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REGULARITY CONDITIONS IN THE REALISABILITY PROBLEM WITH APPLICATIONS TO POINT PROCESSES AND RANDOM CLOSED SETS

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We study existence of random elements with partially specified distributions. The technique relies on the existence of a positive extension for linear functionals accompanied by additional conditions that ensure the regularity of the extension needed for interpreting it as a probability measure. It is shown in which case the extension can be chosen to possess some invariance properties.

The results are applied to the existence of point processes with given correlation measure and random closed sets with given two-point covering function or contact distribution function. It is shown that the regularity condition can be efficiently checked in many cases in order to ensure that the obtained point processes are indeed locally finite and random sets have closed realisations.

1. Introduction. Defining the distribution of a random element ξ in a topological space $\mathcal{X}$ is equivalent to specialising the expected values for all bounded continuous functions $\mathsf{g}(\xi)$. These expected values define a linear functional $\Phi(\mathsf{g}) = \mathbf{E}\mathsf{g}(\xi)$ on the space of bounded continuous functions $\mathsf{g}:\mathcal{X} \mapsto \mathbb{R}$. It is well known that a functional Φ indeed corresponds to a random element if and only if Φ is positive [i.e., $\Phi(\mathsf{g}) \geq 0$ if g is nonnegative] and upper semi-continuous [i.e., $\Phi(\mathsf{g}_n) \downarrow 0$ if $\mathsf{g}_n \downarrow 0$]; see, for example, [36].

Below we consider the case of functional Φ defined only on some functions on $\mathcal{X}$ and address the *realisability* of Φ, that is, the mere existence of a random element ξ such that $\Phi(\mathsf{g}) = \mathbf{E}\mathsf{g}(\xi)$ for g from the chosen family G of functions. The uniqueness is not on the agenda, since typically the family G will not suffice to uniquely specify the distribution of ξ. A classical example of this setting is the existence of a probability distribution with given marginals; see [9]. The present paper focuses on some geometric instances of the problem. We will see that in most cases the answer to the existence problem consists of the two main steps.

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1. (*Positivity*) Checking the positivity condition on Φ—in most cases this requires checking a system of inequalities, which is a serious (but unavoidable) computational burden.

2. (*Regularity*) Ensuring that the extended functional is regular (namely, upper semi-continuous) and so defines a σ-additive measure.

The first step ensures that it is possible to extend functional Φ positively from a certain family of functions to a wider family. In this work, we put the emphasis on the latter step—checking the regularity condition, leaving aside the computational difficulties arising from validating the positivity assumption.

The use of positive extension techniques (that goes back to Kantorovitch) in the framework of stochastic geometry was pioneered by Kuna, Lebowitz and Speer [13] in application to point processes, which greatly inspired the current work. In this paper, we establish the general nature of an idea proposed in [13] and show how it leads to various further realisability results. The new idea is to introduce an additional function, what we call the regularity modulus, and to formulate sufficient and necessary conditions in terms of a positive extension of a functional onto the linear space containing the regularity modulus and requiring only a priori integrability of the regularity modulus.

We concentrate on two basic examples of the realisability problem: the existence of point processes with given correlation (factorial moment) measure and the existence of a random closed set with given two-point coverage probabilities or contact distribution functions. The introduction to the realisability issue for point processes is available in several papers by Kuna, Lebowitz and Speer [12, 13]; see also Section 3 of this paper. The realisability problem for random closed sets has been widely studied in physics and material science literature; see [8, 17, 31, 33, 34] and in particular the comprehensive monograph by Torquato [32] and a recent survey by Quintanilla [24]. If ξ is a random closed set (see Section 4 for formal definitions) in a locally compact metric space $\mathbb{X}$, its *one-point covering functions* is defined by

$$p_x = \mathbf{P}\{x \in \xi\}, \qquad x \in \mathbb{X}.$$

It is easy to characterise all one-point covering functions of random closed sets as follows.

THEOREM 1.1. *A function p_x, $x \in \mathbb{X}$, with values in* $[0, 1]$ *is the one-point covering function of a random closed set if and only if p is upper semi-continuous.*

The upper semi-continuity of the one-point covering function of a random closed set ξ is a straightforward consequence of the upper semi-continuity property of the capacity functional of a random closed set; see [22], Section 1.1.2. Conversely, the function p from the theorem is realised (e.g.) as the one-point covering function of the random set $\xi = \{x : p_x \geq v\}$ where v is a uniformly distributed variable (the details are left to the reader).

It is considerably more complicated to characterise *two-point covering functions*

$$p_{x,y} = \mathbf{P}\{x, y \in \xi\}, \qquad x, y \in \mathbb{X}.$$

In view of applications to modelling of random media, it is often assumed that ξ is a stationary set in $\mathbb{R}^d$, so that the one-point covering function is constant and the two-point covering function $p_{x,y}$ depends only on $x - y$. Since a random closed set can be considered as an upper semi-continuous indicator function, the realisability problem for the two-point covering function can be rephrased as follows:

> Characterise covariance functions of (stationary) upper semi-continuous random functions with values in $\{0, 1\}$.

These covariances are obviously a sub-family of positive semi-definite functions. Without the upper semi-continuity requirement, this problem, of combinatorial nature, was solved by McMillan [21] and Shepp [27, 28] using the extension argument from [9]. More exactly, they normalised indicators by letting them take values $+1$ or -1 and assumed that the mean is zero. Their result does not rely on the topological structure of the underlying space and so does not necessarily lead to an upper semi-continuous indicator function.

EXAMPLE 1.2. Let $p_{x,y} = \frac{1}{4}$ and let $p_x = \frac{1}{2}$ for all $x, y \in \mathbb{R}$. While this two-point covering function corresponds, for example, to the indicator field with independent values, it cannot be obtained as the two-point covering function of a random closed set; see Proposition 4.4.

Even leaving aside the upper semi-continuity property, the McMillan–Shepp condition involves a family of corner-positive matrices, which is poorly understood. As a result, its practical use to check the realisability for random media is rather limited. A number of authors have attempted to come up with simpler (but only necessary) conditions; see, for example, [8, 19, 24, 33]. Another set of conditions for joint distributions of binary random variables is formulated in [26] in terms of the corresponding copulas.

The realisability problem can be also posed for point processes in terms of their moment measures. In case of moment measures of arbitrary order, it has been solved by Lenard [15, 16]. The case of moment measures up to the second order has been studied by Kuna, Lebowitz and Speer [12], whose recent paper [13] contains (among other results) a *complete* solution of this realisability problem for point processes with finite third-order moments and hard-core type conditions with fixed exclusion distance. The results of [13] can be extended to higher order moment measures, as was explicitly indicated there. Again, the positivity condition of [13] is extremely difficult to verify, even more complicated than the original condition for point processes because of new polynomial functionals involved in the positivity condition.

The paper is organised as follows. Section 2 presents a series of general results on regular extensions and also invariant extensions (relevant for the existence of stationary random elements). These results form the theoretical backbone of our study, and are new even in the abstract setting of extending general positive linear functionals.

Section 3 presents a number of realisability conditions for correlation measures of point processes that considerably extend the results of [13] by relaxing the moment and hardcore conditions. One of our most important results is Theorem 3.3 that shows how to split the positivity and regularity conditions, so that the latter can be efficiently checked. The importance of the packing number in relation to realisability conditions for hard-core point processes is also explained.

Section 4 deals with the realisability problem for two-point covering probabilities of random sets. The closedness of the corresponding random set can be ensured by imposing appropriate regularity conditions. Section 5 addresses a further variant of the realisability problem that involves contact distribution functions of random sets.

The notational convention is that the carrier space is denoted as $\mathbb{X}$ (e.g., $\mathbb{R}$, $\mathbb{R}^d$), points in the carrier space are x, y, subsets of carrier spaces are denoted by capitals X, Y, F (while Y is reserved for counting measures identified with corresponding support sets), the families of sets (or families of counting measures) as $\mathcal{X}, \mathcal{N}, \mathcal{F}$ (while in Section 2 $\mathcal{X}$ denotes also a rather general space) and random element in these spaces (random sets or point processes) as ξ, real functions acting on $\mathcal{X}, \mathcal{N}, \mathcal{F}$ are g, v and families of such functions are G, E, V, a functional on G, E, V is denoted by Φ, real numbers are denoted by t, r, λ, while c denotes a generic constant and at the same time the corresponding constant function.

2. Extending positive functionals. Fundamental results about the extension of positive operators form the heart of our main results, and are necessary to understand the machinery of the proofs. Nevertheless, the results of the subsequent sections can be understood without Section 2, with the exception of Definition 2.5.

2.1. *General extension theorems*. Consider a *vector lattice* E, that is a linear space with a partial order and such that for any $\mathsf{v}_1, \mathsf{v}_2 \in \mathsf{E}$ their maximum $\mathsf{v}_1 \vee \mathsf{v}_2$ also belongs to E. The absolute value $|\mathsf{v}|$ of v is defined as the sum of $\mathsf{v} \vee 0$ and $(-\mathsf{v}) \vee 0$.

Let G be a *vector subspace* of E, which is not necessarily a lattice itself, that is G may be not closed with respect to the maximum operation. We say that G *majorises* E if each $\mathsf{v} \in \mathsf{E}$ satisfies $|\mathsf{v}| \leq \mathsf{g}$ for some $\mathsf{g} \in \mathsf{G}$. A real-valued functional Φ defined on E (resp., G) is said to be *positive* if $\Phi(\mathsf{v}) \geq 0$ whenever $\mathsf{v} \geq 0$ and $\mathsf{v} \in \mathsf{E}$ (resp., $\mathsf{v} \in \mathsf{G}$). A functional defined on E is said to be an *extension* of $\Phi : \mathsf{G} \mapsto \mathbb{R}$ if it coincides with Φ on G. The extended Φ is always denoted by the same letter. The following result about extension of positive functionals goes back to Kantorovich.

THEOREM 2.1 (See [1], Theorem 8.12 and [35], Theorem X.3.1). *Assume that* G *is a majorising vector subspace of a vector lattice* E. *Then each positive linear functional on* G *admits a positive extension on the whole* E.

If G is a lattice itself, then it is possible to gain much more control over the extension of Φ, for example, a continuous functional admits a continuous extension, see [35], Section X.5. On the contrary, very little is known about regularity properties of the extension if G is not a lattice.

In the following, we assume that G and E are families of functions g on a certain space $\mathcal{X}$. If G contains constant functions, the positivity of Φ over G can be equivalently formulated as

$$\Phi(\mathsf{g}) \geq \inf_{X\in\mathcal{X}} \mathsf{g}(X). \tag{2.1}$$

This equivalence is a particular case of the following result for $\chi = 0$ [replace g with $-\mathsf{g}$ in (2.2)].

PROPOSITION 2.2. *Assume that vector space* G *contains constant functions and denote by* $\mathsf{G}\setminus\mathbb{R}$ *the family of nonconstant functions from* G. *If* χ *is any nonnegative function on* $\mathcal{X}$, *then a linear functional* Φ *on* G *admits a positive extension on* $\mathsf{G}+\mathbb{R}\chi$ *with* $\Phi(\chi)=r$ *if and only if*

$$r = \sup_{\mathsf{g}\in\mathsf{G},\mathsf{g}\leq\chi} \Phi(\mathsf{g}) = \sup_{\mathsf{g}\in\mathsf{G}\setminus\mathbb{R}} \inf_{X\in\mathcal{X}} \big[\chi(X)-\mathsf{g}(X)\big] + \Phi(\mathsf{g}) < \infty. \tag{2.2}$$

PROOF. Since every element of G can be written $c+\mathsf{g}$ with $\mathsf{g}\in\mathsf{G}\setminus\mathbb{R}$ and $c\in\mathbb{R}$, the left-hand side of (2.2) equals

$$r = \sup_{\mathsf{g}\in\mathsf{G}\setminus\mathbb{R}} \sup_{c\in\mathbb{R}:\, c+\mathsf{g}\leq\chi} c+\Phi(\mathsf{g}) = \sup_{\mathsf{g}\in\mathsf{G}} c_{\mathsf{g}} + \Phi(\mathsf{g}),$$

where $c_{\mathsf{g}} = \inf_{X\in\mathcal{X}}(\chi-\mathsf{g})(X)$ is the largest c such that $c+\mathsf{g}\leq\chi$, which yields the equality in (2.2).

The necessity of (2.2) is straightforward because $r\leq\Phi(\chi)<\infty$. For the sufficiency, assume that (2.2) holds. The proof consists in checking that assigning the value $\Phi(\chi)=r$ yields a positive extension on $\mathsf{G}+\mathbb{R}\chi$. Let us first prove that Φ is positive on G. If some $\mathsf{g}\leq 0$ satisfies $\Phi(\mathsf{g})>0$, then $\Phi(t\mathsf{g})\uparrow\infty$ as $t\to\infty$ whereas $t\mathsf{g}\leq\chi$, which contradicts (2.2).

Let $\mathsf{g}+\lambda\chi\geq 0$ for $\lambda\neq 0$ and $\mathsf{g}\in\mathsf{G}$. If $\lambda>0$, then $-\lambda^{-1}\mathsf{g}\leq\chi$, whence $\Phi(-\lambda^{-1}\mathsf{g})\leq r$ and $\Phi(\mathsf{g}+\lambda\chi)\geq-\lambda r+\lambda\Phi(\chi)=0$. If $\lambda<0$, $-\lambda^{-1}\mathsf{g}\geq\chi$ whence $-\lambda^{-1}\mathsf{g}$ is larger than any $\mathsf{g}'\leq\chi$, and

$$\Phi(-\lambda^{-1}\mathsf{g}) \geq \sup_{\mathsf{g}'\in\mathsf{G},\mathsf{g}'\leq\chi} \Phi(\mathsf{g}') = r$$

by monotonicity of Φ on G. Hence, $\Phi(\mathsf{g}+\lambda\chi)\geq-\lambda r+\lambda\Phi(\chi)=0$. $\square$

The advantage of the latter condition in (2.2) consists in the explicit reference to the space $\mathcal{X}$ where random elements lie instead of checking the inequality $g\leq\chi$.

2.2. *Regularity conditions and distributions of random elements*. Let E be a certain family of functions $\mathsf{v}:\mathcal{X}\mapsto\mathbb{R}$ defined on a space $\mathcal{X}$ with lattice operation being the pointwise maximum and the corresponding partial order.

THEOREM 2.3 (Daniell, see [4], Section 4.5 and [11], Theorem 14.1). *Let a vector lattice E consist of real-valued functions on $\mathcal{X}$ and let E contain constants. If Φ is a positive functional on E such that $\Phi(\mathsf{v}_n)\downarrow 0$ for each sequence $\mathsf{v}_n\downarrow 0$ and $\Phi(1)=1$, then there exists a unique random element ξ in $\mathcal{X}$, measurable with respect to the σ-algebra generated by all functions from E, such that $\Phi(\mathsf{v})=\mathbf{E}\mathsf{v}(\xi)$ for all $\mathsf{v}\in\mathsf{E}$.*

In view of the positivity of Φ, the condition imposed on Φ is equivalent to its upper semi-continuity on E. In this paper, we start with a functional Φ defined on a vector sub-space $\mathsf{G}\subset\mathsf{E}$ and discuss the existence of a random element $\xi\in\mathcal{X}$ such that $\Phi(\mathsf{g})=\mathbf{E}\mathsf{g}(\xi)$ for all $\mathsf{g}\in\mathsf{G}$. In this case, Φ is said to be *realisable* as a probability distribution on $\mathcal{X}$.

ASSUMPTION 2.4. The vector space G of functions on $\mathcal{X}$ contains constants and, for each $\mathsf{g}_1,\mathsf{g}_2\in\mathsf{G}$, there exists a $\mathsf{g}\in\mathsf{G}$ such that $(\mathsf{g}_1\vee\mathsf{g}_2)\le\mathsf{g}$.

From now on assume that $\mathcal{X}$ is a completely regular topological space, that is, each closed set and each singleton disjoint from it can be separated by a continuous function.

DEFINITION 2.5. Given a vector space G of functions on $\mathcal{X}$, a *regularity modulus* on $\mathcal{X}$ is a lower semi-continuous function $\chi:\mathcal{X}\mapsto[0,\infty]$ such that

$$\mathcal{H}_{\mathsf{g}}=\{X\in\mathcal{X}:\chi(X)\le\mathsf{g}(X)\} \tag{2.3}$$

is relatively compact for each $\mathsf{g}\in\mathsf{G}$ (if all $\mathsf{g}\in\mathsf{G}$ are bounded, χ is a regularity modulus if and only if it has compact level sets).

Examples of regularity moduli are given in Sections 3 and 4. A measurable function $\mathsf{v}:\mathcal{X}\mapsto\mathbb{R}$ is said to be *χ-regular* if v is continuous on $\mathcal{H}_{\mathsf{g}}$ for each g in G. Each continuous function is trivially χ-regular. The proof of the following central result is based on the ideas from the proof of [13], Theorem 3.14. It should be noted that our result entails not only the realisability, but also provides a bound for the expected value of the regularity modulus. It also holds on not necessarily completely regular space $\mathcal{X}$ if the regularity modulus is continuous or otherwise without the explicit bound on $\mathbf{E}\chi(\xi)$.

THEOREM 2.6. *Consider a vector space G of functions on $\mathcal{X}$ satisfying Assumption 2.4 and such that each g from G is χ-regular for a regularity modulus χ.*

Let Φ be a linear functional on G *with* $\Phi(1)=1$. *Then, for any given* $r \geq 0$, *there exists a Borel random element* ξ *in* $\mathcal{X}$ *such that*

$$\begin{cases} \mathbf{E}\mathsf{g}(\xi)=\Phi(\mathsf{g}) & \textit{for all } \mathsf{g}\in\mathsf{G}, \\ \mathbf{E}\chi(\xi)\leq r, \end{cases} \tag{2.4}$$

if and only if

$$\sup_{\mathsf{g}\in\mathsf{G},\mathsf{g}\leq\chi} \Phi(\mathsf{g}) \leq r. \tag{2.5}$$

PROOF. Condition (2.5) is necessary because $\mathsf{g}\leq\chi$ implies $\Phi(\mathsf{g})=\mathbf{E}\mathsf{g}(\xi)\leq \mathbf{E}\chi(\xi)\leq r$.

Sufficiency. Let E be the family of all χ-regular functions v that satisfy $\mathsf{v}\leq\mathsf{g}$ for some $\mathsf{g}\in\mathsf{G}$. Each function $\mathsf{v}\in\mathsf{E}$ is Borel measurable. Note that E contains all bounded continuous functions that generate the Baire σ-algebra on $\mathcal{X}$ being in general a sub-σ-algebra of the Borel one. For each $\mathsf{v}_1,\mathsf{v}_2\in\mathsf{E}$, the function $\mathsf{v}_1\vee\mathsf{v}_2$ is χ-regular and is majorised by $\mathsf{g}_1\vee\mathsf{g}_2$, where $\mathsf{g}_1,\mathsf{g}_2\in\mathsf{G}$ majorise v_1 and v_2, respectively. In view of Assumption 2.4, E is a lattice.

Without loss of generality, assume that the supremum in (2.5) equals r. By Proposition 2.2, Φ is positive on G and can be positively extended onto $\mathsf{G}+\mathbb{R}\chi$ with $\Phi(\chi)=r$, and further on to $\mathsf{E}+\mathbb{R}\chi$ by Theorem 2.1. It remains to prove that the obtained extension satisfies conditions of Theorem 2.3. For that, we use an argument similar to that of [13]. First, restrict the obtained functional Φ onto E. Assume that χ is strictly positive. Consider a sequence $\{\mathsf{v}_n, n\geq 1\}\subset\mathsf{E}$ such that $\mathsf{v}_n\downarrow 0$. For each n, let g_n be a function of G such that $\mathsf{v}_n\leq\mathsf{g}_n$. Take $\varepsilon>0$. Then $\mathcal{K}_n=\{X:\mathsf{v}_n(X)\geq\varepsilon\chi(X)\}$ is a subset of relatively compact $\mathcal{H}_{\mathsf{g}_n/\varepsilon}$, since χ is a regularity modulus. Since v_n is continuous on $\mathcal{H}_{\mathsf{g}_n/\varepsilon}$, the set $\mathcal{K}_n$ is closed and, therefore, compact. The pointwise convergence $\mathsf{v}_n\downarrow 0$ yields that $\bigcap_n\mathcal{K}_n=\varnothing$ (recall that χ is strictly positive). Since $\{\mathcal{K}_n\}$ is a decreasing sequence of compact sets, $\mathcal{K}_{n_0}=\varnothing$ for some n_0, whence $\mathsf{v}_n(X)<\varepsilon\chi(X)$ for sufficiently large n. The positivity of Φ on $\mathsf{E}+\mathbb{R}\chi$ implies $\Phi(\mathsf{v}_n)\leq\varepsilon\Phi(\chi)=\varepsilon r$, whence $\Phi(\mathsf{v}_n)\downarrow 0$. Theorem 2.3 yields the existence of a random element ξ in $\mathcal{X}$ such that $\Phi(\mathsf{v})=\mathbf{E}\mathsf{v}(\xi)$ for all $\mathsf{v}\in\mathsf{E}$.

Since χ is lower semi-continuous and $\mathcal{X}$ is completely regular, it can be pointwisely approximated from below by a sequence $\{\mathsf{v}_n\}$ of nonnegative continuous functions; see [2], Chapter 9. Then $\tilde{\mathsf{v}}_n=\min(n,\mathsf{v}_n)$ belongs to E and also approximates χ from below, so that $\mathbf{E}\tilde{\mathsf{v}}_n(\xi)=\Phi(\tilde{\mathsf{v}}_n)\leq\Phi(\chi)=r$, while the monotone convergence theorem yields

$$\mathbf{E}\chi(\xi)=\lim_{n\to\infty}\mathbf{E}\tilde{\mathsf{v}}_n(\xi)\leq r.$$

If χ is not strictly positive, it suffices to apply the above argument to $\chi'=1+\chi$ and use the linearity of Φ. $\square$

Condition (2.5), equivalent to (2.2), is expressed solely in terms of the values taken by Φ on G and, therefore, yields a self-contained solution of the realisability problem. It is not easy to check in general, but if χ can be approximated by functions $\chi_n \in \mathsf{G}$, $n \geq 1$, then it is possible to "split" (2.5) into the positivity condition on Φ and the uniform boundedness of $\Phi(\chi_n)$, $n \geq 1$. This idea is used successfully in several different frameworks, which justify the abstract setting of Theorem 2.6: in Section 3 for point processes (see Theorem 3.1), in Section 4.4 for random closed sets (see Theorem 4.9) and in [6] in the framework of random measurable sets with the regularity modulus being the perimeter of a set.

The realisability problem is particularly simple if $\mathcal{X}$ is compact and G consists of continuous functions. Then, for identically vanishing χ, Theorem 2.6 yields the following result, which is similar to the Riesz–Markov theorem; see [11].

COROLLARY 2.7. *Let $\mathcal{X}$ be a compact space with its Borel σ-algebra. Consider a vector space G containing constants such that each $\mathsf{g} \in \mathsf{G}$ is continuous and a map $\Phi : \mathsf{G} \mapsto \mathbb{R}$ such that $\Phi(1) = 1$. Then there exists a random element ξ in $\mathcal{X}$ such that $\mathbf{E}\mathsf{g}(\xi) = \Phi(\mathsf{g})$ for all $\mathsf{g} \in \mathsf{G}$ if and only if Φ is a linear positive functional on G.*

It should be noted that the complete regularity assumption on $\mathcal{X}$ is not needed if the regularity modulus χ is continuous.

2.3. *Passing to the limit.* The following result shows that the family of all random elements that realise Φ in the sense of (2.4) is weakly compact.

THEOREM 2.8. *Assume that G satisfies Assumption 2.4 and consists of continuous functions on a Polish space $\mathcal{X}$ with regularity modulus χ. Let Φ be a linear positive functional on G. Then the family $\mathfrak{M}$ of all Borel random elements ξ that satisfy (2.4) for any given $r \geq 0$ is compact in the weak topology.*

PROOF. Since χ is a regularity modulus, the set $\mathcal{H}_{r/\varepsilon}$ is compact. By Markov's inequality,

$$\mathbf{P}\{\xi \notin \mathcal{H}_{r/\varepsilon}\} = \mathbf{P}\{\chi(\xi) > r/\varepsilon\} \leq \varepsilon,$$

for all $\xi \in \mathfrak{M}$, so that $\mathfrak{M}$ is tight.

Let $\{\xi_n, n \geq 1\}$ be random elements from $\mathfrak{M}$. Assume that ξ_n converges weakly to some ξ. Without loss of generality, assume that the ξ_n's are defined on the same probability space and converge almost surely to ξ. Since χ is nonnegative, Fatou's lemma yields

$$r \geq \liminf \mathbf{E}\chi(\xi_n) \geq \mathbf{E} \liminf \chi(\xi_n) \geq \mathbf{E}\chi(\lim \xi_n) = \mathbf{E}\chi(\xi),$$

where the lower semi-continuity of χ also has been used.

Take an arbitrary $\mathsf{g} \in \mathsf{G}$ and define $\mathcal{H}_{\lambda \mathsf{g}}$ as in (2.3). Let $\mathsf{g}^+(X) = \max(\mathsf{g}(X), 0)$ be the positive part of g. Then, for $\lambda > 0$,

$$\mathbf{E}\mathsf{g}^+(\xi_n) = \mathbf{E}\mathsf{g}^+(\xi_n)\mathbb{1}_{\xi_n \notin \mathcal{H}_{\lambda \mathsf{g}}} + \mathbf{E}\mathsf{g}^+(\xi_n)\mathbb{1}_{\xi_n \in \mathcal{H}_{\lambda \mathsf{g}}}.$$

Since g is continuous, $\mathcal{H}_{\lambda \mathsf{g}}$ is closed (and compact), so that if $\xi_n \in \mathcal{H}_{\lambda \mathsf{g}}$ for infinitely many n, then also $\xi \in \mathcal{H}_{\lambda \mathsf{g}}$. Furthermore, $\lambda \mathsf{g}$ and also g itself, are continuous and bounded on $\mathcal{H}_{\lambda \mathsf{g}}$, so that Fatou's lemma yields

$$\begin{aligned} \limsup \mathbf{E}\mathsf{g}^+(\xi_n)\mathbb{1}_{\xi_n \in \mathcal{H}_{\lambda \mathsf{g}}} &\le \mathbf{E} \limsup\big(\mathsf{g}^+(\xi_n)\mathbb{1}_{\xi_n \in \mathcal{H}_{\lambda \mathsf{g}}}\big) \\ &\le \mathbf{E}\mathsf{g}^+(\xi)\mathbb{1}_{\xi \in \mathcal{H}_{\lambda \mathsf{g}}} \le \mathbf{E}\mathsf{g}^+(\xi). \end{aligned}$$

Thus,

$$\limsup \mathbf{E}\mathsf{g}^+(\xi_n) \le \mathbf{E}\frac{\chi(\xi_n)}{\lambda} + \mathbf{E}\mathsf{g}^+(\xi) \le \frac{r}{\lambda} + \mathbf{E}\mathsf{g}^+(\xi).$$

Since λ is arbitrary,

$$\limsup \mathbf{E}\mathsf{g}^+(\xi_n) \le \mathbf{E}\mathsf{g}^+(\xi).$$

Since g^+ is nonnegative, Fatou's lemma yields that $\mathbf{E}\mathsf{g}^+(\xi_n) \to \mathbf{E}\mathsf{g}^+(\xi)$. By applying the same argument to the function $(-\mathsf{g})$, $\lim \mathbf{E}\mathsf{g}(\xi_n) = \mathbf{E}\mathsf{g}(\xi)$, so that $\mathbf{E}\mathsf{g}(\xi) = \Phi(\mathsf{g})$ for all $\mathsf{g} \in \mathsf{G}$. Therefore, $\xi \in \mathfrak{M}$. $\square$

The following result concerns realisability of pointwise limits of linear functionals. Special conditions of this type for correlation measures of point processes are given in [13], Section 3.4.

THEOREM 2.9. *Let $\{\Phi_n, n \ge 1\}$ be a sequence of linear positive functionals on a space G that satisfies the assumptions of Theorem 2.8. Assume that*

$$\liminf_n \sup_{\mathsf{g} \in \mathsf{G}, \mathsf{g} \le \chi} \Phi_n(\mathsf{g}) < \infty. \tag{2.6}$$

If $\Phi_n(\mathsf{g}) \to \Phi(\mathsf{g})$ for all $\mathsf{g} \in \mathsf{G}$, then Φ is realisable as a random element ξ satisfying (2.4) and such that ξ is the weak limit of random elements realising Φ_{n_k} for a subsequence n_k.

PROOF. By passing to a subsequence, it suffices to assume that (2.6) holds for the limit instead of the lower limit. Let ξ_n be a random element that realises Φ_n. If r is larger than the limit of (2.6), then $\mathbf{P}\{\xi_n \notin \mathcal{H}_{r/\varepsilon}\} \le \varepsilon$, so that $\{\xi_n\}$ is a tight sequence. Without loss of generality, assume that ξ_n weakly converges to a random element ξ.

The pointwise convergence of Φ_n yields that $\mathbf{E}\mathsf{g}(\xi_n) \to \Phi(\mathsf{g})$ for all $\mathsf{g} \in \mathsf{G}$. Now the arguments from the proof of Theorem 2.8 can be used to show that $\mathbf{E}\mathsf{g}(\xi_n) \to \mathbf{E}\mathsf{g}(\xi)$, so that $\mathbf{E}\mathsf{g}(\xi) = \Phi(\mathsf{g})$ for all $\mathsf{g} \in \mathsf{G}$, that is, ξ indeed satisfies (2.4). $\square$

2.4. *Invariant extension.* Consider an Abelian group Θ of continuous transformations acting on $\mathcal{X}$. For a function v on $\mathcal{X}$, define

$$(\theta \mathsf{v})(X) = \mathsf{v}(\theta X), \qquad \theta \in \Theta, X \in \mathcal{X}.$$

A functional Φ is said to be Θ-invariant if, for each $\theta \in \Theta$ and v from the domain of definition of Φ, $\Phi(\theta \mathsf{v})$ is defined and equal to $\Phi(\mathsf{v})$.

A Borel random element ξ in $\mathcal{X}$ is said to be Θ-*stationary* if, for each $\theta \in \Theta$, $\theta\xi$ has the same distribution as ξ. A variant of the following result for correlation measures of point processes is given in [13], Theorem 4.3.

THEOREM 2.10. *Assume that* G *is a* Θ-*invariant space satisfying Assumption* 2.4 *and consisting of* χ-*regular functions. Furthermore, assume that at least one of the following conditions holds*:

(i) G *consists of continuous functions and* χ *is pointwisely approximated from below by a monotone sequence of functions* $\mathsf{g}_n \in \mathsf{G}$, $n \geq 1$.
(ii) χ *is* Θ-*invariant.*

Let Φ *be a* Θ-*invariant functional on* G. *Then, for every given* $r \geq 0$, *there exists a* Θ-*stationary random element* ξ *in* $\mathcal{X}$ *satisfying* (2.4) *if and only if* (2.5) *holds.*

PROOF. (i) As in [13], Proposition 4.1, the proof consists in checking hypotheses of the Markov–Kakutani fixed-point theorem. Let $\mathfrak{M}$ be the family of random elements ξ that realise Φ on G, and satisfy $\mathbf{E}\chi(\theta\xi) \leq r$ for every $\theta \in \Theta$. The family $\mathfrak{M}$ is easily seen to be convex with respect to addition of measures, it is compact by Theorem 2.8, and Θ-invariant, since Φ is Θ-invariant on G. It remains to prove that $\mathfrak{M}$ is not empty.

In view of (2.5), it is possible to extend Φ positively onto $\mathsf{G} + \mathbb{R}\chi$, so that $\mathbf{E}\chi(\xi) \leq r$. The Θ-invariance of Φ on G together with the monotone convergence theorem imply that $\mathbf{E}\chi(\theta\xi) = \mathbf{E}\chi(\xi) \leq r$, whence $\xi \in \mathfrak{M}$.

(ii) By Proposition 2.2, we can extend Φ positively onto the Θ-invariant vector space $\mathsf{V} = \mathsf{G} + \mathbb{R}\chi$. Since Φ is Θ-invariant on G, we have $\Phi(\theta(\mathsf{g} + t\chi)) = \Phi(\theta\mathsf{g}) + t\Phi(\theta\chi) = \Phi(\mathsf{g} + t\chi)$ for $\mathsf{g} + t\chi$ in V, whence Φ is Θ-invariant on V. According to [29], Theorem 3, Φ admits a positive Θ-invariant extension to the space $\mathsf{E} + \mathbb{R}\chi$, defined like in the proof of Theorem 2.6. The restriction of the obtained functional onto E corresponds to a random element ξ in $\mathcal{X}$ that verifies (2.4) and satisfies $\mathbf{E}(\theta\mathsf{v})(\xi) = \Phi(\theta\mathsf{v}) = \Phi(\mathsf{v}) = \mathbf{E}\mathsf{v}(\xi)$, $\theta \in \Theta$, for v in E. Since E contains all bounded continuous functions on $\mathcal{X}$, $\theta\xi$ and ξ are identically distributed for all $\theta \in \Theta$. $\square$

3. Correlation measures of point processes.

3.1. *Framework and main results.* Let $\mathcal{N}$ be the family of locally finite counting measures on a locally compact complete separable metric space $\mathbb{X}$. We denote the support of $Y \in \mathcal{N}$ by the same letter Y, so that $x \in Y$ means $Y(\{x\}) \geq 1$.

Equip $\mathcal{N}$ with the vague topology, see [3], Chapter 7, so that $\mathcal{N}$ is metric and so completely regular. A random element ξ in $\mathcal{N}$ with the corresponding Borel σ-algebra is called a *point process*. Denote by $\mathcal{N}_0$ the family of *simple* counting measures, that is, those which do not attach mass 2 or more to any given point. If ξ is simple, that is, $\xi \in \mathcal{N}_0$ a.s., then ξ can be identified with a locally finite random set in $\mathbb{X}$, which is also denoted by ξ.

For a real function h on $\mathbb{X} \times \mathbb{X}$ and counting measure $Y = \sum_i \delta_{x_i}$ given by the sum of Dirac measures, define

$$\mathsf{g}_h(Y) = \sum_{x_i, x_j \in Y, i \neq j} h(x_i, x_j),$$

whenever the series absolutely converges, the empty sum being 0. Note that the sum in the right-hand side is taken over all pairs of distinct points from the support of Y, where multiple points appear several times according to their multiplicities. The value $\mathsf{g}_h(Y)$ is necessarily finite if h is bounded and has a bounded support. The value $\mathsf{g}_h(Y)$ is termed in [13] the quadratic polynomial of Y, while polynomials of order $n \geq 1$ are sums of functions of n points of the process, and are constants if $n = 0$.

Let G be the vector space formed by constants and functions g_h for h from the space $\mathscr{C}_\mathrm{o}$ of symmetric continuous functions with compact support. Note that G satisfies Assumption 2.4, since

$$(c_1 + \mathsf{g}_{h_1}) \vee (c_2 + \mathsf{g}_{h_2}) \leq c_1 \vee c_2 + \mathsf{g}_{h_1 \vee h_2} \in \mathsf{G}$$

for all $c_1, c_2 \in \mathbb{R}$ and $h_1, h_2 \in \mathscr{C}_\mathrm{o}$. Furthermore, each g_h is continuous in the vague topology, and so is χ-regular for any regularity modulus χ.

Assume that ξ has locally finite second moment, that is, $\mathbf{E}\xi(A)^2$ is finite for each bounded A. The *correlation measure* ρ (also called the second factorial moment measure) of a point process ξ is a measure on $\mathbb{X} \times \mathbb{X}$ that satisfies

$$\int_{\mathbb{X} \times \mathbb{X}} h(x, y) \rho(dx\, dy) = \mathbf{E}\mathsf{g}_h(\xi) \tag{3.1}$$

for each $h \in \mathscr{C}_\mathrm{o}$; see [3], Section 5.4 and [30], Section 4.3. The left-hand side defines a linear functional $\Phi(\mathsf{g}_h)$ on $\mathsf{g}_h \in \mathsf{G}$.

Let $\mathcal{X}$ be a subset of $\mathcal{N}$, which may be $\mathcal{N}$ itself. Recall that a subset of a completely regular space is completely regular, see [14], Theorem 14.I.2. Given a symmetric locally finite measure ρ on $\mathbb{X} \times \mathbb{X}$, the *realisability problem* amounts to the existence of a point process ξ with realisations from $\mathcal{X}$ and with correlation measure ρ, so that $\Phi(\mathsf{g}_h) = \mathbf{E}\mathsf{g}_h(\xi)$ for all $h \in \mathscr{C}_\mathrm{o}$.

By (2.1), the positivity of Φ means

$$\Phi(\mathsf{g}_h) \geq \inf_{Y \in \mathcal{X}} \mathsf{g}_h(Y) \tag{3.2}$$

for all $h \in \mathscr{C}_\mathrm{o}$. Then it is clear that the positivity of Φ is necessary for its realisability. If $\mathcal{X}$ is compact in the vague topology, then Corollary 2.7 applies and the positivity condition (3.2) is necessary and sufficient for the realisability of ρ.

However, in general the positivity condition alone is not sufficient for the realisability; see [12], Example 3.12. In the following, we find another condition that is not directly related to the positivity, but together with the positivity, is necessary and sufficient for the realisability.

As an introduction, let us present our results for $\mathbb{X}$ being a subset of the Euclidean space $\mathbb{R}^d$. For $\varepsilon \geq 0$, define

$$\chi_\varepsilon(Y) = \sum_{x,y \in Y, x \neq y} \|x - y\|^{-d-\varepsilon}, \qquad Y \in \mathcal{N},$$

which is later acknowledged as being a regularity modulus (see Definition 2.5) if $\varepsilon \neq 0$. Note that $\chi_\varepsilon(Y)$ is infinite if Y has multiple points. The tools developed in this paper enable us to resolve the original realisability problem with a supplementary regularity condition involving χ_ε.

THEOREM 3.1. (i) *Let $\mathbb{X}$ be a compact subset of $\mathbb{R}^d$ without isolated points. A symmetric finite measure $\rho(dx\,dy)$ on $\mathbb{X} \times \mathbb{X}$ is the correlation measure of a simple point process $\xi \subset \mathbb{X}$ such that $\mathbf{E}\chi_0(\xi) < \infty$ if and only if Φ given by the left-hand side of* (3.1) *is positive and*

$$\int_{\mathbb{X}\times\mathbb{X}} \|x - y\|^{-d} \rho(dx\,dy) < \infty.$$

(ii) *Let ρ be a symmetric locally finite measure on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\rho((A + x) \times (B + x)) = \rho(A \times B)$ for all $x \in \mathbb{R}^d$ and measurable sets A and B. Then there exists a simple stationary point process ξ with correlation measure ρ, such that*

$$\mathbf{E}\chi_0(\xi \cap C) < \infty$$

for every compact $C \subset \mathbb{R}^d$, if and only if Φ defined by (3.1) *is positive and*

$$\int_{B\times B} \|x - y\|^{-d} \rho(dx\,dy) < \infty \tag{3.3}$$

for some open set B.

PROOF. The proof relies of several theorems that will appear later in this section. The first statement follows from Theorem 3.5 using the fact that the packing number $P_t(\mathbb{X})$ of $\mathbb{X}$ is bounded by ct^{-d} for all sufficiently small t. For (ii), apply Theorem 3.9(ii) noticing that the imposed condition is equivalent to (3.24). □

In the remainder of this section, one can find a quantification of this result [i.e., how the left-hand member of (3.3) controls the value of $\mathbf{E}\chi_0(X \cap C)$] as well as generalisations for general metric spaces. The main argument used is a splitting method based on Theorem 2.6; the details are made clear in the proof of Theorem 3.3. Note that the packing number of the metric space appears as a crucial quantity to uncouple in this way the realisability problem; see Lemma 3.2.

3.2. *Moment conditions*. The family $\mathcal{X}_k$ of all counting measures with total mass at most k on a compact space $\mathbb{X}$ is compact. Thus, a measure ρ on $\mathbb{X} \times \mathbb{X}$ is realisable as a point process with at most k points if (3.2) holds with $\mathcal{X} = \mathcal{X}_k$.

Assume that Y is a finite counting measure. For $\alpha > 2$, define

$$\chi_\alpha(Y) = Y(\mathbb{X})^\alpha, \qquad Y \in \mathcal{N}.$$

The finiteness of $\mathbf{E}\chi_\alpha(\xi)$ amounts to the finiteness of the moment of order α for the total mass of ξ. Since $h \in \mathscr{C}_0$ is bounded by a constant c' and $\alpha > 2$, the family

$$\{Y \in \mathcal{N} : \chi_\alpha(Y) \le c + \mathsf{g}_h(Y)\} \subset \{Y \in \mathcal{N} : Y(\mathbb{X})^\alpha \le c + c' Y(\mathbb{X})^2\}$$

consists of counting measures with total masses bounded by a certain constant and, therefore, is compact in the space $\mathcal{N}$. Hence, χ_α is a regularity modulus and so Theorem 2.6 yields the realisability condition

$$\sup_{\mathsf{g} \in \mathsf{G}, \mathsf{g} \le \chi_\alpha} \Phi(\mathsf{g}) < \infty \tag{3.4}$$

of ρ by a point process ξ whose total number of points has finite moment of order α. Note that [13], Theorem 3.14, provides a variant of this result assuming the existence of the third factorial moment of the cardinality of ξ (i.e., with $\alpha = 3$) and for the joint realisability of the intensity and the correlation measures. The condition of [13], Theorem 3.14 (reformulated for the correlation measure only) reads in our notation as $c + \Phi(\mathsf{g}_h) + br \ge 0$ whenever $c + \mathsf{g}_h + b\chi_3$ is nonnegative on $\mathcal{N}$. Noticing that $b \ge 0$, this is equivalent to the fact that $c + \Phi(\mathsf{g}_h) \le r$ whenever $c + \mathsf{g}_h \le \chi_3$, being exactly (3.4). If Θ is a group of continuous transformations acting on $\mathbb{X}$ and ρ is Θ-invariant, then the point process ξ can be chosen Θ-stationary by Theorem 2.10(ii).

In order to handle possibly nonfinite point processes ξ, define

$$\chi_{\alpha,\beta}(Y) = \Big(\sum_{x \in Y} \beta(x)\Big)^\alpha, \qquad Y \in \mathcal{N},$$

for a lower semi-continuous strictly positive function $\beta : \mathbb{X} \mapsto \mathbb{R}$ and $\alpha > 2$. By approximating β from below with compactly supported functions, it is easy to see that $\chi_{\alpha,\beta}$ is a regularity modulus. By Theorem 2.6 and (2.2), for any given $r \ge 0$, there is a point process ξ with correlation measure ρ such that $\mathbf{E}\chi_{\alpha,\beta}(\xi) \le r$ if and only if ρ satisfies

$$\inf_{Y \in \mathcal{X}} [\chi_{\alpha,\beta}(Y) - \mathsf{g}_h(Y)] + \int_{\mathbb{X} \times \mathbb{X}} h(x, y)\rho(dx\,dy) \le r, \qquad h \in \mathscr{C}_0. \tag{3.5}$$

For $\alpha = 3$, condition (3.5) is a reformulation of [13], Theorem 3.17, meaning the positivity of Φ on a family of positive polynomials that involve symmetric functions of the support points up to the third order. The realisability condition for Θ-stationary random elements can be obtained by applying Theorem 2.10.

3.3. *Hardcore point processes on a compact space.* Assume that $\mathbb{X}$ is a compact metric space with metric $\mathbf{d}$. Let $\mathcal{N}_\varepsilon$ be the family of ε-*hard-core* point sets in $\mathbb{X}$ (including the empty set), that is, each $Y \in \mathcal{N}_\varepsilon$ attaches unit masses to distinct points with pairwise distances at least ε with a fixed $\varepsilon > 0$. In this case, no multiple points are allowed, that is, $\mathcal{N}_\varepsilon \subset \mathcal{N}_0$.

According to [7, 10], a subset $\mathcal{X}$ of simple counting measures $\mathcal{N}_0$ is relatively compact if and only if $\sup\{Y(K) : Y \in \mathcal{X}\}$ is finite and the infimum over $Y \in \mathcal{X}$ of the minimal distance between two points in $Y \cap K$ is strictly positive for each compact set $K \subset \mathbb{X}$. The hard-core condition yields that the number of points in any compact set is uniformly bounded, and so $\mathcal{N}_\varepsilon$ is indeed compact. By Corollary 2.7, ρ is realisable as the correlation measure of an ε-hard-core point process with given $\varepsilon > 0$ if and only if

$$\Phi(\mathsf{g}_h) \geq \inf_{Y \in \mathcal{N}_\varepsilon} \mathsf{g}_h(Y) \tag{3.6}$$

for all $h \in \mathscr{C}_0$. This result is formulated in [13], Theorem 3.4, which essentially reduces to the positivity of Φ over the family $c + \mathsf{g}_h$ (in our setting).

In this paper, we assume that the hardcore distance is not predetermined and the point process takes realisations from $\bigcup_{\varepsilon>0} \mathcal{N}_\varepsilon$, which coincides with $\mathcal{N}_0$ in case of compact $\mathcal{X}$. Note that (3.6) is stronger than the positivity of Φ on functions g_h defined on the whole family $\mathcal{N}_0$ and formulated as

$$\Phi(\mathsf{g}_h) \geq \inf_{Y \in \mathcal{N}_0} \mathsf{g}_h(Y), \qquad h \in \mathscr{C}_0. \tag{3.7}$$

If $\mathbb{X}$ does not have isolated points, then the infimum in (3.7) can be taken over $\mathcal{N}$. This is seen by approximating a multiple atom with a sequence of simple counting measures supported by points converging to the atom's location.

In the following, we use the (hard-core) regularity modulus of the form

$$\chi_\psi^{\mathrm{hc}}(Y) = \sum_{x_i, x_j \in Y, i \neq j} \psi\big(\mathbf{d}(x_i, x_j)\big), \qquad Y \in \mathcal{N}_0,$$

where $\psi : (0, \infty) \mapsto [0, \infty]$ is a monotone decreasing right-continuous function, such that $\psi(t) \to \infty$ as $t \downarrow 0$. The compactness of $\mathbb{X}$ and the lower semi-continuity of ψ imply that χ_ψ^{hc} is lower semi-continuous on $\mathcal{N}_0$. As shown below χ_ψ^{hc} is a regularity modulus if ψ grows sufficiently fast at zero.

Let $P_t(\mathbb{X})$ be the *packing number* of $\mathbb{X}$, that is, the maximum number of points in $\mathbb{X}$ with pairwise distances exceeding t, see [20], page 78. It is convenient to define the packing number at $t = 0$ as $P_0(\mathbb{X}) = \infty$ if $\mathbb{X}$ is infinite and otherwise let $P_0(\mathbb{X})$ be the cardinality of $\mathbb{X}$.

LEMMA 3.2. *Function χ_ψ^{hc} is a regularity modulus on $\mathcal{N}_0$ if*

$$\psi(t)/P_t(\mathbb{X}) \to \infty \qquad \text{as } t \downarrow 0. \tag{3.8}$$

PROOF. In view of the compactness of $\mathbb{X}$, it is possible to bound $h \in \mathscr{C}_0$ by a constant λ, so that χ_ψ^{hc} is a regularity modulus if

$$\mathcal{H}_\lambda = \{Y \in \mathcal{N}_0 : \chi_\psi^{\mathrm{hc}}(Y) \le \lambda Y(\mathbb{X})^2\}$$

is compact in $\mathcal{N}_0$ for each $\lambda > 0$. For this, it suffices to show that the total mass of all $Y \in \mathcal{H}_\lambda$ is bounded by a fixed number and $\mathcal{H}_\lambda \subset \mathcal{N}_\varepsilon$ for some $\varepsilon > 0$.

Let $\gamma_t(n)$ be the minimal number of pairs (x_i, x_j) with $i \neq j$, such that $x_i, x_j \in Y$ and $\mathbf{d}(x_i, x_j) \le t$ over all counting measures Y of total mass n.

Take t such that $\psi(t)/P_t(\mathbb{X}) > \lambda$. If $Y(\mathbb{X}) \ge n$, then

$$\chi_\psi^{\mathrm{hc}}(Y) \ge \sum_{x_i, x_j \in Y, i \neq j} \psi(t) \mathbb{1}_{\mathbf{d}(x_i, x_j) \le t} \ge \gamma_t(n)\psi(t).$$

Therefore,

$$\mathcal{H}_\lambda \subset \{Y : n^{-2}\gamma_t(n)\psi(t) \le \lambda\}$$

consists of Y with total mass uniformly bounded by fixed number n_λ. Indeed, by Lemma A.1,

$$\lim_{n\to\infty} n^{-2}\gamma_t(n) \ge \lim_{n\to\infty} n^{-2} n\left(\frac{n}{P_t(\mathbb{X})} - 1\right) = P_t(\mathbb{X})^{-1}.$$

Choose $\varepsilon > 0$ so that $\psi(t) \ge \lambda n_\lambda^2$ for $t \le \varepsilon$. For $Y \in \mathcal{H}_\lambda$ and any $x_i, x_j \in Y$,

$$\psi(\mathbf{d}(x_i, x_j)) \le \chi_\psi^{\mathrm{hc}}(Y) \le \lambda n_\lambda^2,$$

whence $\mathbf{d}(x_i, x_j) \ge \varepsilon$. Thus, $\mathcal{H}_\lambda \subset \mathcal{N}_\varepsilon$, so $\mathcal{H}_\lambda$ is relatively compact. □

The following theorem shows that the realisability condition can be split into the positivity condition (3.7) on the linear functional Φ and the regularity condition (3.9) on the correlation measure, so that the latter can be easily checked. Such a split is possible because the regularity modulus χ_ψ^{hc} can be approximated by functions from G.

THEOREM 3.3. *A locally finite measure ρ on $\mathbb{X} \times \mathbb{X}$ is the correlation measure of a simple point process ξ such that $\mathbf{E}\chi_\psi^{\mathrm{hc}}(\xi) \le r$ for some $r \ge 0$ with ψ satisfying* (3.8) *if and only if* (3.7) *holds and*

$$\int_{\mathbb{X}\times\mathbb{X}} \psi(\mathbf{d}(x, y))\rho(dx\,dy) \le r. \tag{3.9}$$

PROOF. *Necessity*. The definition of the correlation measure implies that

$$\int_{\mathbb{X}\times\mathbb{X}} \psi(\mathbf{d}(x, y))\rho(dx\,dy) = \mathbf{E}\chi_\psi^{\mathrm{hc}}(\xi) \le r.$$

Sufficiency. First assume that ψ only takes finite values. The proof consists of checking (2.2), which is equivalent to (2.5).

For each family of positive numbers $\{t_{\mathsf{g}}, \mathsf{g} \in \mathsf{G}\}$,

$$\sup_{\mathsf{g}\in\mathsf{G}} \inf_{Y\in\mathcal{N}_0} \big[\chi(Y) - \mathsf{g}(Y)\big] + \Phi(\mathsf{g}) \le \sup_{\mathsf{g}\in\mathsf{G}} \inf_{Y\in\mathcal{N}_{t_{\mathsf{g}}}} \big[\chi(Y) - \mathsf{g}(Y)\big] + \Phi(\mathsf{g}). \tag{3.10}$$

The crucial step of the proof consists in the careful choice of $t_{\mathsf{g}} > 0$.

Fix $\mathsf{g} \in \mathsf{G}$. For $t > 0$, define $\psi_t(s) = \psi(\max(t, s))$, $s \ge 0$. Since any $Y \in \mathcal{N}_t$ does not contain any two points at distance less than t, $\chi(Y) = \mathsf{g}_{\psi_t}(X)$. Therefore,

$$\inf_{Y\in\mathcal{N}_t} \big[\chi(Y) - \mathsf{g}(Y)\big] = \inf_{Y\in\mathcal{N}_t} (\mathsf{g}_{\psi_t} - \mathsf{g})(Y). \tag{3.11}$$

Our aim is to prove that

$$\inf_{Y\in\mathcal{N}_t} (\mathsf{g}_{\psi_t} - \mathsf{g})(Y) = \inf_{Y\in\mathcal{N}_0} (\mathsf{g}_{\psi_t} - \mathsf{g})(Y), \tag{3.12}$$

because then, since $\mathsf{g}_{\psi_t} \in \mathsf{G}$, the positivity of Φ on G yields that (3.11) is not greater than $\Phi(\mathsf{g}_{\psi_t} - \mathsf{g})$. Thus, (3.10) is bounded above by

$$\sup_{\mathsf{g}\in\mathsf{G}} \Phi(\mathsf{g}_{\psi_t} - \mathsf{g}) + \Phi(\mathsf{g}) \le \sup_t \Phi(\mathsf{g}_{\psi_t}) = \int_{\mathbb{X}\times\mathbb{X}} \psi\big(\mathbf{d}(x, y)\big)\rho(dx\,dy)$$

by the monotone convergence theorem.

The proof of (3.12) relies on the proper choice for t (depending on g). Assume without loss of generality $\mathsf{g} = \mathsf{g}_h$ for $h \in \mathscr{C}_0$ with absolute value bounded by $\lambda > 0$. By (3.8), there exists t_0 such that $\psi(t_0)/P_{t_0}(\mathbb{X}) \ge \lambda + 1$. By Lemma A.1, there is n_0 such that for all Y with mass $n \ge n_0$, the number of pairs of points of Y at distance at most t_0 satisfies

$$\gamma_{t_0}(n) \ge n^2 \frac{1}{P_{t_0}(\mathbb{X})}.$$

Choose $t \le t_0$ so that $\psi(t) > \lambda n_0^2$ and consider any $Y \in \mathcal{N}_0 \setminus \mathcal{N}_t$. If $Y(\mathbb{X}) \le n_0$, then

$$\mathsf{g}_{\psi_t}(Y) \ge \psi(t) > \lambda n_0^2 \ge \mathsf{g}_h(Y),$$

while if $Y(\mathbb{X}) > n_0$, then

$$\mathsf{g}_{\psi_t}(Y) - \mathsf{g}_h(Y) \ge \mathsf{g}_{\psi_{t_0}}(Y) - \mathsf{g}_h(Y) \ge \psi(t_0)\gamma_{t_0}(Y) - \lambda n^2 > 0.$$

Thus for $Y \notin \mathcal{N}_t$, we have $\mathsf{g}_{\psi_t}(Y) - \mathsf{g}_h(Y) > 0$. Therefore, the infimum of $\mathsf{g}_{\psi_t} - \mathsf{g}_h$, which is nonpositive because zero is obtained for $Y = \varnothing$, is reached on $\mathcal{N}_t$, and (3.12) is proved.

Now assume that $\psi(t)$ is infinite for $t \in [0, \delta)$ and finite on (δ, ∞) with $\delta > 0$. If $\psi(t) \to \infty$ as $t \downarrow \delta$, then the above arguments apply with $t_0 > \delta$ chosen such that $\psi(t_0)/P_\delta(\mathbb{X}) > \lambda$.

Assume that $\psi(\delta)$ is finite. Let $\psi_0(t)$ be a function satisfying (3.8) and finite for all $t > 0$, for example, $\psi_0(t) = t^{-1}P_t(\mathbb{X})$. Define $\psi^*(t) = \psi(t)$ for $t \ge \delta$ and let $\psi^*(t) = \psi_0(t) + a$ for $t \in (0, \delta)$ with a sufficiently large a, so that ψ^* is monotone right-continuous, and $\chi_{\psi^*}^{\mathrm{hc}}$ is a regularity modulus. Applying the previous arguments to ψ^* yields that there exists a point process ξ such that $\mathbf{E}\chi_{\psi^*}^{\mathrm{hc}}(\xi) \le r$. Since $r < \infty$, ρ vanishes on $\{(x,y) : \mathbf{d}(x,y) < \delta\}$, and so $\mathbf{E}\chi_{\psi^*}^{\mathrm{hc}}(\xi) = \mathbf{E}\chi_{\psi}^{\mathrm{hc}}(\xi) \le r$. □

The following result is obtained by letting ψ be infinite on $[0, \varepsilon)$ and otherwise setting it to zero.

COROLLARY 3.4. *A measure ρ on $\mathbb{X} \times \mathbb{X}$ is the correlation measure of a point process ξ with $\xi \in \mathcal{N}_\varepsilon$ a.s. if and only if* (3.7) *holds and $\rho(\{(x,y) : \mathbf{d}(x,y) < \varepsilon\}) = 0$.*

The following result yields a direct realisability condition for ρ without mentioning a regularity modulus.

THEOREM 3.5. *Let ρ be a locally finite measure on $\mathbb{X} \times \mathbb{X}$, and fix any $r \ge 0$. Then there exists, for every $r' > r$, a simple point process ξ with correlation measure ρ, such that*

$$\mathbf{E} \sum_{x_i, x_j \in \xi, i \neq j} P_{\mathbf{d}(x_i, x_j)}(\mathbb{X}) \le r', \tag{3.13}$$

if and only if (3.7) *holds and*

$$\int_{\mathbb{X} \times \mathbb{X}} P_{\mathbf{d}(x,y)}(\mathbb{X}) \rho(dx\, dy) \le r. \tag{3.14}$$

PROOF. *Necessity*. Call $h_t(x,y) = \min(t, P_{\mathbf{d}(x,y)}(\mathbb{X}))$ for $x \neq y \in \mathbb{X}$ and $t > 0$. Assume that ξ realises ρ and satisfies (3.13). The monotone convergence theorem yields that

$$\int_{\mathbb{X} \times \mathbb{X}} P_{\mathbf{d}(x,y)}(\mathbb{X}) \rho(dx\, dy) = \lim_{t \to \infty} \mathbf{E}\mathsf{g}_{h_t}(\xi) \le r'$$

for every $r' > r$, whence (3.14) holds.

Sufficiency. Define a measure on $\mathbb{R}_+$ by

$$\rho'\big([a,b)\big) = \rho\big(\{(x,y) \in \mathbb{X} \times \mathbb{X} : a \le \mathbf{d}(x,y) < b\}\big).$$

Fubini's theorem yields that

$$r = \int_{\mathbb{R}_+} P_t(\mathbb{X}) \rho'(dt).$$

Let $\{t_k, k \ge 1\}$ be a strictly decreasing sequence of numbers such that

$$\int_{[0, t_k)} P_t(\mathbb{X}) \rho'(dt) \le 2^{-k}.$$

For $m \geq 1$, the function

$$\psi_m(t) = \begin{cases} k P_t(\mathbb{X}), & \text{if } t_{k+1} \leq t < t_k < t_m, k \geq 1, \\ P_t(\mathbb{X}), & \text{if } t \geq t_m \end{cases}$$

is monotone right-continuous and satisfies $\psi_m(t)/P_t(\mathbb{X}) \to \infty$ as $t \to 0$. Then

$$\begin{aligned}\int_{\mathbb{X}\times\mathbb{X}} \psi_m\big(\mathbf{d}(x, y)\big)\rho(dx\,dy) &= \int_{\mathbb{R}_+} \psi_m(t)\rho'(dt) \\ &\leq \int_{\mathbb{R}_+} P_t(\mathbb{X})\rho'(dt) + \sum_{k\geq m} k2^{-k} \leq r + \sum_{k\geq m} k2^{-k}.\end{aligned}$$

By Theorem 3.3, choosing m sufficiently large yields the realisability of ρ by a point process ξ satisfying

$$\mathbf{E} \sum_{x_i,x_j\in\xi,i\neq j} P_{\mathbf{d}(x_i,x_j)}(\mathbb{X}) \leq \mathbf{E}\chi^{\mathrm{hc}}_{\psi_m}(\xi) \leq r + \sum_{k\geq m} k2^{-k} < r'.$$

□

REMARK 3.6. Let Θ be a group of continuous transformations on $\mathbb{X}$ that leave ρ invariant, that is, $\rho(\theta A \times \theta B) = \rho(A \times B)$ for all $\theta \in \Theta$ and Borel A, B. Since the regularity modulus $\chi^{\mathrm{hc}}_{\psi}$ can be approximated from below by a sequence of functions from G, Theorem 2.10(i) is applicable and so the corresponding point process ξ in Theorems 3.3, 3.5 and Corollary 3.4 can be chosen Θ-stationary. If Θ consists of isometric transformations, then Theorem 2.10(ii) is also applicable.

3.4. *Noncompact case and stationarity.* Assume that $\mathbb{X} = \mathbb{R}^d$ and $\mathbf{d}(x, y) = \|x - y\|$ is the Euclidean metric. Let ψ be a positive right-continuous monotone function on $\mathbb{R}_+$ such that $\psi(t)t^d \to \infty$ as $t \to 0$. Denote by B_n the open ball of radius n centred at 0. Given a known bound for the packing number in the Euclidean space ([20], page 78, Lemma 3.2) implies that $\chi^{\mathrm{hc}}_{\psi}$ is a regularity modulus on every B_n, $n \geq 1$. Define

$$\chi^{\mathrm{hc}}_{\beta\psi}(Y) = \sum_{x_i,x_j\in Y,i\neq j} \beta(x_i, x_j)\psi\big(\|x_i - x_j\|\big) \tag{3.15}$$

for a bounded lower semi-continuous strictly positive on $\mathbb{R}^d \times \mathbb{R}^d$ function β.

THEOREM 3.7. *Let ρ be a locally finite measure on $\mathbb{R}^d \times \mathbb{R}^d$.*

(i) *The measure ρ is realisable as the correlation measure of a point process ξ that satisfies $\mathbf{E}\chi^{\mathrm{hc}}_{\beta\psi}(\xi) \leq r$ if and only if* (3.7) *holds and*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \beta(x, y)\psi\big(\|x - y\|\big)\rho(dx\,dy) \leq r. \tag{3.16}$$

(ii) *Fix* $r \geq 0$, *let*

$$r_n = \int_{B_n \times B_n} \|x - y\|^{-d} \rho(dx\,dy), \qquad n \geq 1, \tag{3.17}$$

and let $\{\beta_n, n \geq 1\}$ *be a sequence of nonincreasing numbers converging to* 0. *Then the following assertions are equivalent.*

(a) *Equation* (3.7) *holds and*

$$\sum_{n \geq 1} \beta_n (r_{n+1} - r_n) \leq r < \infty, \tag{3.18}$$

in particular every r_n, $n \geq 1$, *is finite.*

(b) *For every* $r' > r$, *there exists* ξ *with correlation measure* ρ *and such that*

$$\sum_{n \geq 1} (\beta_n - \beta_{n+1}) \mathbf{E} \sum_{x_i, x_j \in B_n, i \neq j} \|x_i - x_j\|^{-d} \leq r'. \tag{3.19}$$

PROOF. *Sufficiency*. (i) The function $\chi_{\beta\psi}^{\mathrm{hc}}$ is a regularity modulus on $\mathcal{N}_0$, since

$$\mathcal{H}_{c,h} = \{Y \in \mathcal{X} : \chi_{\beta\psi}^{\mathrm{hc}}(Y) \leq c + \mathsf{g}_h(Y)\}, \qquad c \in \mathbb{R}, h \in \mathscr{C}_{\mathrm{o}},$$

is compact in $\mathcal{N}_0$. This follows from Lemma 3.2, which yields the compactness of the restriction of Y from $\mathcal{H}_{c,h}$ onto any compact set C. Indeed, this family of restricted counting measures coincides with the family of simple counting measures supported by C such that $\chi_{\psi}^{\mathrm{hc}}(Y) \leq c/m + \mathsf{g}_{h/m}(Y)$, where $m > 0$ is a lower bound of $\beta(x, y)$ for $x, y \in C$.

In order to apply Theorem 2.6 with the regularity modulus (3.15) and in view of (2.5) it suffices to show that

$$\inf_{Y \in \mathcal{N}_0} [\chi_{\beta\psi}^{\mathrm{hc}}(Y) - \mathsf{g}_h(Y)] + \int_{\mathbb{R}^d \times \mathbb{R}^d} h(x, y) \rho(dx\,dy) \leq r \tag{3.20}$$

for all $h \in \mathscr{C}_{\mathrm{o}}$. Assume that h is supported by a subset of $B_n \times B_n$ for some $n \geq 1$. Then (3.20) holds by the same reasoning as in the proof of Theorem 3.3 applied to the compact space B_n. (One might first consider only $Y \subset B_n$, and then note that the infimum over all $Y \in \mathcal{N}_0$ is necessarily smaller.) By Theorem 2.6, (3.16) implies the existence of a point process ξ with correlation measure ρ that satisfies $\mathbf{E}\chi_{\beta\psi}^{\mathrm{hc}}(\xi) \leq r$.

(ii) Define $\mathbb{Y}_n = (B_n \times B_n) \setminus (B_{n-1} \times B_{n-1})$, $n \geq 1$ (with $B_0 = \varnothing$). For every $n \geq 1$, define the measure

$$\rho'_n([a, b)) = \rho(\{(x, y) \in B_n \times B_n : a \leq \|x - y\| < b\}),$$

and let

$$t_k^n = \sup\left\{t > 0 : \int_{[0,t)} s^{-d} \rho'_n(ds) \leq 2^{-k}\right\}, \qquad k \geq 1.$$

Since $\rho'_{n+1} \ge \rho'_n$ for every $n \ge 1$, for every $k, n \ge 1$ we have $t_k^{n+1} \le t_k^n$. Let $\{m_n, n \ge 1\}$ be a nondecreasing sequence of positive integers so that

$$\sum_{n\ge 1} \beta_n \Bigl(r_n - r_{n-1} + \sum_{k\ge m_n} k 2^{-k} \Bigr) \le r'. \tag{3.21}$$

Now define

$$\psi_n(t) = \begin{cases} k t^{-d}, & \text{if } t_{k+1}^n \le t < t_k^n < t_{m_n}^n, \\ t^{-d}, & \text{if } t \ge t_{m_n}^n. \end{cases}$$

Since $m_n \le m_{n+1}$, $\psi_{n+1} \le \psi_n$ for every $n \ge 1$. Function ψ_n satisfies $\psi_n(t) t^d \to \infty$ as $t \to 0$, whence, for every $n \ge 1$, $\chi_{\psi_n}^{\mathrm{hc}}$ is a regularity modulus on counting measures supported by B_n and

$$\int_{\mathbb{Y}_n} \psi_n(\|x-y\|) \rho(dx\,dy) \le \int_{\mathbb{R}_+} \psi_n(t) \rho''_n(dt),$$

where

$$\rho''_n([a,b)) = \rho(\{(x,y) \in \mathbb{Y}_n : a \le \|x-y\| < b\}) \le \rho'_n([a,b)).$$

Then

$$\int_{\mathbb{Y}_n} \psi_n(\|x-y\|) \rho(dx\,dy) \le \int_{\mathbb{R}_+} t^{-d} \rho''_n(dt) + \int_{t \le t_{m_n}} \psi_n(t) \rho''_n(dt).$$

We have

$$\int_{\mathbb{R}_+} t^{-d} \rho''_n(dt) = \int_{(B_n \times B_n) \setminus (B_{n-1} \times B_{n-1})} \|x-y\|^{-d} \rho(dx\,dy) = r_n - r_{n-1}$$

and

$$\int_{t \le t_{m_n}} \psi_n(t) \rho''_n(dt) \le \int_{t \le t_{m_n}} \psi_n(t) \rho'_n(dt) \le \sum_{k \ge m_n} k 2^{-k},$$

whence

$$\int_{\mathbb{Y}_n} \psi_n(\|x-y\|) \rho(dx\,dy) \le r_n - r_{n-1} + \sum_{k\ge m_n} k 2^{-k}. \tag{3.22}$$

Define $\psi(x,y) = \psi_n(\|x-y\|)$ for $x, y \in \mathbb{Y}_n$. Since $\psi_{n+1} \le \psi_n$ and functions $\psi_n, n \ge 1$, are lower semi-continuous, the function ψ is lower semi-continuous on $\mathbb{R}^d \times \mathbb{R}^d$. Define $\beta(x,y) = \beta_n$ on $\mathbb{Y}_n$. Since $\beta_n, n \ge 1$, decrease, β is a lower semi-continuous function on $\mathbb{R}^d \times \mathbb{R}^d$. Since $\psi_n \le \psi_k$ for every $k \le n$, the restriction of $\chi_{\beta\psi}^{\mathrm{hc}}$ onto sets $Y \subset B_n$ is larger than $\chi_{\beta_n \psi_n}^{\mathrm{hc}}$, whence $\chi_{\beta\psi}^{\mathrm{hc}}$ is a regularity modulus on $\mathcal{N}_0$. By Theorem 2.6, Φ is realised by a point process ξ satisfying

$$\mathbf{E}\chi_{\beta\psi}^{\mathrm{hc}}(\xi) \le \int_{\mathbb{R}^d \times \mathbb{R}^d} \beta(x,y)\psi(x,y)\rho(dx\,dy) = \sum_{n\ge 1} \beta_n \int_{\mathbb{Y}_n} \psi_n(\|x-y\|)\rho(dx\,dy).$$

Since $t^{-d} \le \psi_n(t)$ for each n and $t > 0$,

$$\mathbf{E}\chi^{\mathrm{hc}}_{\beta\psi}(\xi) = \lim_{m\to\infty} \mathbf{E}\sum_{n=1}^{m} \beta_n \Bigg(\chi^{\mathrm{hc}}_{\psi_n}(\xi \cap B_n) - \chi^{\mathrm{hc}}_{\psi_n}(\xi \cap B_{n-1})$$

$$\ge \lim_{m\to\infty} \mathbf{E}\sum_{n=1}^{m} (\beta_n - \beta_{n+1})\chi^{\mathrm{hc}}_{\psi_n}(\xi \cap B_n)\Bigg)$$

$$\ge \sum_{n\ge 1} (\beta_n - \beta_{n+1})\mathbf{E} \sum_{i\ne j, x_i, x_j \in B_n} \|x_i - x_j\|^{-d}.$$

Using successively (3.22) and (3.21),

$$\sum_{n\ge 1} \beta_n \int_{\mathbb{Y}_n} \psi_n(\|x - y\|)\rho(dx\,dy) \le \sum_{n\ge 1} \beta_n \bigg(r_n - r_{n-1} + \sum_{k\ge m_n} k2^{-k}\bigg) \le r',$$

we arrive at (3.19).

Necessity. For (ii), remark first that for $x \ne y \in \mathbb{R}^d$

$$\beta(x, y)\|x - y\|^{-d} = \sum_{n\ge 1} (\beta_n - \beta_{n+1})\mathbb{1}_{x,y\in B_n} \|x - y\|^{-d},$$

where β is defined in the sufficiency part of the proof. The function $\beta\psi$ in (i) and the function $(x, y) \mapsto \beta(x, y)\|x - y\|^{-d}$ in (ii) are lower semi-continuous and can therefore be approximated from below by compactly supported continuous functions. The rest follows from the monotone convergence theorem similarly to the proof of necessity in Theorem 3.5. □

REMARK 3.8. Remark for point (ii) that if each r_n, $n \ge 1$, is finite, there always exists a sequence $\{\beta_n\}$ of sufficiently small numbers such that the right-hand side of (3.18) is finite.

If the distribution of point process ξ is invariant with respect to the group Θ of translations of $\mathbb{R}^d$, then ξ is called *stationary*. Its correlation measure ρ is translation invariant, that is, $\rho((A + x) \times (B + x)) = \rho(A \times B)$ for all $x \in \mathbb{R}^d$ and so

$$\rho(A \times B) = \lambda^2 \int_A \int_{\mathbb{R}^d} \mathbb{1}_{x+y\in B}\,\bar{\rho}(dy)\,dx, \tag{3.23}$$

where λ is the intensity of ξ and $\bar{\rho}$ is a measure on $\mathbb{R}^d$ called the *reduced* correlation measure; see [25], page 76.

THEOREM 3.9. *Let $\bar{\rho}$ be a locally finite measure on $\mathbb{R}^d$, let β be a bounded lower semi-continuous strictly positive function on $\mathbb{R}^d$ satisfying*

$$\bar{\beta}(y) = \int_{\mathbb{R}^d} \beta(x, x + y)\,dx < \infty, \qquad y \in \mathbb{R}^d,$$

and let ψ be a monotone decreasing nonnegative function such that $t^d\psi(t) \to \infty$.

(i) $\bar{\rho}$ *is the reduced correlation measure of a stationary point process* ξ *that satisfies* $\mathbf{E}\chi^{\mathrm{hc}}_{\beta\psi}(\xi) \le r$ *if and only if* (3.7) *holds and*

$$\int_{\mathbb{R}^d} \bar{\beta}(y)\psi(\|y\|)\bar{\rho}(dy) \le r.$$

(ii) $\bar{\rho}$ *is realisable as the reduced correlation measure of a stationary point process* ξ *that satisfies* (3.19) *for some sequence* $\{\beta_n, n \ge 1\}$ *if and only if*

$$\int_B \|y\|^{-d}\bar{\rho}(dy) < \infty \tag{3.24}$$

for some open ball B *containing the origin. If* $\int_{\mathbb{R}^d} \|y\|^{-d}\bar{\rho}(dy)$ *is finite, it is possible to let* $\beta_n = n^{-d-\delta}$, $n \ge 1$, *for any fixed* $\delta > 0$.

PROOF. It suffices to use (3.23) to confirm the conditions imposed in Theorem 3.7, see also Remark 3.8. In order to show that ξ can be chosen stationary, note that $\chi^{\mathrm{hc}}_{\beta\psi}$ can be pointwisely approximated from below by a monotone sequence of functions from G, so Theorem 2.10(i) applies. □

3.5. *Joint realisability of the intensity and correlation.* Recall that the intensity measure ρ_1 of a point process ξ is defined from

$$\mathbf{E}\sum_{x_i \in \xi} h(x_i) = \int h(x)\rho_1(dx), \qquad h \in \mathscr{C}_{\mathrm{o},1},$$

where $\mathscr{C}_{\mathrm{o},1}$ is the family of continuous functions on $\mathbb{X}$ with compact support. A pair (ρ_1, ρ) of locally finite nonnegative measures on $\mathbb{X}$ and $\mathbb{X}\times\mathbb{X}$, respectively, is said to be jointly realisable if there exists a point process ξ with intensity measure ρ_1 and correlation measure ρ.

Let G_1 be the vector space formed by constants and functions

$$\mathsf{g}_{h_1,h}(Y) = \sum_{x\in Y} h(x) + \mathsf{g}_h(Y), \qquad Y \in \mathcal{N},$$

for $h_1 \in \mathscr{C}_{\mathrm{o},1}$ and $h \in \mathscr{C}_{\mathrm{o}}$. It is easy to see that Assumption 2.4 is verified in this case. The pair (ρ_1, ρ) yields a linear functional

$$\Phi(\mathsf{g}_{h_1,h}) = \int_{\mathbb{X}} h_1(x)\rho_1(dx) + \int_{\mathbb{X}\times\mathbb{X}} h(x,y)\rho(dx\,dy). \tag{3.25}$$

The realisability of Φ by a point process ξ means that $\Phi(\mathsf{g}_{h_1,h}) = \mathbf{E}\mathsf{g}_{h_1,h}(\xi)$. Functional Φ is positive on G_1 if and only if

$$\Phi(\mathsf{g}_{h_1,h}) \ge \inf_{Y\in\mathcal{X}} \mathsf{g}_{h_1,h}(Y), \qquad h_1 \in \mathscr{C}_{\mathrm{o},1}, h \in \mathscr{C}_{\mathrm{o}}. \tag{3.26}$$

Similar arguments as before apply and yield the joint realisability conditions. Consider the special case of stationary processes in $\mathbb{X} = \mathbb{R}^d$ with the reduced correlation measure $\bar{\rho}$ [see (3.23)] and intensity $\rho_1(dx) = \lambda\,dx$ proportional to the Lebesgue measure.

THEOREM 3.10. *Let λ be a constant, and let $\bar{\rho}$ be a locally finite measure of $\mathbb{R}^d$. Then there is a stationary point process ξ with intensity $\rho_1(dx) = \lambda\,dx$ and reduced correlation measure $\bar{\rho}$ if Φ given by* (3.25) *satisfies* (3.26) *with $\mathcal{X} = \mathcal{N}_0$ and*

$$\int_B \|z\|^{-d} \bar{\rho}(dz) < \infty$$

for some open set B containing the origin.

PROOF. It suffices to note that $\mathsf{g}_{h_1,h}$ is dominated by $c\mathsf{g}_h$ for a constant c and follow the proof of (ii) in Theorem 3.7. The condition on $\bar{\rho}$ follows from (3.17) and (3.23). □

4. Realisability of covering probabilities for random sets. The nature of realisability problem changes with the choice of the family of subsets of a carrier space $\mathbb{X}$ taken as possible values for a random set. We start with the case when a random set is allowed to be any subset of $\mathbb{X}$, where realisability results are available under minimal conditions, while the obtained random set lacks properties and might even not be measurable. In the remainder of this section, we treat the case of random closed sets, a classical setting in stochastic geometry. Some examples of possible regularity moduli are presented, along with the corresponding realisability results that resemble those of [12] in the point processes setting. The framework of random measurable sets with finite perimeter (in the variational sense), developed in the forthcoming paper [6], provides a compromise between regularity of the random set and the explicitness of conditions.

4.1. *Random binary functions.* Let $\mathcal{X}$ be the family of all subsets of $\mathbb{X}$ identified with their indicator functions. Endow $\mathcal{X}$ with the topology of pointwise convergence and the corresponding σ-algebra. Since $\mathcal{X}$ is compact, Corollary 2.7 yields the following result.

THEOREM 4.1. *Let G be a vector space that consists of continuous functions on $\mathcal{X}$ and includes constants, and let Φ be a map from G to $\mathbb{R}$. Then there exists a random indicator function ξ, such that $\Phi(\mathsf{g}) = \mathbf{E}\mathsf{g}(\xi)$ for all $\mathsf{g} \in \mathsf{G}$ if and only if Φ is a linear positive functional on G and $\Phi(1) = 1$.*

The key issue in applying Theorem 4.1 is the choice of the space G.

EXAMPLE 4.2 (One-point covering function). Let G be generated by constants c and one-point indicator functions $\mathsf{g}_x(F) = \mathbb{1}_{x\in F}$, $F \in \mathcal{X}$, for $x \in \mathbb{X}$. The positivity of a linear functional $\Phi : \mathsf{G} \mapsto \mathbb{R}$ together with $\Phi(1) = 1$ means that $\Phi(\mathsf{g}_x) \in [0, 1]$ for all $x \in \mathbb{X}$. Thus, a function $p_x = \Phi(\mathsf{g}_x)$ is a one-point covering function $\mathbf{P}\{x \in \xi\}$ for a random set ξ if and only if p_x takes values in [0, 1]. Compare with Theorem 1.1, where the extra upper semi-continuity condition ensures that the corresponding random binary function is upper semi-continuous and so ξ is a random *closed* set.

EXAMPLE 4.3 (Covariances of random sets). Consider vector space G generated by constants and functions $\mathsf{g}_{x,y}(F) = \mathbb{1}_{x,y\in F}$ for $x, y \in \mathbb{X}$. The values of a linear functional Φ on G are determined by $p_{x,y} = \Phi(\mathsf{g}_{x,y})$, $x, y \in \mathbb{X}$. By (2.1), Φ is positive on G if and only if

$$\sum_{ij=1}^{n} a_{ij} p_{x_i,x_j} \geq \inf_{F\subset\mathbb{X}} \sum_{ij=1}^{n} a_{ij} \mathbb{1}_{x_i,x_j\in F} \tag{4.1}$$

for all $n \geq 1$ and all matrices $(a_{ij})_{ij=1}^{n}$. In particular, if $a_{ij} = a_i a_j$, then (4.1) implies the nonnegative definiteness of $p_{x,y}$, $x, y \in \mathbb{X}$. Note that the one-point covering probabilities are specified if $p_{x,y}$ are given.

4.2. *The closedness condition.* A *random closed set* ξ in a locally compact metric space $\mathbb{X}$ is a random element that takes values in the family $\mathcal{X} = \mathcal{F}$ of closed subsets of $\mathbb{X}$ equipped with the σ-algebra (called the Effros σ-algebra) generated by families $\{F \in \mathcal{F} : F \cap K \neq \varnothing\}$ for all compact sets K. The distribution of a random closed set ξ is uniquely determined by its *capacity functional*

$$T(K) = \mathbf{P}\{\xi \cap K \neq \varnothing\}$$

for all K from the family of all compact sets in $\mathbb{X}$; see [18] and [22], Theorem 1.1.13.

Theorem 4.1 ensures only the existence of a binary stochastic process with given marginal distributions up to a certain order. However, it is not guaranteed that the constructed stochastic process has upper semi-continuous realisations, which should be the case if this process is the indicator of a random *closed* set in a topological space $\mathbb{X}$. If the carrier space $\mathbb{X}$ is finite (more generally, has a discrete topology), then this problem is avoided, since each random set is closed. Furthermore, the closedness issue can be settled in the following special case of two-point probabilities in the product form (and can be generalised for multi-point covering probabilities). The following result implies, in particular, that the random indicator function from Example 1.2 does not correspond to a random closed set. It also illustrates regularity problems arising in realisability problems for random closed sets.

THEOREM 4.4. *Assume that $\mathbb{X}$ is a separable space. A function*

$$p_{x,y} = \begin{cases} p_x p_y, & \textit{if } x \neq y, \\ p_x, & \textit{if } x = y \end{cases}$$

is a two-point covering function of a random closed set if and only if p_x, $x \in \mathbb{X}$, is an upper semi-continuous function with values in $[0, 1]$ such that each point x with $p_x \in (0, 1)$ has an open neighbourhood U such that $p_y > 0$ only for at most a countable number of $y \in U$ and the sum of p_y for $y \in U$ is finite.

PROOF. *Sufficiency*. Note that the set $L = \{x : p_x = 1\}$ is closed by the upper semi-continuity of p_x. The separability of $\mathbb{X}$ and the condition of theorem imply that the set $M = \{x : p_x \in (0, 1)\}$ is at most countable. The sufficiency is obtained by a direct construction of a random subset Z of M that contains each point x with probability p_x independently of all other points. It remains to show that the random set $\xi = Z \cup L$ is closed. Consider $x \in M$ and its neighbourhood from the condition of theorem. Since $\sum p_y < \infty$, only a finite number of y belong to Z and so they do not converge to x. Thus, x with probability zero appears as a limit of other points from ξ unless $x \in L$ and so belongs to ξ almost surely.

Necessity. The function $p_x = \mathbf{P}\{x \in \xi\}$ is upper semi-continuous, since ξ is a random closed set. The product form of the two-point covering function implies that the capacity functional on two-point set is given by

$$T(\{x, y\}) = p_x + p_y - p_x p_y.$$

The upper semi-continuity property of the capacity functional yields that

$$\limsup_{y \to x} T(\{x, y\}) \le p_x,$$

while the monotonicity implies that $T(\{x, y\}) \to p_x$ as $y \to x$. Thus, $p_y(1 - p_x) \to 0$ as $y \to x$ for all x. Unless $p_x = 1$, we have $p_y \to 0$.

Assume that $p_x > 0$ and $p_{x_n} > 0$, where $x_n \to x$ and $x_n \ne x$ with $\sum p_{x_n} = \infty$. A variant of the lemma of Borel–Cantelli for pairwise independent events (see [5], Lemma 6.2.5) implies that almost surely infinitely many points x_n belong to ξ, so that $x \in \xi$ a.s. by the closedness of ξ and so $p_x = 1$. Thus, the sum of p_{x_n} for each sequence $\{x_n\}$ in a neighbourhood of x is finite. This rules out the existence of uncountably many y with $p_y > 0$ in any neighbourhood of x. Indeed, then $\{y : p_y \ge 1/n\}$ is finite for all n, and so the union of such sets is countable. $\square$

It is known that $\mathcal{F}$ is compact and completely regular in the Fell topology that generates the Effros σ-algebra; see [14], Theorem 17.V.3 and [22]. However, Corollary 2.7 is not applicable, since functions $\mathbb{1}_{x,y \in F}$, $F \in \mathcal{F}$, generating the vector space G, do not generate the Effros σ-algebra on $\mathcal{F}$.

It is known ([22], Theorem 1.2.6) that the σ-algebra generated by G on the family of *regular closed* sets (that coincide with closures of their interiors) coincides with the trace of the Effros σ-algebra on the family of regular closed sets. However, the family of regular closed sets is no longer compact in the Fell topology. Furthermore, indicator functions are not continuous in the Fell topology, so it is again not possible to appeal to Corollary 2.7 or explicitly check the upper semi-continuity condition required in Daniell's theorem.

In order to describe a useful family G of functionals acting on sets, consider a σ-finite measure ν on $\mathbb{X}$ and define

$$\mathsf{g}_h(F) = \int_{F \times F} h(x, y)\nu(dx)\nu(dy) \tag{4.2}$$

for all measurable $F \subset \mathbb{X}$ and h from the family $\mathscr{C}_0$ of symmetric continuous functions with compact support in $\mathbb{X} \times \mathbb{X}$. A function $p_{x,y}$, $x, y \in \mathbb{X}$, generates a functional acting on g_h as

$$\Phi(\mathsf{g}_h) = \int_{\mathbb{X}\times\mathbb{X}} p_{x,y} h(x, y)\nu(dx)\nu(dy). \tag{4.3}$$

The function $p_{x,y}$ is said to be *weakly realisable* as the two-point covering function if there exists a random set ξ (or the corresponding random indicator function) such that ξ is almost surely measurable and $\mathbf{E}\mathsf{g}_h(\xi) = \Phi(\mathsf{g}_h)$ for all $h \in \mathscr{C}_0$. By approximating the atomic masses at two points with continuous functions, it is easy to see that the weak realisability is equivalent to $\Phi(\mathsf{g}_{x,y}) = p_{x,y}$ for $\nu \otimes \nu$-almost all (x, y), in contrast to the *strong realisability* requiring this equality everywhere. The strong and weak realisability do not coincide in general. For instance, a nonpositive function which vanishes almost everywhere, but takes some negative values is weakly realisable by the empty set, but not strongly realisable. Nevertheless, in the case of a *stationary* random regular closed set ξ in $\mathbb{R}^d$ and the Lebesgue measure ν, the strong and weak realisability properties coincide; see Theorem 4.7.

In view of the required continuity property of functions from G, it is essential to ensure that $\mathsf{g}_h(F)$, $F \in \mathcal{F}$, defined in (4.2) is continuous in the Fell topology. Note that it is not the case for most nontrivial measures ν, even for the Lebesgue measure. The continuity holds only on some subfamilies of $\mathcal{F}$ considered in the following sections.

4.3. *Neighbourhoods of closed sets*. For simplicity, in the following consider random sets in the Euclidean space, that is, assume that $\mathbb{X} = \mathbb{R}^d$ with Euclidean metric $\mathbf{d}$.

Let $\mathcal{F}^\varepsilon$ be the family of ε-neighbourhoods of closed sets in $\mathbb{R}^d$, that is, $\mathcal{F}^\varepsilon$ consists of $F^\varepsilon = \{x : \mathbf{d}(x, F) \le \varepsilon\}$ for all $F \in \mathcal{F}$ and also contains the empty set. The vector space G is generated by constants and the functions g_h defined by (4.2) with the Lebesgue measure ν.

LEMMA 4.5. *The space $\mathcal{F}^\varepsilon$ with the Fell topology is compact and, for each $h \in \mathscr{C}_0$, the function g_h is continuous on $\mathcal{F}^\varepsilon$.*

PROOF. Recall that the upper limit of a sequence of sets $\{F_n\}$ is the set of all limits for sequences $\{x_{n_k}\}$ such that $x_{n_k} \in F_{n_k}$ for all k, while the lower limit is the set of all limits for convergent sequences $\{x_n\}$ such that $x_n \in F_n$ for all n. The sequence of closed sets converges in the Fell topology if its upper and lower limits coincide.

If $F_n = F^\varepsilon_{n,0} \in \mathcal{F}^\varepsilon$ converges to F in the Fell topology, then we can assume without loss of generality (by passing to subsequences) that $F_{n,0}$ converges to F_0, so that $F = F^\varepsilon_0$ and $F \in \mathcal{F}^\varepsilon$. Thus, $\mathcal{F}^\varepsilon$ is a closed subset of $\mathcal{F}$ and so is compact, since $\mathcal{F}$ is compact itself.

Consider a nonnegative $h \in \mathscr{C}_0$ supported by a ball B_R centred at the origin with sufficiently large radius R. If $F_n \to F$ in the Fell topology, then the upper limit of $(F_n \cap B_R)$ is a subset of $(F \cap B_R)$. Thus, $\mathsf{g}_h(F) = \mathsf{g}_h(F_n \cap B_R) \le \mathsf{g}_h((F \cap B_R)^\delta)$ for any $\delta > 0$ and sufficiently large n, so that g_h is upper semi-continuous on $\mathcal{F}$ and so on $\mathcal{F}^\varepsilon$.

In order to prove the lower semi-continuity of g_h on $\mathcal{F}^\varepsilon$ assume $F_n = F_{n,0}^\varepsilon \to F = F_0^\varepsilon$ and $F_{n,0} \to F_0$. Fix $\delta > 0$. Then the lower limit of $F_{n,0} \cap B_{r+\delta}$ includes $F_0 \cap B_R$. Indeed, if $x \in F_0 \cap B_R$, then $x_n \to x$ for $x_n \in F_{n,0}$ and so $x_n \in B_{R+\delta}$ for all sufficiently large n. Thus, for sufficiently large n, we have

$$(F_0 \cap B_R) \subset (F_{n,0} \cap B_{R+\delta})^\delta.$$

Taking $(\varepsilon - \delta)$-neighbourhoods of the both sides yields that

$$(F_0 \cap B_R)^{\varepsilon-\delta} \subset (F_{n,0} \cap B_{R+\delta})^\varepsilon \subset (F_n \cap B_{R+\delta+\varepsilon}).$$

If $x \in F_0^{\varepsilon-\delta} \cap B_{R-\varepsilon+\delta}$, then there is a point $y \in F_0$ with $\mathbf{d}(x, y) \le \varepsilon - \delta$, in particular $y \in B_R$. Thus,

$$\left(F_0^{\varepsilon-\delta} \cap B_{R-\varepsilon+\delta}\right) \subset (F_{n,0} \cap B_R)^{\varepsilon-\delta}.$$

Taking r sufficiently large yields that $\mathsf{g}_h(F_n) \ge \mathsf{g}_h(F_0^{\varepsilon-\delta})$. Since the interior of F equals $\bigcup_{\delta>0} F_0^{\varepsilon-\delta}$, the Lebesgue theorem yields that $\mathsf{g}_h(F_0^{\varepsilon-\delta}) \to \mathsf{g}_h(F)$ as $\delta \to 0$, that is, g_h is lower semi-continuous on $\mathcal{F}^\varepsilon$.

For a nonpositive function h with compact support, the result follows by applying the above argument to its positive and negative parts. □

THEOREM 4.6. *A function $p_{x,y}$, $x, y \in \mathbb{R}^d$, is weakly realisable by a random closed set ξ with realisations in $\mathcal{F}^\varepsilon$ for given $\varepsilon > 0$ if and only if*

$$\Phi(\mathsf{g}_h) \ge \inf_{F \in \mathcal{F}^\varepsilon} \mathsf{g}_h(F), \qquad h \in \mathscr{C}_0,$$

where $\Phi(\mathsf{g}_h)$ is given by (4.3).

PROOF. In view of the continuity of g_h established in Lemma 4.5, it suffices to refer to Corollary 2.7. □

In order to handle random sets with realisations from the space $\mathcal{F}^0 = \bigcup_{\varepsilon>0} \mathcal{F}^\varepsilon$, we need the regularity modulus $\chi(F)$ defined as the infimum of $\varepsilon > 0$ such that $F \in \mathcal{F}^{1/\varepsilon}$ and $\chi(F) = \infty$ if $F \notin \mathcal{F}^0$.

THEOREM 4.7. *For any given $r > 0$, a function $p_{x,y}$, $x, y \in \mathbb{R}^d$, is weakly realisable by a random closed set ξ such that $\mathbf{E}\chi(\xi) \le r$ if and only if*

$$\inf_{F \in \mathcal{F}^0} \left[\chi(F) - \mathsf{g}_h(F)\right] + \Phi(\mathsf{g}_h) \le r, \qquad h \in \mathscr{C}_0, \tag{4.4}$$

where $\Phi(\mathsf{g}_h)$ is given by (4.3). *If, additionally, $p_{x,y}$ is an even continuous function of $x - y$, then $p_{x,y}$ is strongly realisable by a stationary random closed set ξ.*

PROOF. Function χ is lower semi-continuous, since $\{F \in \mathcal{F}: \chi(F) \le c\} = \mathcal{F}^{1/c}$ is closed for all $c > 0$. Furthermore,

$$\{F \in \mathcal{F}: \chi(F) \le \mathsf{g}_h(F)\} \subset \{F \in \mathcal{F}: \chi(F) \le c\}, \tag{4.5}$$

where $c = \int |h(x, y)|\nu(dx)\nu(dy)$ is a finite upper bound for $\mathsf{g}_h(F)$. The left-hand side of (4.5) is compact, since g_h is continuous on $\mathcal{F}^{1/c}$ by Lemma 4.5 and the right-hand side of (4.5) is compact. Thus, χ is a regularity modulus and the result follows from Theorem 2.6 and (2.2).

Note that the regularity modulus χ is invariant for the group Θ of translations of $\mathbb{R}^d$. By Theorem 2.10(ii), ξ can be chosen to be stationary. In order to confirm the strong realisability, it remains to show that the covariance function of a stationary regular closed random set is continuous.

Since $\chi(\xi)$ is integrable, $\xi \in \mathcal{F}^0$, so that ξ is almost surely regular closed and its boundary $\partial\xi$ has a.s. vanishing Lebesgue measure. By Fubini's theorem, almost every point x belongs to the boundary of ξ with probability zero, and so $\mathbf{P}\{x \in \partial\xi\} = 0$ for all x in view of the stationarity property.

Let $\mathbf{P}\{x, y \in \xi\}$ be the covariance function of ξ. Take $x, y \in \mathbb{R}^d$, and (x_n, y_n) that converges to (x, y). Since with probability 1, x does not belong to $\partial\xi$, $\mathbb{1}_{x\in\xi}$ is almost surely equal to $\mathbb{1}_{x\in\mathrm{Int}(\xi)}$ for the interior $\mathrm{Int}(\xi)$ of ξ and so $\mathbb{1}_{x_n\in\xi}$ almost surely converges to $\mathbb{1}_{x\in\xi}$. Similarly, $\mathbb{1}_{y_n\in\xi} \to \mathbb{1}_{y\in\xi}$ a.s., whence the product converges too $\mathbb{1}_{x_n\in\xi, y_n\in\xi} \to \mathbb{1}_{x\in\xi, y\in\xi}$. The Lebesgue theorem yields that $\mathbf{P}\{x_n, y_n \in \xi\} \to \mathbf{P}\{x, y \in \xi\}$. Since $p_{x,y}$ and $\mathbf{P}\{x, y \in \xi\}$ are both continuous and coincide almost surely, they are equal everywhere. The continuity of $\mathbf{P}\{x, y \in \xi\}$ can be also obtained by referring to a result of [23] saying that the capacity functional of each stationary regular closed random set is continuous in the Hausdorff metric. □

4.4. *Convexity restrictions.* The family $\mathcal{C}$ of convex closed sets in $\mathbb{R}^d$ (including the empty set) is closed in the Fell topology and it is easy to see that the function g_h given by (4.2) is continuous on $\mathcal{C}$. Corollary 2.7 yields that $p_{x,y}$ is weakly realisable for a convex random closed set if and only if

$$\Phi(\mathsf{g}_h) \ge \inf_{F\in\mathcal{C}} \mathsf{g}_h(F)$$

for the functional $\Phi(\mathsf{g}_h)$ given by (4.3).

Let $\mathcal{P}$ be the *convex ring* in $\mathbb{R}^d$, that is, the family of finite unions of compact convex subsets of $\mathbb{R}^d$. For $F \in \mathcal{P}$, let $\chi(F)$ be the smallest number k, such that F can be represented as the union of k convex compact sets.

THEOREM 4.8. *Let Φ be linear functional defined by* (4.3). *Fix any $r > 0$. Then there is a random closed set ξ with realisations in $\mathcal{P}$ such that $\mathbf{E}\mathsf{g}_h(\xi) = \Phi(\mathsf{g}_h)$ for all $h \in \mathscr{C}_0$ and $\mathbf{E}\chi(\xi) \le r$ if and only if*

$$\inf_{F\in\mathcal{P}}[\chi(F) - \mathsf{g}_h(F)] + \Phi(\mathsf{g}_h) \le r, \qquad h \in \mathscr{C}_0.$$

PROOF. The family $\mathcal{P}_k$ of unions of at most k convex compact sets is closed in $\mathcal{F}$ and so is compact, whence χ is lower semi-continuous. It is easily seen that g_h is continuous on convex compact sets, and so is also continuous on $\mathcal{P}_k$. Thus, g_h is χ-regular and Theorem 2.6 applies. $\square$

If $\mathbb{X} = [0, 1]$, then $\mathcal{P}$ is be the family of finite unions of segments in $[0, 1]$. The number of convex components of $F \subset [0, 1]$ is the variation of its indicator function,

$$\chi(F) = \sup \sum_{i=0}^{n-1} |\mathbb{1}_{t_i \in F} - \mathbb{1}_{t_{i+1} \in F}|,$$

where the supremum is taken over partitions $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$. The quantity

$$\mathsf{v}(F) = \sup_{\varphi \in \mathscr{C}^1, 0 \leq \varphi \leq 1} \int_F \varphi'(x)\, dx,$$

where $\mathscr{C}^1$ is the family of differentiable functions on $[0, 1]$, captures the number of components of F with nonempty interiors, in particular $\mathsf{v}(F) \leq \chi(F)$. Remark that v is not a regularity modulus, because a set F with small $\mathsf{v}(F)$ can contain an arbitrarily large number of isolated singletons.

THEOREM 4.9. *If $p_{x,y}$ is a function of $x, y \in [0, 1]$ such that*

$$\sup_{\varphi \in \mathscr{C}^1, 0 \leq \varphi \leq 1} \int_{\mathbb{X} \times \mathbb{X}} p_{x,y} \varphi'(x) \varphi'(y)\, dx\, dy = \infty, \tag{4.6}$$

then there is no random closed set ξ satisfying $\mathbf{E}\chi(\xi)^2 < \infty$ having $p_{x,y}$ as its two-point covering function.

PROOF. Let H be the family of functions $h(x, y) = \varphi'(x)\varphi'(y)$ for $\varphi \in \mathscr{C}^1$ with $0 \leq \varphi \leq 1$. Then

$$\mathsf{v}(F)^2 = \sup_{h \in \mathsf{H}} \int_{\mathbb{X} \times \mathbb{X}} \mathbb{1}_{x,y \in F} h(x, y)\, dx\, dy = \sup_{h \in \mathsf{H}} \mathsf{g}_h(F).$$

Theorem 2.6 implies that Φ is realisable by a random closed set ξ with $\mathbf{E}\chi(\xi)^2 < \infty$ if and only if

$$\sup_{h \in \mathscr{C}_0} \Big[\inf_{F \in \mathcal{X}} [\chi(F)^2 - \mathsf{g}_h(F)] + \Phi(\mathsf{g}_h) \Big] < \infty.$$

It implies in particular

$$\sup_{h \in \mathsf{H}} \Big[\inf_{F \in \mathcal{X}} [\chi(F)^2 - \mathsf{g}_h(F)] + \Phi(\mathsf{g}_h) \Big] < \infty.$$

Since $\chi(F)^2 \geq \mathsf{v}(F)^2 \geq \mathsf{g}_h(F)$ for $h \in \mathsf{H}$, this condition would imply that

$$\sup_{h \in \mathsf{H}} \Phi(\mathsf{g}_h) < \infty,$$

contradicting (4.6). Thus Φ is not realisable. $\square$

Further results on realisability of random sets can be found in [6], where it is shown that by relaxing the closedness assumption it is possible to split the positivity and regularity conditions as it was the case in Section 3.3.

5. Contact distribution functions for random sets. Results from Section 4 concern realisability of the two-point covering probabilities, which are closely related to the values of the capacity functional (hitting probabilities) on two-point sets. Here, we consider the realisability problem for a capacity functional defined on the family of balls in $\mathbb{R}^d$. If T is the capacity functional of a random closed set ξ, then

$$T\big(B_R(x)\big) = \mathbf{P}\big\{\xi \cap B_R(x) \neq \varnothing\big\}$$

is closely related to the spherical contact distribution function $\mathbf{P}\{\mathbf{d}(x, \xi) \leq R | x \notin \xi\}$, $R \geq 0$, which is the cumulative distribution function of the distance between ξ and x given that $x \notin \xi$.

THEOREM 5.1. *A function $\tau_x(R)$, $R \geq 0$, $x \in A \subset \mathbb{R}^d$, is realisable as $T(B_R(x))$ for a random closed set ξ if and only if*

$$\Phi(\mathsf{g}) = \sum_{i=1}^{m} a_i \tau_{x_i}(R_i) \geq 0 \tag{5.1}$$

for all $m \geq 1$, $x_1, \dots, x_m \in A$ and $R_1, \dots, R_m \geq 0$, such that the function

$$\mathsf{g}(F) = \sum_{i=1}^{m} a_i \mathbb{1}_{B_{R_i}(x_i) \cap F \neq \varnothing} \geq 0, \qquad F \in \mathcal{F} \tag{5.2}$$

is nonnegative.

PROOF. The necessity is evident.

Sufficiency. Let G be the vector space generated by constants and functions $\mathsf{g}_{h,x}(F) = h(\mathbf{d}(x, F))$, $F \in \mathcal{F}$, where $\mathbf{d}(x, F)$ is the distance from $x \in \mathbb{R}^d$ to the nearest point of F, and h is a continuous function on $\mathbb{R}$ with bounded support. The functions $\mathsf{g}_{h,x}$ are all continuous in the Fell topology, since the Fell topology in $\mathbb{R}^d$ coincides with the topology of pointwise convergence of distance functions $\mathbf{d}(x, F)$ for $x \in \mathbb{R}^d$; see [22], Theorem B.12.

It suffices to show that Φ is positive on G. Let $\mathsf{g}(F) = \sum_{i=1}^m a_i h_i(\mathbf{d}(x_i, F))$. Uniform approximation of $h_1, \dots, h_m$ by step functions on their supports yields a function $\hat{\mathsf{g}}$ of the form (5.2) so that $\hat{\mathsf{g}}(F) \geq -\varepsilon$ for some $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ and using (5.1) yield that

$$\Phi(\mathsf{g}) = \sum_{i=1}^{m} a_i \int h_i(t)\, d\tau_{x_i}(t) \geq 0. \qquad \square$$

If $\tau_x(R) = \tau(R)$ does not depend on x, it may be possible to realise it as the contact distribution function of a stationary random closed set. If the argument x of $\tau_x(R)$ takes only a single value, then the necessary and sufficient condition on $\tau_x(\cdot)$ is that it is a nondecreasing right-continuous function with values in [0, 1], that is, the cumulative distribution function of a sub-probability measure on $\mathbb{R}_+$. The following result concerns the case of x taking two possible values.

THEOREM 5.2. *Let $x_1, x_2 \in \mathbb{R}^d$, with $l = \|x_1 - x_2\|$, and let τ_{x_1} and τ_{x_2} be cumulative distribution functions of two sub-probability measures on $\mathbb{R}_+$. Then there exists a random closed set ξ such that $\tau_{x_i}(R) = T(B_R(x_i))$ for $r \geq 0$ and $i = 1, 2$ if and only if for all $r \geq 0$*

$$\tau_{x_1}\big(\max(R - l, 0)\big) \leq \tau_{x_2}(R) \leq \tau_{x_1}(R + l). \tag{5.3}$$

PROOF. *Necessity*. Let ξ be a random closed set with $\tau_{x_i}(R) = T(B_R(x_i))$. Let a_1 and a_2 be random points such that $a_1, a_2 \in \xi$ a.s. and $R_i = \mathbf{d}(x_i, a_i) = \mathbf{d}(x_i, \xi)$, $i = 1, 2$, have cumulative distribution functions τ_{x_1} and τ_{x_2}, respectively. Then $|R_1 - R_2| \leq l$. Indeed, if, for instance, $R_1 > R_2 + l$, then a_2 is nearer to x_1 than a_1 contrary to the assumption. Thus $R_1 \leq R$ implies $R_2 \leq R + l$, so that $\tau_{x_1}(R) \leq \tau_{x_2}(R + l)$. The symmetry argument with x_1 and x_2 interchanged yields (5.3).

Sufficiency. Define two random variables R_1 and R_2 as inverse functions to τ_{x_1} and τ_{x_2} applied to a single uniform random variable, so that (5.3) yields that $|R_1 - R_2| \leq l$ a.s. This means that none of the balls $B_{R_1}(x_1)$ and $B_{R_2}(x_2)$ lies in the interior of the other one. Now construct random closed set ξ consisting of two points: a_1 on the boundary of $B_{R_1}(x_1)$ but outside of the interior of $B_{R_2}(x_2)$ and a_2 on the boundary of $B_{R_2}(x_2)$ but outside of the interior of $B_{R_2}(x_1)$. Then a_1 is nearest to x_1 and a_2 is nearest to x_2 with given distributions of the distance. $\square$

APPENDIX: A COMBINATORIAL LEMMA

Recall that $P_t(\mathbb{X})$ denotes the *packing number* of $\mathbb{X}$ with metric $\mathbf{d}$, that is, the maximum number of points in the space $\mathbb{X}$ with pairwise distance exceeding t; see [20], page 78.

LEMMA A.1. *If $Y = \sum \delta_{x_i}$ is a counting measure of total mass n, then for all $t > 0$,*

$$\sum_{i \neq j} \mathbb{1}_{\mathbf{d}(x_i, x_j) \leq t} \geq n\left(\frac{n}{P_t(\mathbb{X})} - 1\right).$$

PROOF. Denote

$$n(Y, x_i) = Y\big(B_t(x_i)\big) - 1,$$

where $B_t(x_i)$ is the closed ball of radius t centred at x_i. Furthermore, denote

$$\mathsf{g}_{h_t}(Y) = \sum_{i \neq j} \mathbb{1}_{\mathbf{d}(x_i, x_j) \le t}.$$

Then

$$\begin{aligned}
\mathsf{g}_{h_t}(Y - \delta_{x_i}) &= \mathsf{g}_{h_t}(Y) - 2n(Y, x_i),\\
\mathsf{g}_{h_t}(Y + \delta_{x_i}) &= \mathsf{g}_{h_t}(Y) + 2n(Y, x_i) + 2.
\end{aligned}$$

Let x_i and x_j be two distinct points from the support of Y with $\mathbf{d}(x_i, x_j) \le t$. Assume that $n(Y, x_i) < n(Y, x_j)$ or $n(Y, x_i) = n(Y, x_j)$ with $i < j$ and define

$$Y' = Y - \delta_{x_j} + \delta_{x_i}$$

obtained from Y by transferring a mass 1 from x_j to x_i. Call $Y'' = Y - \delta_{x_j}$. Remark that $n(Y'', x_i) = n(Y, x_i) - 1$ because $\mathbf{d}(x_i, x_j) \le t$. Since $n(Y, x_j) \ge n(Y, x_i)$,

$$\begin{aligned}
\mathsf{g}_{h_t}(Y') &= \mathsf{g}_{h_t}(Y'') + 2n(Y'', x_i) + 2\\
&= \mathsf{g}_{h_t}(Y) - 2n(Y, x_j) + 2n(Y'', x_i) + 2\\
&= \mathsf{g}_{h_t}(Y) - 2n(Y, x_j) + 2n(Y, x_i) - 2 + 2\\
&\le \mathsf{g}_{h_t}(Y).
\end{aligned}$$

Furthermore, $n(Y', x_i) = n(Y, x_i)$ because the transferred mass remains in the ball with centre x_i and radius t, and $n(Y', x_j) = n(Y, x_j)$ as well. Thus, $n(Y', x_i) \le n(Y', x_j)$. Repeat the mass transfer from x_j to x_i until the mass at x_j disappears. Call the resulting counting measure Y_1.

Apply the same construction to Y_1 and repeat it until there are no more distinct points at distance at most t. This happens in a finite time because the cardinality of the support of Y strictly decreases at each step.

The obtained counting measure $\widehat{Y}$ is supported by a set of points $\{y_1, \dots, y_q\}$ with pairwise distances exceeding t. Thus,

$$\mathsf{g}_{h_t}(Y) \ge \mathsf{g}_{h_t}(\widehat{Y}) = \sum_{i=1}^{q} m_i(m_i - 1),$$

where $m_i = \widehat{Y}(\{y_i\})$. Under the restriction $\sum_{i=1}^{q} m_i = n$, the minimal value $\sum_i m_i(m_i - 1)$ is reached for $m_i = n/q$, whence

$$\mathsf{g}_{h_t}(Y) \ge n\left(\frac{n}{q} - 1\right).$$

It remains to note that $q \le P_t(\mathbb{X})$. $\square$

It is also possible to define a counting measure by placing masses from the interval $[n/q, n/q + 1]$ at the points forming the packing net of $\mathbb{X}$. Thus, there exists a counting measure Y such that

$$\mathsf{g}_{h_t}(Y) \le n\left(\frac{n}{P_t(\mathbb{X})} + 1\right).$$

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