \sigma^\mu \bar{\epsilon}_2) \epsilon_{1\alpha} - 2i(\partial_\mu \psi \sigma^\mu \bar{\epsilon}_1) \epsilon_{2\alpha}. \end{aligned} \quad (2.9)$$

The first term is as expected, but the second only vanishes if the spinor solves the Dirac equation $\partial_\mu \psi \sigma^\mu = 0$ implied by the lagrangian. Hence, variations (2.7) only close the supersymmetry algebra for on-shell fields. This is certainly a problem if one wishes to construct more complicated, interacting lagrangians with nonlinear field equations. One must then simultaneously invent the lagrangian and the corresponding supersymmetry variations (which become nonlinear as well).

However, modify the variation of the spinor:

$$\delta\psi_\alpha = -\sqrt{2}f\epsilon_\alpha - \sqrt{2}i\partial_\mu z(\sigma^\mu\bar{\epsilon})_\alpha \quad (2.10)$$

where f is a complex scalar field. The new term adds

$$-\sqrt{2}\delta_1 f \epsilon_{2\alpha} + \sqrt{2}\delta_2 f \epsilon_{1\alpha}$$

to $[\delta_1, \delta_2]\psi_\alpha$ and choosing then

$$\delta f = -\sqrt{2}i(\partial_\mu \psi \sigma^\mu \bar{\epsilon}) \quad (2.11)$$

leads to the expected algebra

$$\begin{aligned} [\delta_1, \delta_2]\psi_\alpha &= -2i(\epsilon_2\sigma^\mu\bar{\epsilon}_1 - \epsilon_1\sigma^\mu\bar{\epsilon}_2)\partial_\mu\psi_\alpha, \\ [\delta_1, \delta_2]f &= -2i(\epsilon_2\sigma^\mu\bar{\epsilon}_1 - \epsilon_1\sigma^\mu\bar{\epsilon}_2)\partial_\mu f \end{aligned} \quad (2.12)$$

for all three fields z , ψ and f . The modification of the spinor variation also adds a new contribution to the variation of \mathcal{L}_0 :

$$-f\delta\bar{f} - \bar{f}\delta f + \frac{i}{\sqrt{2}}\partial_\mu[f\epsilon\sigma^\mu\bar{\psi} - \bar{f}\psi\sigma^\mu\bar{\epsilon}].$$

This in turn imposes to modify the lagrangian to

$$\mathcal{L} = (\partial_\mu\bar{z})(\partial^\mu z) + \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi} + \bar{f}f, \quad (2.13)$$

with field equations

$$\square z = 0, \quad \partial_\mu\psi\sigma^\mu = 0, \quad f = 0 \quad (2.14)$$

and the scalar f is *auxiliary*: it describes $n_B = 2$ off-shell fields and $n_B = 0$ on-shell states. Since ψ includes $n_F = 4$ off-shell and $n_F = 2$ on-shell degrees of freedom while for z $n_B = 2$ on-shell and off-shell, the equality $n_B = n_F$ is verified on-shell *and* off-shell in the supermultiplet (z, ψ, f) . On shell, $\delta f = 0$ and one returns to the original expressions (2.6) and (2.7).

In general, the equality of the number of bosonic and fermionic physical (on-shell) degrees of freedom is imposed by the supersymmetry algebra while a mismatch in the numbers of bosonic and fermionic off-shell fields suggests that adding auxiliary fields is necessary to obtain an off-shell representation, if possible at all.

The canonical dimensions (in energy unit) of z , ψ and f are respectively 1, 3/2 and 2 and the parameter ϵ has dimension $-1/2$. Hence f must transform in a field with dimension 5/2, which is then a derivative of ψ . This suggests a method to construct supersymmetric lagrangians: starting with an off-shell supermultiplet like (z, ψ, f) , combine supermultiplets into a new supermultiplet (*tensor calculus*) and take its component with the highest dimension as a lagrangian term: it necessarily transforms as a derivative. The simplest example is

$$Z = z^2, \quad \Psi = 2z\psi, \quad F = 2fz + \psi\psi. \quad (2.15)$$

One easily verifies that (Z, Ψ, F) and (z, ψ, f) have identical transformations. Hence, since the variation of F is a derivative,

$$\begin{aligned} \mathcal{L}_m = & (\partial_\mu \bar{z})(\partial^\mu z) + \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi} + \bar{f}f \\ & - m[fz + \frac{1}{2}\psi\psi] - m[\bar{f}\bar{z} + \frac{1}{2}\bar{\psi}\bar{\psi}] \end{aligned} \quad (2.16)$$

is supersymmetric. Eliminating f with its field equation $f = m\bar{z}$ leads to

$$\mathcal{L}_m = (\partial_\mu \bar{z})(\partial^\mu z) - m^2 z\bar{z} + \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi} - \frac{m}{2}[\psi\psi + \bar{\psi}\bar{\psi}], \quad (2.17)$$

with a common mass m for z and ψ , and to the supersymmetry variations

$$\delta z = \sqrt{2}\epsilon\psi, \quad \delta\psi_\alpha = -\sqrt{2}m\bar{z}\epsilon_\alpha - \sqrt{2}i\partial_\mu z(\sigma^\mu\bar{\epsilon})_\alpha. \quad (2.18)$$

Contrary to variation (2.10), $\delta\psi_\alpha$ now depends on the lagrangian parameter m . Note that with Dirac equation $i\partial_\mu\psi\sigma^\mu = -m\bar{\psi}$, the on-shell variation of the auxiliary field is

$$\delta f = -\sqrt{2}i(\partial_\mu\psi\sigma^\mu\bar{\epsilon}) = m\sqrt{2}\epsilon\bar{\psi} = m\delta\bar{z}, \quad (2.19)$$

as indicated by $f = m\bar{z}$.

Similarly, a renormalizable interaction would follow from the observation that

$$\begin{aligned} Z &= \frac{m}{2}z^2 + \frac{\lambda}{3}z^3, \equiv W(z), & \Psi_\alpha &= (mz + \lambda z^2)\psi_\alpha, \\ F &= (mz + \lambda z^2)f + \frac{1}{2}(m + 2\lambda z)\psi\psi \end{aligned} \quad (2.20)$$

is a chiral multiplet. The holomorphic function W of z only is the *superpotential*. The supersymmetric lagrangian

$$\begin{aligned} \mathcal{L}_{m,\lambda} = & (\partial_\mu \bar{z})(\partial^\mu z) + \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi} + \bar{f}f \\ & - (mz + \lambda z^2)f - (m\bar{z} + \lambda\bar{z}^2)\bar{f} - \frac{m}{2}[\psi\psi + \bar{\psi}\bar{\psi}] - \lambda z\psi\psi - \lambda\bar{z}\bar{\psi}\bar{\psi} \\ = & (\partial_\mu \bar{z})(\partial^\mu z) - V(z, \bar{z}) \\ & + \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi} - \frac{m}{2}[\psi\psi + \bar{\psi}\bar{\psi}] - \lambda z\psi\psi - \lambda\bar{z}\bar{\psi}\bar{\psi}, \end{aligned} \quad (2.21)$$

using $\bar{f} = mz + \lambda z^2$ in the second expression, includes the scalar potential

$$V(z, \bar{z}) = |f|^2 = |mz + \lambda z^2|^2 = \left| \frac{d}{dz}W(z) \right|^2. \quad (2.22)$$

It is a renormalizable quantum field theory, and supersymmetry holds to all orders of perturbation theory. This result shows how supersymmetry relates all three scalar and Yukawa interactions and how the scalar potential is related to the spinor variations

$$\delta\psi_{i\alpha} = -\sqrt{2}\mathcal{A}_i(\varphi_i)\epsilon_\alpha + \partial_\mu(\dots) + \dots \quad \longleftrightarrow \quad V = \sum_i |\mathcal{A}_i(\varphi_i)|^2, \quad (2.23)$$

$$\mathcal{A} = f = \frac{d\bar{W}(\bar{z})}{d\bar{z}}.$$

In the first line, the index i would label the various spinor and scalar fields φ_i in the theory. The second line refers to our example of a single chiral multiplet with superpotential W . This relation between the potential and spinor variations is a universal property, even in theories and supermultiplets for which auxiliary fields (and then the second line) do not exist [24]. Notice that the on-shell spinor variation is not linear in an interacting theory.

The generalization of the "square" (2.15) of a chiral multiplet is as follows. Consider a set of chiral multiplets (z^i, ψ^i, f^i) and an arbitrary superpotential function $W(z^i)$ of the scalar fields z^i . Then,

$$Z = W(z^i), \quad \Psi_\alpha = \frac{\partial W}{\partial z^i} \psi_\alpha^i, \quad F = \frac{\partial W}{\partial z^i} f^i + \frac{1}{2} \frac{\partial^2 W}{\partial z^i \partial z^j} \psi^i \psi^j \quad (2.24)$$

are the components of a chiral multiplet. Another important chiral multiplet is called *kinetic*. Its component fields are

$$Z = \bar{f}, \quad \Psi_\alpha = i(\sigma^\mu \partial_\mu \bar{\psi})_\alpha, \quad F = -\square \bar{z}, \quad (2.25)$$

where (z, ψ, f) is a chiral multiplet. Multiplying then the kinetic multiplet with (z, ψ, f) using the tensor product rule (2.24) leads to the kinetic lagrangian (2.13):

$$zF + fZ + \psi\Psi = -z\square\bar{z} + i\psi\sigma^\mu\partial_\mu\bar{\psi} + \bar{f}f = \mathcal{L} + \partial_\mu \left[\frac{i}{2} \psi\sigma^\mu\bar{\psi} - z\partial^\mu\bar{z} \right]. \quad (2.26)$$

The method of tensor calculus [1] can be systematically applied to construct lagrangians invariant under global supersymmetry. It has found a beautiful synthesis, at least for the case of $\mathcal{N} = 1$ supersymmetry (in four dimensions), in *superspace* and *superfield* techniques [8, 9], building on the idea that supersymmetry generators act like "square roots of translations", as suggested by the superalgebra (2.5). The operators Q_α are realized in terms of derivatives (like translations) acting in a superspace extended with fermionic, Grassmann (fictitious) coordinates. A tensor calculus also exists for conformal supersymmetry (gauge theories of the superconformal algebra). It probably offers the most efficient procedure to construct supergravity theories, with local supersymmetry.⁴

A similar discussion could be made for the supermultiplet with helicities $\pm 1, \pm 1/2$, which is realized by a gauge field A_μ and a Majorana spinor λ_α , the *gaugino*. Since the gaugino includes

⁴ See section 3.5.

four off-shell fields while the gauge field has three, the off-shell supermultiplet includes one real scalar auxiliary field D . Again, the Yang-Mills and Dirac lagrangians, with their non-abelian covariantizations, provide a supersymmetric theory: the *super-Yang-Mills* (SYM) lagrangian is then simply

$$\mathcal{L}_{SYM} = -\frac{1}{4}F_{\mu\nu}^A F^{A\mu\nu} + \frac{i}{2}\lambda^A \sigma^\mu D_\mu \bar{\lambda}^A - \frac{i}{2}D_\mu \lambda^A \sigma^\mu \bar{\lambda}^a + \frac{1}{2}D^A D^A, \quad (2.27)$$

where D_μ and $F_{\mu\nu}^a$ are the usual covariant derivative and field-strength tensor of a non-abelian gauge theory⁵, and $D^A = 0$ by its field equation. The SYM theory can be coupled to chiral multiplets in an anomaly-free representation of the gauge group, to give the supersymmetric extension of gauge theories. This is the framework of the *minimal supersymmetric Standard Model* (MSSM), and of its variations.

It is then not surprising that the supermultiplet with helicity states $\pm 2, \pm 3/2$ would lead to a field theory combining the Einstein-Hilbert lagrangian of general relativity and the Rarita-Schwinger lagrangian for the helicities $\pm 3/2$: this leads to $\mathcal{N} = 1$ supergravity.

2.3. Breaking supersymmetry

With respect to standard “bosonic” symmetries, spontaneous breaking of supersymmetry [5, 6] is peculiar and difficult to achieve. Breaking a local or global symmetry is usually “parameter-controlled”, in the sense that the scalar potential which defines the ground state depends on parameters and the various possible phases correspond in general to sizeable domains in the parameter space of the theory. Selecting values of parameters selects the phase. The spontaneous breaking of supersymmetry is “algebra-controlled”: the scalar potential is a sum of positive terms, each term proportional to the square of an auxiliary field, either f_i for chiral superfields or D^A for gauge multiplets. In the renormalizable theory,

$$V = \sum_i |f_i|^2 + \frac{1}{2} \sum_A D^A D^A. \quad (2.28)$$

By their algebraic field equations, the auxiliary fields are functions of the chiral scalars z^i and if equations

$$f_i(z_i) = D^A(z_i, \bar{z}_i) = 0 \quad (2.29)$$

have a solution, this solution is the true ground state of the theory and supersymmetry is not broken.⁶ We then have an algebraic condition for supersymmetry breaking, that these equations cannot be solved.

Spontaneous supersymmetry breaking has two undesired consequences. Firstly, it generates a massless spin 1/2 Goldstone particle, the Goldstino. This can be seen for instance in the

⁵ All fields are in the adjoint representation.

⁶ The potential vanishes then at a supersymmetric minimum. But since general relativity is absent, the value of the potential at the ground state, sometimes called *vacuum energy*, does not have any physical significance.

variation of ψ_α , eq. (2.10). If f acquires a vacuum expectation value $\langle f \rangle$

$$\delta\psi_\alpha = \sqrt{2}\langle f \rangle\epsilon_\alpha + \dots \quad (2.30)$$

and the inhomogeneous term is typical of Goldstone particles. Secondly, if spontaneous supersymmetry breaking is able to lift fermion–boson mass degeneracies, it in general moves the mass of some spin zero states below their fermionic partner, in contradiction with observations. These obstructions can be avoided if the spontaneously broken supersymmetry is local, *i.e.* in a theory with gauged supersymmetry. This is a first motivation, from a low-energy perspective, for supergravity, with the idea that the residual, effective effects of the breaking will produce the mass terms necessary in a realistic particle spectrum. Models realising this idea are actually easy to construct, and they are at the origin of the various supersymmetric extensions of the Standard Model under test at LHC experiments.

3. Supergravity

Supergravity is the theory of gauged supersymmetry. The spin 1/2 parameter ϵ_α is then local, $\epsilon_\alpha(x)$, and since translation generators P_μ appear in the supersymmetry algebra (2.5), translations are local as well. This calls for coordinate diffeomorphisms (general coordinate transformations, GCT) and then for general relativity and gravitation. The theory of gauged supersymmetry is then a field theory of gravitation.

A local symmetry requires a gauge field (a connection) to construct tensors and invariants involving derivatives, needed in lagrangians and dynamical field equations. Since supercharges Q_α and parameters ϵ_α are Lorentz spinors, the gauge field of supersymmetry is a vector-spinor field, $\psi_{\alpha\mu}$, the *gravitino*, and the physical (massless) states will have helicities $\pm 3/2$. To construct a supersymmetric lagrangian, we first need a kinetic lagrangian for the gravitino: the Rarita-Schwinger lagrangian. We may then add this term to the Einstein-Hilbert lagrangian for the metric tensor and maybe find supersymmetry variations leaving the action invariant and closing the supersymmetry algebra for solutions of the field equations, on-shell.

Or we may try to directly obtain an off-shell representation of supersymmetry including the metric tensor, the gravitino and, if needed, auxiliary fields. Let us count off-shell degrees of freedom:

- The metric tensor $g_{\mu\nu}$ has ten components, four can be removed by gauge transformations (local translations) to remain with six fields, 6_B .
- The gravitino is a Majorana vector–spinor with four local gauge supersymmetries. It includes then $4 \times 4 - 4 = 12_F$ component fields.⁷
- With $6_B + 12_F$ propagating fields, $(6 + n)_B + n_F$ non-propagating auxiliary fields are then needed to construct an off-shell representation.

⁷ Alternatively, the supercharge Q_α includes four operators and one gauge field (3_F) is needed for each of them.

It turns out that several possibilities exist.⁸ Minimal supergravities have 6_B auxiliary fields ($n = 0$). *Old minimal* supergravity [26, 27] has a complex scalar (2_B) and a vector field not associated with a gauge symmetry (4_B), *new minimal* supergravity [28] has an antisymmetric tensor $B_{\mu\nu}$ and a vector field A_μ with gauge symmetries

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \quad (6-3)_B, \quad \delta A_\mu = \partial_\mu \Lambda \quad (4-1)_B. \quad (3.1)$$

And there are *non-minimal* versions with $n \neq 0$, including some versions which have supplementary propagating fields.⁹ In pure supergravity, with $\pm 3/2$ and ± 2 physical states, the formulation does not matter much since auxiliary fields anyway vanish. They play however a role when coupling supergravity to other supermultiplets: they define classes of admissible interactions which depend directly on the choice of supergravity auxiliary fields [25].

3.1. Spinors and the vierbein

From here on, x^μ denotes coordinates of a space-time with metric tensor $g_{\mu\nu}(x)$ and line element $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$.

Spinor fields live in the flat minkowskian space tangent at each point x . This tangent space would have local coordinates $\zeta^a(x)$ with line element¹⁰

$$ds^2 = \eta_{ab} d\zeta^a d\zeta^b = \eta_{ab} (\partial_\mu \zeta^a) (\partial_\nu \zeta^b) dx^\mu dx^\nu. \quad (3.2)$$

Hence, the transition from the curved space-time to the flat tangent space is given by the sixteen fields $e_\mu^a(x) = \partial_\mu \zeta^a(x)$, in other words, we can define a *vierbein* e_μ^a and its inverse e_a^μ (since the metric has an inverse) such that

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \quad e_a^\mu e_\nu^\mu = \delta_\nu^a, \quad e_\mu^a e_b^\mu = \delta_b^a. \quad (3.3)$$

At each point x , a Lorentz algebra acts in the tangent space. This local Lorentz symmetry allows to eliminate six components of the vierbein and since ds^2 in (3.2) is Lorentz invariant, the ten remaining components are the ten components of $g_{\mu\nu}$. It also acts on spinors:

$$\delta\psi(x) = \frac{1}{2}\omega_{ab}\sigma^{ab}\psi(x), \quad \sigma^{ab} = \frac{1}{2}\gamma^{ab}, \quad \gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]. \quad (3.4)$$

Covariant derivatives of spinors are provided by the local Lorentz gauge field, the *spin connection* $\omega_\mu^{ab} = -\omega_\mu^{ba}$:

$$D_\mu\psi = \partial_\mu\psi + \frac{1}{2}\omega_\mu^{ab}\sigma^{ab}\psi \quad (3.5)$$

and the Dirac lagrangian takes then the form

$$e^{-1}\mathcal{L}_\psi = i\bar{\psi}\gamma^\mu D_\mu\psi \quad (3.6)$$

where $e = \sqrt{|\det g_{\mu\nu}|} = \det e_\mu^a$ and $\gamma^\mu = e_a^\mu \gamma^a$. It is invariant under GCT and local Lorentz.

⁸ For a review and a comparison of different choices in the superconformal approach, see [25].

⁹ “ $16_B + 16_F$ ” supergravity [29, 30, 31] for instance is related to string theory compactifications [32], or to supercurrent structures [33, 34].

¹⁰ $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ is the flat Minkowski metric.

3.2. The gravitino and the Rarita-Schwinger action

The Rarita-Schwinger action describes the propagation of a vector-spinor field $\psi_{\alpha\mu}$ in the background defined by the vierbein e_μ^a and the spin connection ω_μ^{ab} . Its form is dictated by invariance requirements and reduction to the relevant helicity components only. Under the Lorentz algebra, the field $\psi_{\alpha a} = e_a^\mu \psi_{\alpha\mu}$ (α is a spinor index) transforms in the reducible representation

$$\begin{array}{ccccccc} \text{spinor} & \otimes & \text{vector} & = & \text{gravitino} & \oplus & \text{spinor}, \\ [(2, 1) \oplus (1, 2)] & \otimes & (2, 2) & = & [(3, 2) \oplus (2, 3)] & \oplus & [(1, 2) \oplus (2, 1)]. \end{array}$$

The second equation indicates the representations of the Lorentz algebra $SO(1, 3) \sim Sl(2, \mathbb{C})$, the numbers are the dimensions of $Sl(2, \mathbb{R})$ representations. The spinor part of $\psi_{\alpha a}$ is $\gamma^a \psi_{\alpha a}$ and the gravitino part is then isolated by the condition

$$(\gamma^a \psi_a)_\alpha = (\gamma^\mu \psi_\mu)_\alpha = 0 \quad \Longrightarrow \quad \tilde{\psi}_{\alpha a} = \psi_{\alpha a} - \frac{1}{4}(\gamma_a \gamma^b \psi_b)_\alpha. \quad (3.7)$$

An action for the gravitino should in principle include this projection condition in its field equations.

Consider then the following free lagrangian density, in Minkowski space (coordinates ζ^a and metric η_{ab}):

$$\mathcal{L}_0 = \frac{1}{2\kappa^2} \bar{\psi}_a \gamma^{abc} \partial_b \psi_c \quad (3.8)$$

where κ is a constant with dimension $(\text{mass})^{-1}$ and $\gamma^{abc} = \gamma^{[a} \gamma^b \gamma^c] = \frac{1}{6} \gamma^a \gamma^b \gamma^c \pm 5$ terms. The gravitino ψ_a is Majorana and \mathcal{L}_0 is hermitian. It implies the field equation

$$\gamma^{abc} \partial_b \psi_c = 0. \quad (3.9)$$

Invariance under the gauge transformation $\delta\psi_a = \partial_a \lambda$, with an arbitrary Majorana spinor λ , can be used to impose the projection condition (3.7) by solving $\gamma^a \partial_a \lambda = -\gamma^a \psi_a$. This leaves a residual gauge symmetry $\delta\psi_a = \partial_a \tilde{\lambda}$ with $\tilde{\lambda}$ solution of the massless Dirac equation $\gamma^a \partial_a \tilde{\lambda} = 0$. In the gauge $\gamma^a \psi_a = 0$, the field equation reduces to

$$\gamma^a \partial_b \psi^b = \gamma^b \partial_b \psi^a \quad (3.10)$$

and multiplication by γ_a leads to

$$\gamma^a \psi_a = 0 \quad (\text{gauge choice}), \quad \partial_b \psi^b = 0, \quad \gamma^b \partial_b \psi_a = 0 \quad (\text{Dirac}). \quad (3.11)$$

The Dirac equation indicates that the field is massless and the count of physical degrees of freedom is as follows. Starting with 16_F fields, the gauge choice and $\partial^a \psi_a = 0$ remove two spinors (8_F), the massless Dirac equation removes four of the 8_F remaining fields and finally the residual gauge symmetry eliminates one of the massless Dirac spinor (2_F) to leave only two degrees of freedom, which turn out to have helicities $\pm 3/2$.¹¹

¹¹ Plane waves $\epsilon_{\alpha a}(k)e^{-ikx}$ can be used to see this.

Coupling theory (3.8) to the background described by the vierbein e_ν^a leads to the Rarita-Schwinger lagrangian

$$e^{-1}\mathcal{L}_{RS} = \frac{1}{2\kappa^2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \tilde{D}_\nu \psi_\rho, \quad (3.12)$$

where $\gamma^{\mu\nu\rho} = e_a^\mu e_b^\nu e_c^\rho \gamma^{abc}$. In principle, since the gravitino field $\psi_{\alpha\mu}$ is a space-time vector (index μ), its covariant derivative should be

$$D_\mu \psi_{\alpha\rho} = \partial_\mu \psi_{\alpha\rho} - \Gamma_{\mu\rho}^\sigma(g) \psi_{\alpha\sigma} + \frac{1}{2} \omega_\mu^{ab} (\sigma_{ab} \psi_\rho)_\alpha \quad (3.13)$$

with affine connection

$$\Gamma_{\nu\rho}^\mu(g) = \frac{1}{2} g^{\mu\sigma} [\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}]. \quad (3.14)$$

But the antisymmetry of $\gamma^{\mu\nu\rho}$ removes the symmetric affine connection and

$$\tilde{D}_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab} \psi_\nu \quad (3.15)$$

appears in the Rarita-Schwinger lagrangian (3.12).

3.3. Simple $\mathcal{N} = 1$ supergravity

The need for spinor fields imposes a formulation of general relativity in terms of the vierbein. Since the Rarita-Schwinger also uses the spin connection, it is natural to use the *first order* (or Palatini) formalism in which e_μ^a and ω_μ^{ab} are independent fields, their relation being a field equation. One then introduces the curvature tensor of the spin connection,

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac} \omega_\nu{}^b{}_c - \omega_\nu{}^{ac} \omega_\mu{}^b{}_c = -R_{\nu\mu}{}^{ab} = -R_{\mu\nu}{}^{ba}, \quad (3.16)$$

the curvature scalar

$$R = R_{\mu\nu}{}^{ab} e_a^\mu e_b^\nu, \quad (3.17)$$

and the gravity lagrangian

$$\mathcal{L}_{grav.} = \frac{1}{2\kappa^2} e R. \quad (3.18)$$

All quantities are diffeomorphism and Lorentz tensors or scalars and the gravitational coupling constant is $\kappa = \sqrt{8\pi} M_P^{-1}$ in terms of the Planck scale $M_P \simeq 1.2 \times 10^{19}$ GeV. Under a variation of the vierbein,

$$\delta e_a^\mu \frac{\delta}{\delta e_a^\mu} \mathcal{L}_{grav.} = \frac{1}{\kappa^2} e \left[R_{\mu\nu}{}^{ab} e_b^\nu - \frac{1}{2} e_a^\mu R \right] \delta e_a^\mu \quad (3.19)$$

leads to Einstein equation, after the elimination of the spin connection.

Since the action (3.18) is quadratic in the spin connection and linear in its first derivative, the Euler-Lagrange equation for ω_μ^{ab} is algebraic only. To calculate this field equation, rewrite

$$\begin{aligned} eR &= e(e_a^\mu e_b^\nu - e_b^\mu e_a^\nu) (\partial_\mu \omega_\nu{}^{ab} + \omega_\mu{}^{ac} \omega_\nu{}^b{}_c) \\ &= -\omega_\nu{}^{ab} \partial_\mu [e(e_a^\mu e_b^\nu - e_b^\mu e_a^\nu)] + e(e_a^\mu e_b^\nu - e_b^\mu e_a^\nu) \omega_\mu{}^{ac} \omega_\nu{}^b{}_c + \text{derivative} \end{aligned} \quad (3.20)$$

and the field equation leads to

$$\begin{aligned}\omega_{\mu cd} &= -\frac{1}{2}(\partial_{\mu}e_{\nu c} - \partial_{\nu}e_{\mu c})e_d^{\nu} + \frac{1}{2}(\partial_{\mu}e_{\nu d} - \partial_{\nu}e_{\mu d})e_c^{\nu} - \frac{1}{2}e_c^{\rho}e_d^{\nu}(\partial_{\rho}e_{\nu a} - \partial_{\nu}e_{\rho a})e_{\mu}^a \\ &\equiv \omega_{\mu cd}(e).\end{aligned}\quad (3.21)$$

In terms of $g_{\mu\nu}$ and of the symmetric $\Gamma_{\mu\nu}^{\lambda}(g)$ (3.14), the Ricci tensor $R_{\mu\nu}$ corresponding to definitions (3.16) and (3.17) is

$$R_{\mu\nu} = \partial_{\rho}\Gamma_{\mu\nu}^{\rho}(g) - \partial_{\nu}\Gamma_{\rho\mu}^{\rho}(g) - \Gamma_{\sigma\mu}^{\rho}(g)\Gamma_{\rho\nu}^{\sigma}(g) + \Gamma_{\mu\nu}^{\rho}(g)\Gamma_{\rho\sigma}^{\sigma}(g) = R_{\nu\mu} \quad (3.22)$$

and $R = g^{\mu\nu}R_{\mu\nu}$.

Next, we combine the gravity and Rarita-Schwinger lagrangians:

$$\mathcal{L}_{ERS}(e_{\mu}^a, \psi_{\alpha\mu}, \omega_{\mu ab}) = \frac{1}{2\kappa^2} e \left(R + \bar{\psi}_{\mu}\gamma^{\mu\nu\rho}\tilde{D}_{\nu}\psi_{\rho} \right). \quad (3.23)$$

After elimination of the spin connection, using its algebraic field equation, we will obtain an interacting theory for the propagating vierbein and gravitino. Since the gravitino lagrangian also includes a term linear in ω_{μ}^{ab} in the Lorentz covariant derivative \tilde{D}_{μ} , its field equation and its solution are modified. As a consequence, the spin connection acquires *contorsion*,

$$\omega_{\mu}^{ab} = \omega_{\mu}^{ab}(e) + \kappa_{\mu}^{ab}, \quad (3.24)$$

and the contorsion tensor is quadratic in the gravitino field:

$$\kappa_{\mu ab} = -\frac{1}{4} [\bar{\psi}_{\mu}\gamma_a\psi_b - \bar{\psi}_{\mu}\gamma_b\psi_a + \bar{\psi}_a\gamma_{\mu}\psi_b] = -\kappa_{\mu ba}. \quad (3.25)$$

In the Rarita-Schwinger lagrangian,

$$\tilde{D}_{\mu}\psi_{\nu} - \tilde{D}_{\nu}\psi_{\mu} = \hat{D}_{\mu}\psi_{\nu} - \hat{D}_{\nu}\psi_{\mu} + 2S_{\mu\nu}^{\lambda}\psi_{\lambda}, \quad \hat{D}_{\mu}\psi_{\nu} = \partial_{\mu}\psi_{\nu} + \frac{1}{2}\omega_{\mu}^{ab}(e)\sigma_{ab}\psi_{\nu}, \quad (3.26)$$

with torsion tensor

$$S_{\mu\nu}^{\lambda} = -\frac{1}{4}\bar{\psi}_{\mu}\gamma^{\lambda}\psi_{\nu} \quad (3.27)$$

defined as the antisymmetric part of the affine connection $\Gamma_{\nu\rho}^{\mu} = \Gamma_{\nu\rho}^{\mu}(g) + S_{\nu\rho}^{\mu}$, $S_{\nu\rho}^{\mu} = -S_{\rho\nu}^{\mu}$.

Useful formulas are obtained by inserting the decomposition (3.24) into the gravity lagrangian, after some partial integrations:

$$\begin{aligned}eR &= eR(\omega(e)) - e[\kappa_{ac}^a\kappa_b^{bc} - \kappa_{abc}\kappa^{cba}] + \text{derivative}, \\ eR(\omega(e)) &= e[\omega_{ac}^a(e)\omega_b^{bc}(e) - \omega_{abc}(e)\omega^{cba}(e)] + \text{derivative},\end{aligned}\quad (3.28)$$

where $\omega_{abc}(e) = e_a^{\mu}\omega_{\mu bc}(e)$ and $\kappa_{abc} = e_a^{\mu}\kappa_{\mu bc}$ and the derivatives can be dropped in the lagrangian.

Inserting the spin connection (3.24) with contorsion (3.25) in \mathcal{L}_{ERS} leads to the lagrangian density of $\mathcal{N} = 1$ pure supergravity, as a function of e_μ^a and ψ_μ only (*second order formalism*):

$$\begin{aligned} \mathcal{L} = & \frac{1}{2\kappa^2} eR(\omega(e)) + \frac{1}{2\kappa^2} e\bar{\psi}_\mu \gamma^{\mu\nu\rho} \widehat{D}_\nu \psi_\rho \\ & + \frac{e}{32\kappa^2} \left[4(\bar{\psi}^\mu \gamma_\mu \psi_\rho)(\bar{\psi}^\nu \gamma_\nu \psi^\rho) - (\bar{\psi}_\mu \gamma_\nu \psi_\rho)(\bar{\psi}^\mu \gamma^\nu \psi^\rho) - 2(\bar{\psi}_\mu \gamma_\nu \psi_\rho)(\bar{\psi}^\mu \gamma^\rho \psi^\nu) \right], \end{aligned} \quad (3.29)$$

with now $\widehat{D}_\nu \psi_\rho = \partial_\nu \psi_\rho + \frac{1}{2} \omega_{\nu ab}(e) \sigma^{ab} \psi_\rho$.

With some efforts,¹² one can show that \mathcal{L} transforms with a derivative under the local supersymmetry variations

$$\begin{aligned} \delta e_\mu^a &= -\frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu, & \delta e_\mu^a &= \frac{1}{2} \bar{\epsilon} \gamma^\mu \psi_a, \\ \delta \psi_\mu &= D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{2} \omega_{\mu ab} \sigma^{ab} \epsilon, & \delta \bar{\psi}_\mu &= D_\mu \bar{\epsilon} = \partial_\mu \bar{\epsilon} - \frac{1}{2} \omega_{\mu ab} \bar{\epsilon} \sigma^{ab}, \end{aligned} \quad (3.30)$$

in the first-order formalism. Using eqs. (3.24) and (3.25), the gravitino supersymmetry variation acquires a fermionic nonlinear contribution through the contorsion tensor:

$$\delta \psi_\mu = \widehat{D}_\mu \epsilon - \frac{1}{8} (2\bar{\psi}_\mu \gamma_a \psi_b + \bar{\psi}_a \gamma_\mu \psi_b) \sigma^{ab} \epsilon, \quad \widehat{D}_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{2} \omega_{\mu ab}(e) \sigma^{ab} \epsilon. \quad (3.31)$$

The gravitino transforms as the gauge field of supersymmetry: the first term is the derivative of the transformation parameter ϵ .

One easily obtains the algebra

$$[\delta_1, \delta_2] e_\mu^a = \delta_1 \left[-\frac{1}{2} \bar{\epsilon}_2 \gamma^a \psi_\mu \right] - \delta_2 \left[-\frac{1}{2} \bar{\epsilon}_1 \gamma^a \psi_\mu \right] = -\frac{1}{2} D_\mu (\bar{\epsilon}_2 \gamma^a \epsilon_1), \quad (3.32)$$

the covariant derivative acting on the Lorentz vector $\bar{\epsilon}_2 \gamma^a \epsilon_1$:

$$D_\mu (\bar{\epsilon}_2 \gamma^a \epsilon_1) = \partial_\mu (\bar{\epsilon}_2 \gamma^a \epsilon_1) + \omega_\mu{}^{ab} (\bar{\epsilon}_2 \gamma_b \epsilon_1).$$

The quantity $\xi^\mu = e_\mu^a (\bar{\epsilon}_2 \gamma^a \epsilon_1)$ is then the parameter of the infinitesimal coordinate transformation predicted by the supersymmetry algebra. But as earlier mentioned, we do not expect with $6_B + 12_F$ off-shell fields that the supersymmetry algebra closes without the field equations. Auxiliary fields would be needed and, in the case of $\mathcal{N} = 1$ supergravity, exist and are not unique.

3.4. Anti-de Sitter supergravity: the cosmological constant

The natural background geometry of the supergravity lagrangian (3.29) is flat Minkowski space. Since the matter energy-momentum tensor is entirely generated by the gravitino, it vanishes in a Lorentz-invariant background.¹³ But we know that Poincaré supersymmetry is a limit case

¹² Using properties of Majorana spinors, Fierz rearrangements and partial integrations.

¹³ There could be gravitino condensates [35], recently reviewed in [36].

of the more general supersymmetry in anti-de Sitter (AdS) space-time. A supergravity theory with natural AdS background geometry should then exist.

We use the following standard definition of the cosmological constant Λ : it should contribute to Einstein equations as

$$\begin{aligned} R_{\mu\nu}^{\alpha\beta} e_b^\nu - \frac{1}{2} e_\mu^a R &= -\Lambda e_\mu^a + \text{contributions from other fields,} \\ R &= 4\Lambda + \text{contributions from other fields.} \end{aligned} \quad (3.33)$$

A positive (negative) Λ leads to de Sitter (anti-de Sitter) space-time. The field equation (3.33) follows from the lagrangian density

$$\frac{e}{\kappa^2} \left[\frac{1}{2} R - \Lambda \right]. \quad (3.34)$$

The situation is somewhat similar to the introduction of mass in the chiral supermultiplet theory discussed in section 2.2. Keeping the supersymmetry variation δe_μ^a unchanged, the gravitino variation is modified to

$$\delta\psi_\mu = D_\mu\epsilon - \frac{1}{2} M \gamma_\mu\epsilon, \quad (3.35)$$

with a real number M .¹⁴ The modified variation of the gravitino kinetic term requires the presence in \mathcal{L} of a quadratic, mass-like term for the gravitino, and the variation of this term implies the existence of a negative cosmological constant proportional to M^2 . The resulting lagrangian is then:¹⁵

$$\mathcal{L}_{AdS} = \frac{e}{\kappa^2} \left[\frac{1}{2} R + \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \tilde{D}_\nu \psi_\rho + \frac{M}{2} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu + 3M^2 \right]. \quad (3.36)$$

The gravitino mass-like term in the lagrangian density (3.36) does not mean that the gravitino is massive: the theory is supersymmetric, the graviton is massless, the gravitino must then be massless. Actually, the cosmological constant $\Lambda = -3M^2$ in Einstein equation (3.33) propagates graviton waves along light-like curves in the anti-de Sitter geometry. Similarly, the gravitino mass-like term is precisely the contribution required to propagate gravitino waves on light-like curves in this geometry. Again, we find that a positive cosmological constant is not compatible with supersymmetry, as the basic superalgebra already indicates.

3.5. The superconformal derivation, old minimal supergravity

In section 2.2, we have constructed the (globally) supersymmetric theory of a chiral multiplet (z, ψ, f) , to illustrate the role of auxiliary fields. In this paragraph, we outline a similar approach, in the context of superconformal symmetry, to construct $\mathcal{N} = 1$ supergravity with the old minimal set of auxiliary fields. The reason to consider this construction here is that this

¹⁴ Reality follows from the Majorana property of ψ_μ .

¹⁵ In terms of $\omega_\mu^{ab} = \omega_\mu^{ab}(e) + \kappa_\mu^{ab}$.

procedure generalizes very well to theories describing generic $\mathcal{N} = 1$ supersymmetric gauge theories coupled to supergravity [37, 38], however at the price of a considerable increase in technical complexity.

The Poincaré and Anti-de Sitter $\mathcal{N} = 1$ superalgebras are subalgebras of the $\mathcal{N} = 1$ superconformal algebra $SU(2, 2|1)$. Its bosonic sector $SU(2, 2) \times U(1)_R$ includes the conformal algebra $SU(2, 2) \sim SO(2, 4)$ and $U(1)_R$ symmetry, and $SO(2, 4) \supset SO(1, 3) \times SO(1, 1)$, where $SO(1, 3)$ is Lorentz algebra and $SO(1, 1)$ generates dilatation or scale or Weyl transformations. A supermultiplet of Poincaré supersymmetry is also a representation of the superconformal algebra once a $SO(1, 1)$ Weyl weight w and a $U(1)_R$ chiral charge q have been assigned to all fields and with the appropriate symmetry and supersymmetry variations. There are restrictions on these quantum numbers. For instance $w = q$ for (the lowest component of) a chiral multiplet.¹⁶

To construct $\mathcal{N} = 1$ Poincaré supergravity, one uses a chiral supermultiplet S_0 with Weyl and $U(1)_R$ weights $w = q = 1$ and component fields z_0 ($w = q = 1$), ψ_α ($w = 3/2$, $q = -1/2$) and f_0 ($w = 2$, $q = -2$).¹⁷ Its conjugate antichiral \bar{S}_0 , with weights $w = -q = 1$ has components \bar{z}_0 ($w = -q = 1$), $\bar{\psi}_{\dot{\alpha}}$ ($w = 3/2$, $q = 1/2$) and \bar{f}_0 ($w = 2$, $q = 2$). The chiral kinetic multiplet of S_0 , denoted by $T(S_0)$, analogous to expressions (2.25), has components

$$\begin{aligned} Z_0 &= \bar{f}_0 & (w = q = 2), \\ \Psi_{0\alpha} &= i(\sigma^\mu D_\mu^C \bar{\psi}_0)_\alpha & (w = 5/2, q = 1/2), \\ F_0 &= -\square^C \bar{z}_0 & (w = 3, q = -1). \end{aligned} \quad (3.37)$$

The derivatives D_μ^C and \square^C are covariant under the full local superconformal algebra. Gauge fields are:

$$\begin{aligned} e_\mu^a & \quad (\text{vierbein, translations}), & \omega_\mu^{ab} & \quad (\text{spin connection, Lorentz}), \\ \psi_{\alpha\mu} & \quad (\text{gravitino, supersymmetry}), & & \\ f_\mu^a & \quad (\text{conformal boosts}), & \phi_{\alpha\mu} & \quad (\text{special supersymmetry}), \\ b_\mu & \quad (\text{dilatation}), & A_\mu & \quad (U(1)_R). \end{aligned} \quad (3.38)$$

Constraints lead to algebraic expressions for ω_μ^{ab} (as earlier), $\phi_{\alpha\mu}$ and f_μ^a , leaving bosonic gauge fields e_μ^a , b_μ , A_μ ($6 + 3 + 3 = 12_B$) and the gravitino $\psi_{\alpha\mu}$ (12_F). The idea is then to write a superconformal lagrangian for the chiral multiplet S_0 and then to reduce the symmetry to local Poincaré symmetries, by applying appropriate gauge fixing conditions for conformal boosts,

¹⁶ Two conventions, different by the normalization of $U(1)_R$, exist in the literature: either $w = q$ as used here, or $q = \frac{2}{3}w$.

¹⁷ The weights of a supermultiplet are the weights of its “lowest” component, in our case z_0 . Apart from minor differences (metric sign, two-component spinors), we use the notation of ref. [39].

dilatation, $U(1)_R$ and special supersymmetry. These conditions assign values to

$$\begin{aligned} z_0 &: && \text{dilatation and } U(1)_R, \text{ (modulus and phase of } z_0), \\ \psi_0 &: && \text{special supersymmetry,} \\ b_\mu &: && \text{conformal boosts.} \end{aligned} \tag{3.39}$$

We are then left with the propagating fields of Poincaré supergravity, e_μ^a and $\psi_{\alpha\mu}$ and the auxiliary field A_μ (the gauge field of gauge-fixed $U(1)_R$) and f_0 (in S_0).

In global supersymmetry, we obtain invariant lagrangians by combining supermultiplets into other supermultiplets (tensor calculus), or by multiplying superfields, and by selecting the highest-dimensional component which transforms with a derivative. In the superconformal case, there are two (related) possibilities to produce invariant action terms. Firstly, we can combine supermultiplets to obtain a real supermultiplet with weights $w = 2$, $q = 0$ and take the D -density formula. For instance, symbolically,

$$\left[S_0 \bar{S}_0 \right]_D.$$

Secondly, we can combine chiral multiplets into another chiral multiplet with weights $w = q = 3$ and use the F -density formula. In our case,

$$\left[S_0 T(S_0) \right]_F \quad \text{or} \quad \left[S_0^3 \right]_F.$$

Up to conventions (and partial integration), $[S_0 \bar{S}_0]_D$ and $[S_0 T(S_0)]_F$ are equivalent.

To obtain the $\mathcal{N} = 1$ Poincaré supergravity, start then with the superconformal lagrangian

$$\mathcal{L} = -\frac{3}{2} \left[S_0 \bar{S}_0 \right]_D + \lambda \left[S_0^3 \right]_F \tag{3.40}$$

calculated using superconformal tensor calculus and the density formulas [39]. Applying the gauge fixing conditions

$$b_\mu = 0, \quad \psi_0 = 0, \tag{3.41}$$

but retaining z_0 for a moment, the superconformal lagrangian reads

$$\begin{aligned} e^{-1} \mathcal{L} &= \frac{1}{2} z_0 \bar{z}_0 \left[R(\omega(e)) + \bar{\psi}_\mu \gamma^{\mu\nu\rho} \widehat{D}_\nu \psi_\rho \right] \\ &\quad - \frac{3}{2} \left[2(D_\mu \bar{z}_0)(D^\mu z_0) + 2f_0 \bar{f}_0 \right] + \lambda \left[3z_0^2 f_0 + 3\bar{z}_0^2 \bar{f}_0 \right] + \dots \end{aligned} \tag{3.42}$$

The covariant derivative is $D_\mu z_0 = \partial_\mu z_0 - \frac{i}{2} q A_\mu z_0$ ($q = 1$) and some gravitino interactions have been omitted. With the dilatation and $U(1)_R$ gauge choice $z_0 = \kappa^{-1}$,

$$\begin{aligned} e^{-1} \mathcal{L} &= \frac{1}{2\kappa^2} \left[R(\omega(e)) + \bar{\psi}_\mu \gamma^{\mu\nu\rho} \widehat{D}_\nu \psi_\rho \right] - \frac{3}{4\kappa^2} A_\mu A^\mu - 3f_0 \bar{f}_0 \\ &\quad + \lambda \left[3z_0^2 f_0 + 3\bar{z}_0^2 \bar{f}_0 \right] + \dots \end{aligned} \tag{3.43}$$

The first line displays the auxiliary fields A_μ and f_0 of old minimal supergravity. Eliminating them leads finally to

$$e^{-1}\mathcal{L} = \frac{1}{\kappa^2} \left[\frac{1}{2}R(\omega(e)) + \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\widehat{D}_\nu\psi_\rho + 3\frac{\lambda^2}{\kappa^2} \right] + \dots, \quad (3.44)$$

with an Anti-de Sitter cosmological constant $\Lambda = -3\lambda^2\kappa^{-2}$ induced by the F -density. The related gravitino mass-like term required by supersymmetry¹⁸ is actually generated by the omitted gravitino term

$$\frac{\lambda}{4}z_0^3\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + \text{h.c.}$$

omitted in the F -density $\lambda[S_0^3]_F$.

3.6. Four-dimensional supergravities for all \mathcal{N}

Massless supermultiplets of \mathcal{N} -extended supersymmetry fall in three categories: matter multiplets with $|\text{helicities}| \leq 1/2$, gauge or Yang-Mills multiplets with $|\text{helicities}| \leq 1$ and supergravity multiplets with $|\text{helicities}| \leq 2$. The following table indicates, as a function of the number \mathcal{N} of supersymmetries:¹⁹

- The number of (on-shell) helicity states in supergravity, gauge and matter multiplets.
- Supermultiplets with scalar fields. These theories are potentially able, at ground states with nonzero scalar expectation values, to offer various patterns of symmetry and supersymmetry breakings (indicated by *).
- That while Yang-Mills multiplets can gauge all symmetry groups, chirality of the fermion representation can only be obtained in the matter (chiral) multiplet of $\mathcal{N} = 1$ supersymmetry.

SUSY	Supergravity	$ \text{Hel.} \leq 1$	$ \text{Hel.} \leq 1/2$	Chirality
$N = 1$	$2_B + 2_F$	$2_B + 2_F$	$2_B + 2_F$ *	✓
$N = 2$	$4_B + 4_F$	$4_B + 4_F$ *	$4_B + 4_F$ *	-
$N = 3$	$8_B + 8_F$	$8_B + 8_F$ *	-	-
$N = 4$	$16_B + 16_F$ *	$8_B + 8_F$ *	-	-
$N = 5$	$32_B + 32_F$ *	-	-	-
$N = 6$	$64_B + 64_F$ *	-	-	-
$N = 8$	$128_B + 128_F$ *	-	-	-

Supergravity field theories with $\mathcal{N} > 1$ are much harder to construct. The number of fields of all helicities increases fast with \mathcal{N} and off-shell representations do not exist in general. The flexibility in the choice of gauge group and matter representation decreases fast with increasing \mathcal{N} . Arbitrary representations are allowed with $\mathcal{N} = 1$ only, arbitrary non-chiral representations with $\mathcal{N} \leq 2$, and for $\mathcal{N} \geq 3$, only the adjoint representation is admitted. Arbitrary gauge

¹⁸ As in eq. (3.36).

¹⁹ The $\mathcal{N} = 7$ theory does not exist: the eighth supersymmetry arises automatically and cannot be decoupled.

groups are allowed for $\mathcal{N} \leq 4$, while for higher \mathcal{N} , gauged supergravities exclusively depend on the vector fields present in the supergravity multiplet.

The number of vector fields in the supergravity multiplet is $\mathcal{N}(\mathcal{N} - 1)/2$. The choice of possible gaugings increases then rapidly with \mathcal{N} , and also taking advantage of electric-magnetic duality. These gauged algebras cannot be identified with the compact Lie algebras used in the Standard Model or its extensions, but their breaking patterns is a subject of interest due to relations with properties found in superstrings. This is an area where fundamental developments of supergravity theories is a subject of present researches.

Chirality of the $SU(3)_c \times SU(2)_L \times U(1)_Y$ fermion representation in the Standard Model, associated with parity violation by weak interactions, appears to be a fundamental property. This has given a particular importance to $N = 1$ supergravity coupled to a Yang-Mills multiplet gauging any symmetry algebra, and allowing any anomaly-free representation of this gauged symmetry. The most general form of this theory has been derived in 1982 [37, 17].

4. $\mathcal{N} = 1$ supergravity–matter couplings

The most general interaction of chiral, gauge and supergravity $\mathcal{N} = 1$ multiplets is defined by two ingredients. Firstly, the choice of a gauge group G and of the representation R of chiral supermultiplets. The only constraint would be the absence of chiral anomaly, even if supergravity is not a quantum field theory. The representation can be chiral and one can then couple the Standard Model to $\mathcal{N} = 1$ supergravity, adding only a sector in which local supersymmetry is spontaneously broken (the *super-higgs* mechanism [40, 41, 42]²⁰) which should also mediate breaking contributions into the supersymmetric Standard Model (generation of *soft breaking terms*). Secondly, the choice of three gauge-invariant (or gauge-covariant) functions of the scalar fields in chiral supermultiplets. The first function, the real Kähler potential \mathcal{K} , defines the kinetic lagrangian of chiral superfields. The holomorphic superpotential W defines the interactions of chiral supermultiplets and the holomorphic \mathcal{F} defines the gauge kinetic (super-Yang-Mills) lagrangian.

These ingredients are known from Poincaré global supersymmetry: the most general gauged nonlinear supersymmetric sigma model is defined in terms of identical ingredients. There is however a subtlety related to the AdS case. In global supersymmetry in an AdS space-time with cosmological constant $-3M^2$, the theory depends on the combination $M\mathcal{K} + W + \overline{W}$ [43]. Then, since supergravity naturally describes AdS *and* the limiting Minkowski case, a similar phenomenon would not be a surprise: one actually finds that the supergravity theory depends on²¹

$$\mathcal{G} = \mathcal{K} + \ln(W\overline{W}), \quad (4.1)$$

²⁰ I use the lower case “higgs” for Higgs-Brout-Englert ...

²¹ This combination always used in recent literature corresponds to $-\mathcal{G}$ in ref. [37].

and of its derivatives (if the superpotential does not vanish). This fact can be loosely traced to the fact that Kähler (or R) and dilatation symmetries in the superconformal algebra are not compatible (do not commute) with the AdS superalgebra.

The best procedure to derive the lagrangian is probably²² to start from the observation that all supermultiplets of $\mathcal{N} = 1$ Poincaré or AdS supersymmetry are also representations of the $\mathcal{N} = 1$ superconformal symmetry. The method is described in full detail in the book recently published by Dan Freedman and Toine Van Proeyen [38]. We only give here a symbolic explanation and focus on the gravitino and scalar sectors. Very schematically, it is as follows:

- Consider all supermultiplets denoted as Φ^i (chiral, helicities $\pm 1/2, 0, 0$) and \mathcal{W}_α (gauge, helicities $\pm 1, \pm 1/2$) as representations of the superconformal algebra. A Weyl weight and a $U(1)_R$ charge are then associated with each supermultiplet. Supergravity fields e_μ^a and $\psi_{\alpha\mu}$ are part of superconformal gauge fields.
- Add a compensating supermultiplet which is used to gauge fix the unwanted superconformal symmetries. Here: we symbolically describe old minimal supergravity with a chiral compensating multiplet S_0 . It provides the most general coupling (up to two derivatives and up to some generalizations of minor importance) to supergravity [25].
- Use tensor calculus methods, as explained in [38, 39], to generate the locally superconformal lagrangian.
- Gauge fix superconformal symmetries absent in the Poincaré or AdS symmetries and eliminate all auxiliary fields. In this step, a gravity frame (Einstein, Jordan, string) is chosen, see below.
- Identify the ground state(s) of the theory from the analysis of the scalar potential. It defines the background geometry (the cosmological constant) and decides if supersymmetry or symmetries in general are spontaneously broken.

Symbolically, the superconformal lagrangian is represented by

$$\mathcal{L} = -\frac{3}{2} \left[S_0 \bar{S}_0 \exp \left\{ -\frac{1}{3} \mathcal{K}(\Phi^i, \bar{\Phi}_i e^A) \right\} \right]_D + \left[S_0^3 W(\Phi^i) + \frac{1}{4} \mathcal{F}(\Phi^i) \mathcal{W} \mathcal{W} \right]_F \quad (4.2)$$

where $[\dots]_D$ and $[\dots]_F$ denote the real and chiral invariant densities expressed in terms of the supermultiplet components and the superconformal gauge fields [38, 39]. The Weyl weights (scale dimensions) of the supermultiplets are $w = 1, 0, 3/2$ for S_0, Φ^i, \mathcal{W} respectively and the D and F densities apply to supermultiplets with weights 2 and 3: this (with reality and chirality) dictates the occurrences of S_0 .

²² *I.e.* in my opinion.

4.1. The scalar sector

The bosonic part of the lagrangian density (4.2) also depends on the bosonic gauge fields of the superconformal algebra, some of them being algebraic (like the spin connection) or gauge-fixed (the dilatation gauge field for instance). The Poincaré theory retains the vierbein e_μ^a or metric tensor $g_{\mu\nu}$, the gravitino ψ_μ and 6_B auxiliary fields: the gauge field A_μ of the (gauge-fixed) $U(1)_R$ superconformal symmetry and the complex scalar f_0 in the chiral compensator S_0 .

After the elimination of all auxiliary fields, a convenient expression for the scalar part of this theory is

$$e^{-1}\mathcal{L}_{scalar.} = \frac{1}{2}(z_0\bar{z}_0\mathcal{H})R - \frac{3}{4}(z_0\bar{z}_0\mathcal{H})[\partial_\mu\log(z_0\bar{z}_0\mathcal{H})]^2 + (z_0\bar{z}_0\mathcal{H})\mathcal{K}_j^i(\partial_\mu z^j)(\partial^\mu\bar{z}_i) - V_0 \quad (4.3)$$

with $\mathcal{H} = \exp[-\mathcal{K}/3]$ a function of the scalar fields z^i, \bar{z}_i and with

$$\mathcal{K}_j^i = \frac{\partial^2}{\partial z^j \partial \bar{z}_i} \mathcal{K}. \quad (4.4)$$

We have kept the complex compensating scalar z_0 with scale dimension $w = 1$: its value fixes the dilatation and $U(1)_R$ gauges. As we can see in the first term, $z_0\bar{z}_0\mathcal{H}$ defines the gravity frame, and the Einstein frame is the gauge condition

$$\frac{1}{\kappa^2} = z_0\bar{z}_0\mathcal{H} = z_0\bar{z}_0 \exp[-\mathcal{K}/3]. \quad (4.5)$$

In the Einstein frame,

$$e^{-1}\mathcal{L}_{scalar.} = \frac{1}{2\kappa^2}R + \frac{1}{\kappa^2}\mathcal{K}_j^i(\partial_\mu z^j)(\partial^\mu\bar{z}_i) - V_0. \quad (4.6)$$

The scalar fields z^i are then Kähler coordinates: their kinetic metric \mathcal{K}_j^i derives from the *Kähler potential* \mathcal{K} . Notice that this is only true in the Einstein frame.

The scalar potential is generated by the elimination of auxiliary fields f^i (chiral), D^A (gauge) and f_0 (in compensator S_0). The auxiliary fields are:²³

$$\begin{aligned} f^i &= -(z_0\bar{z}_0\mathcal{H})^{-1}\bar{z}_0^3(\mathcal{K}^{-1})_j^i\left[\bar{W}^j(\bar{z}_i) + \mathcal{K}^j\bar{W}(\bar{z}_i)\right], & W_i &= \frac{\partial W}{\partial z^i}, \\ D^A &= -[\text{Re}\mathcal{F}(z^i)]^{-1}(z_0\bar{z}_0\mathcal{H})\bar{z}_i(T_R^A)^i{}_j\mathcal{K}^j, & & \\ \tilde{f} &= f_0 - \frac{1}{3}z_0\mathcal{K}_i f^i = e^{\mathcal{K}/3}\bar{z}_0^2\bar{W}(\bar{z}_i). \end{aligned} \quad (4.7)$$

And the scalar potential reads

$$V_0 = (z_0\bar{z}_0\mathcal{H})K_j^i\bar{f}_i f^j + \frac{1}{2}\text{Re}\mathcal{F}D^A D^A - 3\mathcal{H}\tilde{f}^*\tilde{f}. \quad (4.8)$$

As in global supersymmetry, each auxiliary field f^i of a chiral or D^A of a gauge multiplet produces a positive contribution. A nonzero value at the vacuum state of the potential would

²³ T_R^A : generators of the representation R of chiral multiplets. Possible Fayet-Iliopoulos terms are omitted.

spontaneously break supersymmetry. But, in contrary to the case of global supersymmetry, the supergravity auxiliary field f^0 produces, via \tilde{f} , a negative contribution which drives the theory into Anti-de Sitter space. It is then much easier to create a potential with a ground state breaking supersymmetry, and even at zero cosmological constant. Notice once again that *unbroken supersymmetry* is always associated with either Minkowski or AdS geometry.

The final form of the scalar potential, before fixing the gravity frame is

$$V_0 = (z_0 \bar{z}_0 \mathcal{H})^2 \left\{ e^{\mathcal{K}} \mathcal{K}_j^i [W_i + \mathcal{K}_i W] [\bar{W}^j + \mathcal{K}^j \bar{W}] + \frac{1}{2 \operatorname{Re} \mathcal{F}} \sum_A [\bar{z}_i (T_R^A)^i_j \mathcal{K}^j]^2 - 3 e^{\mathcal{K}} W \bar{W} \right\}, \quad (4.9)$$

inserting expressions (4.7) into the original form (4.8). In the Einstein frame (4.5), the prefactor $(z_0 \bar{z}_0 \mathcal{H})^2$ is simply κ^{-4} .

But we may as well choose another gravity (Jordan) frame with

$$\frac{e^{-2\varphi}}{\kappa^2} = z_0 \bar{z}_0 \mathcal{H} = z_0 \bar{z}_0 \exp[-\mathcal{K}/3]. \quad (4.10)$$

One should understand φ as one of the scalar fields in the theory. The supergravity lagrangian reads then

$$\begin{aligned} \mathcal{L} = & \frac{e^{-2\varphi}}{\kappa^2} e \left[\frac{1}{2} R - 3(\partial_\mu \varphi)(\partial^\mu \varphi) + \mathcal{K}_j^i (\partial_\mu z^j)(\partial^\mu \bar{z}_i) \right] \\ & - \frac{e^{-4\varphi}}{\kappa^4} e \left[e^{\mathcal{K}} \mathcal{K}_j^i [W_i + \mathcal{K}_i W] [\bar{W}^j + \mathcal{K}^j \bar{W}] - 3 e^{\mathcal{K}} W \bar{W} \right. \\ & \left. + \frac{1}{2 \operatorname{Re} \mathcal{F}} \sum_A [\bar{z}_i (T_R^A)^i_j \mathcal{K}^j]^2 \right] + \text{gauge and fermion contributions.} \end{aligned} \quad (4.11)$$

Obviously, different frames are related by rescalings of the vierbein field.

In general, one can show that if the scalar potential has a supersymmetric stationary point, with values $\langle f^i \rangle = \langle D^A \rangle = 0$ at this point, it is then stable under small field fluctuations: a supersymmetric vacuum is stable. This does not apply to non supersymmetric stationary points. We will briefly consider some examples with spontaneously broken supersymmetry below.

4.2. The gravitino sector

In the case of pure $\mathcal{N} = 1$ supergravity, we found that a deformation of the Poincaré theory leads to a negative cosmological constant term associated with an arbitrary energy scale parameter M . These are eqs. (3.36) and (3.35).

Let us write these gravitino terms as

$$e^{-1} \mathcal{L}_{3/2} = \frac{1}{\kappa^2} \left[\frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \tilde{D}_\nu \psi_\rho + \frac{1}{2} m_{3/2} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu + 3 m_{3/2}^2 \right], \quad (4.12)$$

where $m_{3/2}$ is the quantity appearing in the mass-like term and in the negative or zero cosmological constant

$$\Lambda = -3m_{3/2}^2. \quad (4.13)$$

As explained earlier, the relative coefficient 3 is imposed by supersymmetry and ensures that gravitino waves have “light-like” propagation in the AdS geometry. In simple Anti-de Sitter supergravity, $m_{3/2}$ is the constant M appearing in the variation $\delta\psi_\mu = -\frac{1}{2}M\gamma_\mu\epsilon + \dots$

The supergravity theory coupled to gauge and matter multiplets considered previously and defined by the superconformal expression (4.2) actually contains a mass-like term for the gravitino with a field-dependent

$$m_{3/2} = \kappa^2|z_0|^3W = \frac{1}{\kappa}e^{\mathcal{K}/2}W, \quad (4.14)$$

if the theory is formulated in the Einstein frame. For a supersymmetric ground state, the expectation value of the scalar potential and the induced cosmological constants are then

$$\langle V \rangle = -\frac{3}{\kappa^4}e^{\mathcal{K}}W\bar{W} \quad \Lambda = -\kappa^2\langle e^{-1}\mathcal{L} \rangle = \kappa^2\langle V \rangle = -3\langle |m_{3/2}|^2 \rangle, \quad (4.15)$$

as required. A violation at the vacuum state of these relations would indicate spontaneous supersymmetry breaking, and generate a physical mass for the gravitino.

5. A no-scale model, dilaton supergravity

Breaking spontaneously supersymmetry in supergravity is easy. A superpotential is first of all needed. Here is an example taken in the class of “no-scale” models [44, 45]. Consider a theory describing two chiral supermultiplets with scalar fields S and T , defined by

$$\begin{aligned} \mathcal{K} &= -n\ln(T + \bar{T}) + \widehat{\mathcal{K}}(S, \bar{S}) && \text{Kähler potential,} \\ W &= W(S) && \text{Superpotential.} \end{aligned} \quad (5.16)$$

The scalar potential reads (using $\kappa = 1$ in Einstein frame)

$$V = (T + \bar{T})^{-n} e^{\widehat{\mathcal{K}}} \left[\widehat{\mathcal{K}}_{S\bar{S}}^{-1} |W_S + \widehat{\mathcal{K}}_S W|^2 + (n-3)W\bar{W} \right]. \quad (5.17)$$

The value $n = 3$ is particular: the scalar potential is positive or zero, a solution of

$$W_S + \widehat{\mathcal{K}}_S W = 0, \quad (5.18)$$

if it exists, is an absolute minimum and a stable ground state in Minkowski geometry. This minimum condition fixes in general the value of $\langle S \rangle$ and cancels the auxiliary field f_S : supersymmetry is not broken by S . But the value of $\langle T \rangle$ remains arbitrary and the auxiliary field

$$f_T = (T + \bar{T})^{-1/2} e^{\widehat{\mathcal{K}}/2} \bar{W} \quad (5.19)$$

does not vanish if the superpotential is not zero at the ground state. In this case, supersymmetry is broken by T and, since the value of $\langle T \rangle$ is not fixed by the potential, the scale of supersymmetry breaking is arbitrary and unrelated to any fixed scale of the theory. The gravitino mass

$$m_{3/2} = \langle e^{\mathcal{K}/2} W \rangle = \langle (T + \bar{T})^{-3/2} e^{\hat{\mathcal{K}}/2} W \rangle \quad (5.20)$$

is the order parameter of supersymmetry breaking. Since supersymmetry is broken with zero cosmological constant, $m_{3/2}$ is the true gravitino mass.

The Kähler potential $-3 \ln(T + \bar{T})$ commonly appears in compactifications from ten dimensions, T being the volume modulus of the compact space. The S field in a superstring context could arise from the dilaton scalar (the string coupling field) partner of the metric $g_{\mu\nu}$ and of an antisymmetric tensor $B_{\mu\nu}$. In this case however, the superpotential does not depend on S and eq. (5.18) cannot be solved. The consequence is in general the absence of a ground state. Non-perturbative corrections are necessary to create the dependence on S and a Minkowskian ground state.

It is maybe of interest to examine “dilaton supergravity” more precisely. In four space-time dimensions, an antisymmetric tensor $B_{\mu\nu}$ with gauge invariance and (free) wave equation

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad \partial^\mu H_{\mu\nu\rho} = 0, \quad H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]} \quad (5.21)$$

describes 3_B off-shell (since gauge invariance removes three fields) and 1_B on-shell states. The single massless on-shell state has of course helicity zero. Combined with a real scalar C and a Majorana spinor χ_α , the three fields form an off-shell representation of supersymmetry without any auxiliary field. Actually, at the level of global $\mathcal{N} = 1$ supersymmetry, the variations

$$\begin{aligned} \delta C &= i\epsilon\chi - i\bar{\epsilon}\bar{\chi}, & \delta B_{\mu\nu} &= \frac{i}{2\sqrt{2}} \left(\bar{\epsilon}[\bar{\sigma}^\mu, \sigma^\nu]\bar{\chi} - \epsilon[\sigma^\mu, \bar{\sigma}^\nu]\chi \right), \\ \delta\chi_\alpha &= -i(\sigma^\mu\bar{\epsilon})_\alpha \left(\frac{1}{\sqrt{2}} \epsilon_{\mu\nu\rho\sigma} H^{\nu\rho\sigma} - i\partial_\mu C \right) \end{aligned} \quad (5.22)$$

close the supersymmetry algebra without using field equations. This *linear* multiplet can be coupled to supergravity and is also a representation of the superconformal algebra with Weyl weight $w = 2$ for C . The variations (5.22) indicate that a constant C does not break supersymmetry and that the spinor χ cannot be a Goldstino spinor. Hence a linear multiplet is not a source for supersymmetry breaking and it does not contribute to the scalar potential.

A duality transformation can always, in principle, transform the antisymmetric tensor with gauge symmetry into a real scalar τ with shift symmetry. Schematically,

$$H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]} \quad \longleftrightarrow \quad \epsilon_{\mu\nu\rho\sigma} \partial^\sigma \tau \quad (5.23)$$

with symmetry $\delta\tau = \text{constant}$. In other words, the dual chiral theory has a Kähler potential $\mathcal{K}(S + \bar{S})$, $\tau = \text{Im } S$.

On one side of the duality, the linear L does not have an auxiliary field and cannot break supersymmetry. On the other side, the chiral dual S has an auxiliary f_S in principle able to induce supersymmetry breaking. In global supersymmetry, what happens is that either f_S is identically zero or, if other chiral multiplets are present, f_S is a linear combination of the other auxiliary fields f^i . In any case, f_S does not provide an *independent* source of supersymmetry breaking. Hence, models with supersymmetry breaking induced exclusively by the chiral S dual to the string dilaton and the antisymmetric tensor $B_{\mu\nu}$ do not exist.

In supergravity with a linear multiplet, the situation is different. If the superpotential is constant (this is meaningless in global supersymmetry), there is always a scalar potential generated by the supergravity auxiliary field f_0 (which however cannot break supersymmetry). In the dual version with the chiral S , the auxiliary field f_S is proportional to f_0 , but f_S is now in principle able to break supersymmetry, since $\bar{f}_S \sim \mathcal{K}_T W$ is not zero in general. What happens now if that the potential is in general unstable if $W \neq 0$: this is the runaway behaviour naturally expected from the dilaton, which in turn raises the problem of its stabilisation. Notice that supergravity with the Kähler potential $-3\ln(T + \bar{T})$, which has identically zero potential and broken supersymmetry if $W \neq 0$, *cannot* be transformed into a linear superfield: the supersymmetric duality transformation between S and L does not exist for precisely this Kähler potential. If other chiral multiplets are present, f_S is a linear combination of the chiral auxiliary fields f^i and of the supergravity f_0 . Again, f_S is not an independent source of breaking. But stability remains a non simple issue.

The simplest Calabi-Yau compactifications of heterotic superstrings with $\mathcal{N} = 1$ four-dimensional supersymmetry provide a concrete realization of the mechanisms described in this section. Retaining the overall (complex) volume modulus T in a chiral multiplet and the dilaton-antisymmetric tensor supermultiplet, we have two dual descriptions,

$$L \text{ (linear)} \quad \text{and} \quad T \quad \iff \quad S \text{ (chiral)} \quad \text{and} \quad T.$$

In the chiral version, at lowest order of string perturbation theory, the Kähler potential defining the effective supergravity is of no-scale type [46, 47, 48, 49],

$$\mathcal{K} = -\ln(S + \bar{S}) - 3\ln(T + \bar{T}). \quad (5.24)$$

There are two primary sources for a superpotential [46, 48]. Firstly, at the perturbative level, ten-dimensional sixteen-supercharge supergravity has a gauge-invariant three-form H_{MNP} . It generates $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ in the dilaton sector and an order parameter H_{ijk} , leading to a constant superpotential $W = \langle H \rangle$, since the Calabi-Yau space has a holomorphic three-form. At this stage, the minimum equation (5.18) cannot be solved and the potential

$$V = (T + \bar{T})^{-3}(S + \bar{S})W\bar{W} \quad (5.25)$$

is unstable (“run-away behaviour”), as expected from dilaton supergravity. The second source of superpotential is nonperturbative gaugino condensation in a hidden gauge sector, and it is described to a good approximation by the addition to the constant $\langle H \rangle$ of a term of the form ae^{bS} :

$$W(S) = \langle H \rangle + ae^{bS}. \quad (5.26)$$

The equation $f_S = 0$ can now be solved and supersymmetry breaks in the T sector, with Minkowski geometry and order parameter (4.14) controlled by the arbitrary value of $T + \bar{T}$.

It should however be observed that heterotic string perturbation theory is organized in powers of the linear supermultiplet L [50], with its scalar C directly related to the string dilaton, and not as an expansion in S . Since the superpotential cannot depend on L , the description of gaugino condensation uses then a different effective lagrangian, with almost identical phenomenology as long as supersymmetry breaking and scales are concerned [51]. In any case, the $S \sim L$ duality is a useful tool in the effective description of the universal string dilaton sector.

6. Final words

After (almost) forty years, supergravity has certainly found its way into the toolbox of theoretical physicists. Its development is far from complete and gauge-gravity dualities, in particular, have recently suggested new directions and research projects. In the context of superstring compactifications, finding methods to classify supergravity gaugings, and the corresponding symmetry and supersymmetry breaking patterns would allow a better control of flux compactifications, with supergravity providing in this case the “bottom-up” approach to select candidate fluxes from specific low-energy properties. In the unification programme, supergravity cannot claim to be the fundamental theory. But it is certainly on the “supersymmetric path” to quantum gravity. And there are open questions concerning the quantum status of the maximal $\mathcal{N} = 8$ supergravity. Supersymmetric theories have exceptional ultraviolet properties. The maximal ($\mathcal{N} = 4$) super-Yang-Mills theory is known to be finite. Brilliant works have shown that divergences plausible in $\mathcal{N} = 8$ arise in perturbation theory at higher orders than expected. This suggests that the $\mathcal{N} = 8$ theory could maybe display ingredients of a consistent quantum gravity, in a much simpler theoretical framework than superstrings.

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