arXiv:1410.2523v2 [math.PR] 30 Apr 2015

A generalisation of the fractional Brownian field based on non-Euclidean norms

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Abstract

We explore a generalisation of the Lévy fractional Brownian field on the Euclidean space based on replacing the Euclidean norm with another norm. A characterisation result for admissible norms yields a complete description of all self-similar Gaussian random fields with stationary increments. Several integral representations of the introduced random fields are derived.

In a similar vein, several non-Euclidean variants of the fractional Poisson field are introduced and it is shown that they share the covariance structure with the fractional Brownian field and converge to it. The shape parameters of the Poisson and Brownian variants are related by convex geometry transforms, namely the radial pth mean body and the polar projection transforms.

Keywords: fractional Brownian field, fractional Poisson field, radial p-th mean body, polar projection body, Minkowski space, star body, norm
2010 MSC: 60G22, 60G55, 60G60, 52A21

1. Introduction

The *multiparameter fractional Brownian motion* or the *Lévy fractional Brownian field* (fBf) with Hurst index $H \in (0, 1)$ is a centred Gaussian random field $X(z)$, $z \in \mathbb{R}^d$, with the covariance function

$$\mathbf{E}[X(z_1)X(z_2)] = \frac{1}{2}\left[\|z_1\|^{2H} + \|z_2\|^{2H} - \|z_1 - z_2\|^{2H}\right], \qquad (1)$$

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where $\|z\|$ is the Euclidean norm of z. If $h = \frac{1}{2}$, this yields the Lévy Brownian motion on $\mathbb{R}^d$. If $d = 1$, one recovers the classical univariate fractional Brownian motion (fBm), see [25]. This random field was introduced by A.M. Yaglom [35] as a model of turbulence in fluid mechanics. Various proofs showing that (1) defines a valid covariance function are given in [11, 25, 29, 30]. Further results including series expansions and a general functional limit theorem can be found in [24]. The two most important integral representations of the Lévy fBf are the moving average representation using the integral with respect to the white noise and the harmonisable representation as an integral with respect to the Fourier transform of the white noise, see [7, 15, 23, 30].

Istas [17] defined the fractional Brownian motion B on a metric space by assuming that the square of its increment $B(x)-B(y)$ is normally distributed with the variance given by the $2H$-power of the metric distance between x and y. The existence of fractional Brownian motions on the Euclidean sphere and on the hyperbolic space is verified for $H \in (0, \frac{1}{2}]$. These constructions have been extended to stable random fields in [18]. See also the recent monograph [7] for a number of results on general self-similar random fields.

Biermé et al. [3] considered a random field generated by a Poisson random measure on $\mathbb{R}^d \times [0, \infty)$ and proved that it shares the same covariance function (1) with the Lévy fBf. Such a field may be called a fractional Poisson field, noticing that other definitions of fractional Poisson fields are available in the literature, see e. g. [27] for the univariate case and [21] for a multivariate generalisation.

In this paper we introduce a generalisation of the fBf based on replacing the Euclidean norm in (1) with a non-Euclidean one. The space $\mathbb{R}^d$ with such norm is called the Minkowski space [33], so that we term our generalisation the Minkowski fractional Brownian field (MfBf). Section 2 introduces necessary concepts from convex geometry. In Section 3 we establish that the norms giving rise to valid covariance functions are generated by L_p-balls related to the isometric embeddability of the Minkowski space into $L_p([0, 1])$ for $p = 2H$. Furthermore, we derive several integral representations of the introduced random field. In addition to conventional integral representations based on integrating the white noise or its Fourier transform, we derive novel representations based on sums of series of Lévy fBf's and integrals of univariate fractional Brownian motions. Furthermore, we relate the ordering of expected supremum of MfBf with the Banach–Mazur distance between normed spaces. The key idea is the equivalence relation on the family of MfBf up to non-degenerate linear transformations of their arguments.

Section 4 introduces random fields based on Poisson point processes that share the covariance function with the MfBf for $H \in (0, \frac{1}{2})$. The construction follows the ideas from [3] and [34], and is also related to random balls models [6] and the studies of micropulses [26]. In difference to the previous works, we emphasise the role of the shape parameters of the corresponding fields. The main result provides a relationship between the shape parameters of the Poisson random field and its Gaussian counterpart. This relationship is given by the radial pth mean body transformation introduced in [12]. The convergence to the Brownian field with $H = \frac{1}{2}$ using different normalisations of the Poisson model is considered in Section 5. These results are new even in the case of Lévy fBf. The shape parameters are related by the polar projection body transform known from the convex geometry [31]. Finally Section 6 presents other constructions of the fractional Poisson fields that share the covariance structure with the MfBf.

2. Norms and star bodies

A closed bounded set F in $\mathbb{R}^d$ is called a *star body* if for every $u \in F$ the interval $\{tu : \ 0 \le t < 1\}$ is contained in the interior of F and the *Minkowski functional* (or the gauge function) of F defined by

$$\|u\|_F = \inf\{s \ge 0 : \ u \in sF\}$$

is a continuous function of $u \in \mathbb{R}^d$. The set F can be recovered from its Minkowski functional by

$$F = \{u : \ \|u\|_F \le 1\} ,$$

while the *radial function* $\rho_F(u) = \|u\|_F^{-1}$ provides the polar coordinate representation of the boundary of F for u from the unit *Euclidean sphere* $\mathbb{S}^{d-1}$. In the following we mainly consider origin-symmetric star-shaped sets and call them *centred* in this case. If the star body F is centred and *convex*, then $\|u\|_F$ becomes a convex norm on $\mathbb{R}^d$ and $(\mathbb{R}^d, \|\cdot\|_F)$ is called a *Minkowski space*, see [33]. We also keep the same notation $\|u\|_F$ if the norm is not convex. Further $\|x\|$ (without subscript) denotes the *Euclidean norm* of $x \in \mathbb{R}^d$. By $\mathrm{V}_d(K)$ we denote the d-dimensional *Lebesgue measure* of a measurable set K.

The *p-sum* of two star bodies F_1 and F_2 is the star body F such that

$$\|u\|_F^p = \|u\|_{F_1}^p + \|u\|_{F_2}^p ,$$

see [20].

The *support function* of a convex set K in $\mathbb{R}^d$ is defined by

$$h(K,u) = \sup\{\langle x,u\rangle :\ x \in K\}\,, \quad u \in \mathbb{R}^d\,.$$

Note that the support function may take infinite values if K is not bounded. The *polar set* to a convex set K containing the origin is defined by

$$K^* = \{u :\ h(K,u) \le 1\}\,.$$

An L_p-*ball* for $p > 0$ is a star body F such that $(\mathbb{R}^d, \|\cdot\|_F)$ is isometrically embeddable into $L_p([0,1])$, see [20, Lemma 6.4]. The star body F is an L_p-ball if and only if

$$\|z\|_F^p = \int_{\mathbb{S}^{d-1}} |\langle z,u\rangle|^p \sigma(du) \tag{2}$$

for a finite even measure σ on the unit Euclidean sphere $\mathbb{S}^{d-1}$, see [14, Lemma 4.8]. The L_p-balls are necessarily convex for $p \in [1,2]$.

Example 2.1. A set F is an L_2-ball if and only if

$$\|z\|_F^2 = \int_{\mathbb{S}^{d-1}} |\langle z,u\rangle|^2 \sigma(du) = \langle Az, z\rangle\,,$$

where the matrix A is symmetric non-negative definite with entries given by

$$a_{ij} = \int_{\mathbb{S}^{d-1}} u_i u_j \sigma(du)\,, \quad i,j = 1,\dots,d\,.$$

Thus, the family of L_2-balls coincides with the family of ellipsoids.

3. Minkowski fractional Brownian field

3.1. Definition and existence

Definition 3.1. A centred Gaussian random field $X_F(z)$, $z \in \mathbb{R}^d$, with the covariance function $C_F(z_1,z_2) = \mathbf{E}[X_F(z_1)X_F(z_2)]$ given by

$$C_F(z_1,z_2) = \frac{1}{2}\left[\|z_1\|_F^{2H} + \|z_2\|_F^{2H} - \|z_1 - z_2\|_F^{2H}\right], \qquad z_1, z_2 \in \mathbb{R}^d\,, \tag{3}$$

is called the *Minkowski fractional Brownian field* (MfBf) with the Hurst parameter $H \in (0,1]$ and the *associated star body* F.

Since in dimension $d = 1$ all Minkowski functionals are identical up to a constant, a non-trivial generalisation is only possible if the dimension d is at least 2.

The increment of the MfBf is centred Gaussian with the variance

$$\mathbf{E}(X_F(z_1) - X_F(z_2))^2 = \|z_1 - z_2\|_F^{2H} .$$

Thus, the MfBf is a Gaussian field with stationary increments. It follows from [10] that the family of MfBf coincides with the family of all Gaussian random fields with stationary increments.

The following well-known result is useful to establish the positive definiteness of the covariance (3). Note that the positive definiteness is understood in the non-strict sense.

Lemma 3.2. *Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be an even function. The function $e^{-cf(z)}$, $z \in \mathbb{R}^d$, is positive definite for all $c > 0$ if and only if the function*

$$A_f(z_1, z_2) = f(z_1) + f(z_2) - f(z_1 - z_2)$$

is positive definite.

Proof. If $t_1, \dots, t_k \in \mathbb{R}$ and $z_1, \dots, z_k \in \mathbb{R}^d$, then

$$\sum_{i,j=1}^{k} A_f(z_i, z_j) t_i t_j = - \sum_{i,j=0}^{k} f(z_i - z_j) t_i t_j ,$$

where $t_0 = -\sum_{i=1}^k t_i$ and $z_0 = 0$ is the origin. Thus, A_f is positive definite if and only if f is negative definite. The latter is equivalent to the positive definiteness of e^{-cf} for all $c > 0$, see [2, Th. 2.2]. □

Proposition 3.3. *The MfBf exists if and only if $H \in (0, 1]$ and F is an L_p-ball with $p = 2H$.*

Proof. It is known [20, Th. 6.6] that $e^{-c\|z\|_F^p}$ is positive definite if and only if $(\mathbb{R}^d, \|\cdot\|_F)$ isometrically embeds in $L_p([0, 1])$, meaning that F is an L_p-ball. By considering $e^{-c|t|^p\|z\|_F}$ as a function of $t \in \mathbb{R}$ with a fixed z, it is easily seen that $p \in (0, 2]$. The rest follows from Lemma 3.2. □

Since the associated star body F is an L_p-ball, the norm $\|u\|_F$ admits representation (2), and the measure σ from (2) is called the *spectral measure* of the corresponding MfBf $X_F(\cdot)$.

Example 3.4 (Lévy fBf). If σ is the rotation invariant measure on the unit sphere with the total mass

$$\sigma(\mathbb{S}^{d-1}) = \frac{\Gamma(H+\frac{d}{2})}{2\pi^{(d-1)/2}\Gamma(H+\frac{1}{2})},$$

then F is the unit Euclidean ball, and we recover the Lévy fBf.

If F is an L_p-ball with $p \in (0,2]$, then F is also an L_r-ball for all $r \in (0,p]$, see [20, Cor. 6.7]. Thus, if the MfBf with the associated star body F exists for some Hurst index H, then it exists for all Hurst indices $H' \in (0,H]$.

Example 3.5. Let

$$F = \{x = (x_1,\dots,x_d) \in \mathbb{R}^d : \ |x_1|^p + \dots + |x_d|^p \le 1\},$$

be the centred ℓ_p-ball in $\mathbb{R}^d$ with $d \ge 2$ for $p \in (0,2]$. The MfBf with the associated star body F exists if and only if $H \in (0, \frac{1}{2}p]$. For $d = 2$, the MfBf exists for any convex centred star body F and $H \in (0, \frac{1}{2}]$, see also [19]. Indeed, it is well known [31, Cor. 3.5.7] that all centred convex bodies in the plane are L_1-balls and so L_p-balls for $p \in (0,1]$.

Example 3.6. If $H = 1$, then F is an L_2-ball, so F is necessarily an ellipsoid that corresponds to the quadratic form determined by matrix A, see Example 2.1. In this case $X_F(z) = \langle z, \xi\rangle$ for centred normally distributed random vector ξ with the covariance matrix A. If $H \in (0,1)$ and $F = \{z : \ \langle Az, z\rangle \le 1\}$ is an ellipsoid with a strictly positive definite matrix A, then $X_F(A^{-1/2}z)$ is the Lévy fBf with the covariance given by (1).

Example 3.7. The family of L_1-balls is the family of polar bodies to zonoids, well-known from convex geometry [31, Sec. 3.5]. Thus, the family of all Minkowski Brownian fields (that appear for $H = \frac{1}{2}$) corresponds to the family of zonoids.

Since F is an L_p-ball with $p = 2H$, (2) implies that $\|z\|_F^{2H} \le c\|z\|^{2H}$ for a constant c, i.e. the variance of the increments of the MfBf is bounded (up to a constant) by that of the Lévy fBf. Therefore, the MfBf inherits the local properties from the Lévy fBf with the same Hurst parameter, in particular it is a.s. continuous.

3.2. Integral representations

Unless F is the Euclidean ball, it is not possible to use the arguments based on the rotational invariance (like in [15, 23]) to derive integral representations of the MfBf.

Let m be the measure on $\mathbb{R}^d$ whose polar representation has the directional component $\sigma(du)$ (being the spectral measure from (2)) and the radial component $(2\pi)^{-d/2}r^{d-1}dr$. Consider a Gaussian measure W_σ on $\mathbb{R}^d$ with the control measure m, so that, for each square integrable function f,

$$\begin{aligned}\mathbf{E}\left(\int_{\mathbb{R}^d} f(x)W_\sigma(dx)\right)^2 &= \int_{\mathbb{R}^d} f(x)m(dx)\\ &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{S}^{d-1}}\int_0^\infty f(ru)r^{d-1}dr\sigma(du)\,.\end{aligned}$$

If σ is the surface area measure on the unit sphere (the $(d-1)$-dimensional Hausdorff measure), then W_σ is the conventional Brownian random measure as considered in [7, Sec. 2.1.6.1] (also called the white noise) up to a multiplicative constant. The Fourier transform $\hat{W}_\sigma$ of W_σ is defined in the sense of generalised functions as in [7, Def. 2.1.16].

Theorem 3.8. *The MfBf with the spectral measure σ is given by*

$$X_F(z) = a_{H,d}\int_{\mathbb{R}^d}\frac{e^{\imath\langle z,y\rangle}-1}{\|y\|^{H+d/2}}\hat{W}_\sigma(dy)\,, \tag{4}$$

where

$$a_{H,d} = 2(2\pi)^{\frac{d-1}{4}}(H\Gamma(2H)\sin(H\pi))^{\frac{1}{2}}\,.$$

Proof. By passing to the polar coordinates and noticing that the measure σ is even,

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{1-e^{\imath\langle z,y\rangle}}{\|y\|^{2H+d}}m(dy) &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{S}^{d-1}}\int_0^\infty\left(1-e^{\imath t\langle z,u\rangle}\right)t^{-2H-1}dt\sigma(du)\\ &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{S}^{d-1}}\int_0^\infty(1-\cos(t\langle z,u\rangle))t^{-2H-1}dt\sigma(du)\\ &= \frac{c_H}{(2\pi)^{d/2}}\int_{\mathbb{S}^{d-1}}|\langle z,u\rangle|^{2H}\sigma(du)\,,\end{aligned}$$

where

$$c_H = \int_0^\infty(1-\cos t)t^{-2H-1}dt = \frac{\pi}{4H\Gamma(2H)\sin(H\pi)}\,,$$

see [30, p. 329]. The rest of the proof is carried over similarly to [15, Prop. 2]. The normalising constant $a_{H,d}$ is derived from the condition that $a_{H,d}^2c_H(2\pi)^{-d/2} = \frac{1}{2}$. □

Corollary 3.9. *The MfBf can be represented as*

$$X_F(z) = a_{H,d} b_{H,d} \int_{\mathbb{R}^d} [\|z-u\|^{H-d/2} - \|u\|^{H-d/2}] W_\sigma(du) \tag{5}$$

for

$$b_{H,d} = 2^{-H} \frac{\Gamma\left(\frac{1}{2}(-H+d/2)\right)}{\Gamma\left(\frac{1}{2}(H+d/2)\right)}.$$

Proof. It suffices to note that the integrand from (5) is the Fourier transform of the integrand from (4) up to the constant $b_{H,d}$, see [13] noticing that the Fourier transform is defined with the factor $(2\pi)^{-d/2}$. $\square$

Example 3.10. Let the spectral measure σ attach the masses $\frac{1}{2}$ to the points $\{\pm v\}$ for a fixed $v \in \mathbb{S}^{d-1}$. Then $\|z\|_F = |\langle z, v\rangle|$. The corresponding covariance function

$$C_v(z_1, z_2) = \frac{1}{2}\left[|\langle z_1, v\rangle|^{2H} + |\langle z_2, v\rangle|^{2H} - |\langle z_1 - z_2, v\rangle|^{2H}\right]$$

is positive definite for all $H \in (0,1]$ and defines the MfBf Y_v. It is easy to see that $Y_v(z) = B_H(\langle z, v\rangle)$ for the univariate fBm B_H.

Proposition 3.11. *Let $B_H(t)$ be the fBm on the real line and let M be an independent of B_H Gaussian white noise on $\mathbb{S}^{d-1}$ with the control measure σ. Then*

$$X_F(z) = \int_{\mathbb{S}^{d-1}} B_H(\langle z, v\rangle) M(dv) \tag{6}$$

is the MfBf with the spectral measure σ.

Proof. The covariance of $X_F(z)$ is

$$\begin{aligned}
\mathbf{E}[X_F(z_1)X_F(z_2)] &= \mathbf{E}[\mathbf{E}[X_F(z_1)X_F(z_2)|B_H]] \\
&= \mathbf{E}\int_{\mathbb{S}^{d-1}} B_H(\langle z_1, v\rangle) B_H(\langle z_2, v\rangle)\sigma(dv) \\
&= \frac{1}{2}\int_{\mathbb{S}^{d-1}} \left[|\langle z_1, v\rangle|^{2H} + |\langle z_2, v\rangle|^{2H} - |\langle z_1 - z_2, v\rangle|^{2H}\right]\sigma(dv)
\end{aligned}$$

and so coincides with (3). $\square$

In particular, if σ is the rotation invariant measure on $\mathbb{S}^{d-1}$ from Example 3.4, then (6) can be viewed as the analogue of the plain-wave expansion of the norm [13, Sec. I.3.10]. Using (6), the results for the univariate fractional Brownian motion can be extended to the multivariate setting.

3.3. Associated star bodies

Consider here the properties of the MfBf in relation to their associated star bodies.

Proposition 3.12. *If $\{F_n, n \geq 1\}$ is a sequence of L_p-balls with $p = 2H$, such that $\|u\|_{F_n} \to \|u\|_F$ for a star body F, then F is an L_p-ball and the finite-dimensional distributions of X_{F_n} converge to those of X_F. If, additionally, there exists $\varepsilon > 0$ such that $F_n \supset \varepsilon B_2^d$ for all sufficiently large n, then X_{F_n} weakly converges to X_F in the space of continuous functions on any compact subset of $\mathbb{R}^d$.*

Proof. The convergence of finite dimensional distributions follows from the convergence of covariance functions, and the positive definiteness of the limiting covariance implies that F is an L_p-ball. Notice that

$$\mathbf{E}(X_{F_n}(z_1) - X_{F_n}(z_2))^{2k} = c_1\|z_1 - z_2\|_{F_n}^{2Hk} \leq c_2\|z_1 - z_2\|^{2Hk}.$$

for all z_1, z_2 from a compact subset of $\mathbb{R}^d$, constants c_1, c_2, and some $k \geq 1$ such that $2kH > d$. The weak convergence follows from the tightness condition from [16], see also [9, Th. 2]. □

Proposition 3.13. *For two MfBf's X_{F_1} and X_{F_2}, we have*

$$\mathbf{E}\sup_{z\in D}|X_{F_1}(z)| \geq \mathbf{E}\sup_{z\in D}|X_{F_2}(z)| \tag{7}$$

for each compact set $D \subset \mathbb{R}^d$ if and only if $F_1 \subset F_2$.

Proof. The sufficiency follows from the Sudakov–Fernique inequality [1, Th. 2.2.3] noticing that

$$\mathbf{E}(X_{F_1}(z_1)-X_{F_1}(z_2))^2 = \|z_1-z_2\|_{F_1}^{2H} \geq \|z_1-z_2\|_{F_2}^{2H} = \mathbf{E}(X_{F_2}(z_1)-X_{F_2}(z_2))^2\,.$$

For the reverse implication, consider $D = \{z\}$. Then $\mathbf{E}|X_{F_1}(z)| \geq \mathbf{E}|X_{F_2}(z)|$ implies $\mathbf{E}X_{F_1}(z)^2 \geq \mathbf{E}X_{F_2}(z)^2$ and it remains to notice that the increments are stationary. □

Let $A \in \mathrm{GL}(d)$ be an invertible matrix. Then $X_{AF}(z)$ is a version of $X_F(A^{-1}z)$, $z \in \mathbb{R}^d$. The MfBf's obtained by such transformations may be regarded as equivalent in a certain sense. In particular, all ellipsoids F can be transformed to the unit ball in this way, so that all MfBf's with elliptical associated star bodies can be considered equivalent to the Lévy fBf.

In view of Proposition 3.13, the infimum of $t \geq 1$ such that

$$\mathbf{E}\sup_{z\in D}|X_{F_2}(z)| \geq \mathbf{E}\sup_{z\in D}|X_{F_1}(Az)| \geq \frac{1}{t}\mathbf{E}\sup_{z\in D}|X_{F_2}(z)|$$

for some $A \in \mathrm{GL}(d)$ and for all compact sets D in $\mathbb{R}^d$ equals the infimum of $t > 0$ such that $F_2 \subset AF_1 \subset tF_2$ for some $A \in \mathrm{GL}(d)$, which is the *Banach–Mazur distance* between the normed spaces $(\mathbb{R}^d, \|\cdot\|_{F_1})$ and $(\mathbb{R}^d, \|\cdot\|_{F_2})$, see [22, Sec. 2.1]. Since the Banach–Mazur distance between $(\mathbb{R}^d, \|\cdot\|_F)$ and the Euclidean space is at most $\sqrt{d}$ (see [22]), we deduce that for each MfBf X_F there is the Lévy fBf X such that

$$\mathbf{E}\sup_{z\in D}|X(z)| \geq \mathbf{E}\sup_{z\in D}|X_F(Az)| \geq \frac{1}{\sqrt{d}}\mathbf{E}\sup_{z\in D}|X(z)|$$

for some $A \in \mathrm{GL}(d)$ and all compact sets $D \subset \mathbb{R}^d$.

Equation (7) provides a possible ordering of Gaussian processes. Another ordering (which is stronger than (7) for fields that vanish at the origin) is the *convex ordering* of all finite-dimensional distributions meaning that

$$\mathbf{E}g(X_{F_1}(z_1), \ldots, X_{F_1}(z_n)) \geq \mathbf{E}g(X_{F_2}(z_1), \ldots, X_{F_2}(z_n))$$

for all $z_1, \ldots, z_n \in \mathbb{R}^d$, $n \geq 1$, and all convex functions $g : \mathbb{R}^n \to \mathbb{R}$, see [28]. In the case of centred Gaussian processes, this is equivalent to the fact that the difference of covariance matrices of finite-dimensional distributions of X_{F_1} and X_{F_2} is positive definite, see [28, Sec. 3.13]. By Lemma 3.2, this holds if and only if $\exp\{-c(\|z\|_{F_1}^{2H} - \|z\|_{F_2}^{2H})\}$ is positive definite for all $c > 0$.

Proposition 3.14. *The MfBf X_{F_1} is greater than or equal to the MfBf X_{F_2} in the convex ordering if and only if $F_1 = F_2 +_p M$ for an L_p-ball M and $p = 2H$, equivalently, if $\sigma_1 = \sigma_2 + \nu$ for a non-negative measure ν, where σ_i is the spectral measure of F_i, $i = 1, 2$.*

Proof. Note that $\|z\|_{F_1}^{2H} - \|z\|_{F_2}^{2H}$ is a homogenous function that can we written as $\|z\|_M^{2H}$ with $F_1 = F_2 +_p M$ by the definition of the p-sum, and the p-sum of two L_p-balls corresponds to the arithmetic addition of their spectral measures. $\square$

Sums of independent MfBf's can be interpreted as follows.

Proposition 3.15. *Let X_{F_1} and X_{F_2} be two independent MfBf's with associated star bodies F_1 and F_2. Then $X_{F_1}+X_{F_2}$ is the MfBf with the associated star body $F = F_1 +_p F_2$ being the p-sum of F_1 and F_2 for $p = 2H$.*

Corollary 3.16. *Each MfBf with $H \in [\frac{1}{2}, 1]$ can be represented as the weak limit (on each compact subset of $\mathbb{R}^d$) of the sums of $X_i(A_i z)$, $i \geq 1$, where $\{X_i, i \geq 1\}$ are i.i.d. Lévy fBf's with covariance* (1) *and $\{A_i, i \geq 1\}$ are positive definite matrices.*

Proof. The result follows from Proposition 3.15 and [14, Th. 6.13] saying that each L_p-ball F with $p \geq 1$ can be represented as the limit (in the Hausdorff metric) for the p-sum of ellipsoids. The MfBf with the associated star body being an ellipsoid can be represented as $X(Az)$, $z \in \mathbb{R}^d$, where X is the Lévy fBf. The finite dimensional distributions converge, since the convergence of sets in the Hausdorff metric yields the convergence of the corresponding norms. Finally, the representation as the sum of ellipsoids guarantees that at least one summand contains a neighbourhood of the origin, so that Proposition 3.12 applies. □

3.4. Sub-fractional fields

Following the definition of the sub-fractional Brownian motion from [4], it is possible to define its Minkowski analogue as the centred Gaussian random field with the covariance

$$C_F^{\mathrm{sub}}(z_1, z_2) = \|z_1\|_F^{2H} + \|z_2\|_F^{2H} - \frac{1}{2}[\|z_1 + z_2\|_F^{2H} + \|z_1 - z_2\|_F^{2H}] . \qquad (8)$$

Since

$$C_F^{\mathrm{sub}}(z_1, z_2) = C_F(z_1, z_2) + C_F(z_1, -z_2) ,$$

(8) defines a valid covariance function if $H \in (0, 1]$ and F is an L_p-ball. The corresponding random field is given by $\frac{1}{2}(X_F(z) + X_F(-z))$ for the MfBf X_F.

The random field $\tilde{X}_F(z) = X_F(z) - X_F(-z)$, $z \in \mathbb{R}^d$, is a Gaussian random field with the covariance $\|z_1 + z_2\|_F^{2H} - \|z_1 - z_2\|_F^{2H}$, whose univariate version was considered in [5].

4. Fractional Poisson fields

4.1. Definition and scaling property

For $H \in (0, \frac{1}{2})$, let $N_H = \{(x_i, r_i), i \geq 1\}$ be the *Poisson point process* (identified with the corresponding counting measure) on $\mathbb{R}^d \times (0, \infty)$ with the *intensity measure*

$$\nu_H(dx, dr) = dx\, r^{-d-1+2H} dr\,. \tag{9}$$

Let K be a convex body in $\mathbb{R}^d$ with non-empty interior.

Definition 4.1. The *fractional Poisson field* with Hurst index H and the shape parameter K is the random field

$$\xi(z) = \int_{\mathbb{R}^d\times(0,\infty)} (\mathbb{1}_{z\in x+rK} - \mathbb{1}_{0\in x+rK})\, N_H(dx, dr)\,. \tag{10}$$

Sometimes we write ξ_K or $\xi_{K,H}$ to emphasise the shape parameter of the field and its Hurst exponent.

The random field (10) for K being the unit Euclidean ball was considered in [3]. The factor λ in front of the intensity of ν_H in [3] can be incorporated into Definition 4.1 using a rescaled variant of K.

Note that

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathbb{1}_{z\in x+rK} - \mathbb{1}_{0\in x+rK}| dx &= \mathrm{V}_d((z+rK) \triangle rK)\\ &\le \min(r^d\, \mathrm{V}_d(K), r^{d-1} b_K(z))\,,\end{aligned}$$

where $\triangle$ denotes the symmetric difference, $b_K(z) = \mathrm{V}_{d-1}(\mathrm{pr}_{z^\perp} K)$ is the $(d-1)$-dimensional volume of the projection of K onto the hyperplane $z^\perp$ orthogonal to z. Therefore, the integrand in (10) belongs to $L^1(\mathbb{R}^d\times(0,\infty), \nu_H)$, so that the integral (10) is well defined. Since the absolute difference of two indicator functions takes values 0 or 1, the integrand in (10) also belongs to $L^2(\mathbb{R}^d \times (0,\infty), \nu_H)$ and

$$\begin{aligned}\mathbf{E}\xi(z)^2 &= \int_{\mathbb{R}^d\times(0,\infty)} (\mathbb{1}_{z\in rK+x} - \mathbb{1}_{0\in rK+x})^2\, \nu_H(dx, dr)\\ &= \int_0^\infty \mathrm{V}_d((z+rK) \triangle rK) r^{-d-1+2H}\, dr\,.\end{aligned}$$

Since $\xi(z_1) - \xi(z_2)$ coincides in distribution with $\xi(z_1 - z_2)$,

$$\mathbf{E}[\xi(z_1)\xi(z_2)] = \frac{1}{2}\big[\mathbf{E}\xi(z_1)^2 + \mathbf{E}\xi(z_2)^2 - \mathbf{E}\xi(z_1 - z_2)^2\big]\,. \tag{11}$$

By computing the probability generating functional (see [8]) of the Poisson process N_H, it is easy to see that the finite-dimensional distributions of ξ have the following characteristic function

$$\mathbf{E}\exp\left\{\sum_{j=1}^{k} \imath t_j \xi_K(z_j)\right\} = \exp\left\{\int_{\mathbb{R}^d\times(0,\infty)} \left(\cos\Big(\sum_{j=1}^{k} t_j \big(\mathbb{1}_{z_j\in rK+x} - \mathbb{1}_{0\in rK+x}\big)\Big) - 1\right) r^{-d-1+2H} dx\, dr\right\} \quad (12)$$

Lemma 4.2. *For all $a>0$, the random fields $\xi_K(az)$, $z\in\mathbb{R}^d$, and $\xi_{bK}(z)$, $z\in\mathbb{R}^d$, with $b=a^{\frac{2H}{d-2H}}$ have identical finite-dimensional distributions. The finite-dimensional distributions of $\xi_K(az)$ equal the a^{2H}-convolution power of those of $\xi_K(z)$.*

Proof. The fractional Poisson field with the shape parameter bK equals the fractional Poisson field with the shape parameter K generated by the point process $\{(x_i, br_i):\ (x_i,r_i)\in N_H\}$. The intensity measure of this transformed process is

$$\mathbf{E}\sum_i \mathbb{1}_{x_i\in D, br_i\geq t} = \nu_H(D\times[b^{-1}t,\infty)) = a^{2H}\nu_H(D\times[t,\infty))$$

for all Borel $D\subset\mathbb{R}^d$ and $t>0$. Thus, ξ_{bK} equals in distribution the fractional Poisson field with the shape parameter K and the intensity measure $a^{2H}\nu_H$. This corresponds to a superposition of independent Poisson processes and so to the convolution power of the distribution. $\square$

4.2. Relation to the MfBf

It is shown in [3] that, if K is the unit Euclidean ball, the covariance (11) of the fractional Poisson field ξ coincides (up to a multiplicative constant) with the covariance function (1) of the Lévy fBf. The case of a general convex body K cannot any longer be handled by the rotational symmetry argument as in [3] and, for this, we need to recall some further concepts from convex geometry. If K is a convex body in $\mathbb{R}^d$, then its *radial pth mean body* R_pK is defined for $p>-1$ by

$$\|u\|_{R_pK} = \left(\frac{1}{\mathrm{V}_d(K)}\int_K \rho_K(x,u)^p dx\right)^{-1/p},$$

where $\rho_K(x,u)=\max\{t:\ x+tu\in K\}$ is the representation of the boundary of K in the spherical coordinates with the origin located at x, see [12].

Theorem 4.3. *The covariance function of the fractional Poisson field ξ given by* (10) *coincides with that of MfBf X with the associated star body*

$$F=\left(\frac{H}{\mathrm{V}_d(K)}\right)^{1/2H} R_{-2H}K\,. \tag{13}$$

Proof. Let $u^\perp$ denote the linear space orthogonal to the non-trivial vector $u\in\mathbb{R}^d$, and let $\ell_{u,K}(y)$ be the length of the segment obtained as the intersection of K with the line $\{y+tu:\ t\in\mathbb{R}\}$. Then

$$\mathrm{V}_d((u+K)\triangle K)=2\int_{u^\perp}\min(\|u\|,\ell_{u,K}(y))dy\,.$$

If $\|z\|=1$, then

$$\begin{aligned}\mathbf{E}\xi(z)^2&=\int_0^\infty \mathrm{V}_d((z+rK)\triangle rK)r^{-d-1+2H}\,dr\\&=\int_0^\infty \mathrm{V}_d((zs+K)\triangle K)s^{-1-2H}ds\\&=2\int_0^\infty\int_{z^\perp}\min(s,\ell_{z,K}(y))s^{-1-2H}dyds\\&=\frac{1}{H(1-2H)}\int_{z^\perp}\ell_{z,K}(y)^{-2H+1}dy\,.\end{aligned}\tag{14}$$

It is shown in [12, Lemma 2.1] that, for $p>-1$,

$$\int_K\rho_K(x,u)^pdx=\frac{1}{p+1}\int_{u^\perp}\ell_{u,K}(y)^{p+1}dy\,.$$

Thus, for $p=-2H$, we have

$$\begin{aligned}\mathbf{E}\xi(z)^2&=\frac{1}{H(1-2H)}(-2H+1)\int_K\rho_K(x,z)^{-2H}dx\\&=\frac{1}{H}\mathrm{V}_d(K)\|z\|^{2H}_{R_{-2H}K}\,.\end{aligned}$$

It remains to note that $\mathbf{E}\xi(tz)^2=t^{2H}\mathbf{E}\xi(z)^2$ for $t>0$. $\square$

Corollary 4.4. *The radial pth mean body R_pK of any convex body K is an L_p-ball for each $p \in (-1, 0)$.*

It is not known if the radial pth mean body is convex for $p \in (-1, 0)$ and if two different (up to a translation) convex bodies share the same radial pth mean body for any single $p \in (-1, 0)$, see [12]. An inverse to the transform R_p is not yet found.

Below we present a limit theorem that yields the MfBf as a limit when the intensity of the fractional Poisson field ξ grows and its argument is rescaled.

Proposition 4.5. *The finite-dimensional distributions of $a^{-H}\xi_K(az)$ converge as $a \to \infty$ to those of the MfBf $X_F(z)$, $z \in \mathbb{R}^d$, with the associated star body F given by (13). Furthermore, $a^{-H}\int_L \xi_K(az)\,dz$ converges in distribution as $a \to \infty$ to $\int_L X_F(z)\,dz$ for each bounded Borel set L.*

Proof. By Lemma 4.2, if $a^{2H} = m$ is an integer, then $a^{-H}\xi(az)$ is the sum of m i. i. d. copies of $\xi(z)$ normalised by $\sqrt{m}$. By the central limit theorem, it converges to the Gaussian random field that shares the same covariance structure with ξ, so the MfBf X. A standard argument completes the proof of convergence in distribution along an arbitrary sequence $a \to \infty$. The weak convergence of integrals follows from the central limit theorem and the fact that the variances of $\int_L \xi(z)\,dz$ and $\int_L X_F(z)\,dz$ coincide. ☐

5. Convergence to MfBf with $H = \frac{1}{2}$

The results from Section 4.2 concern the case of the Hurst parameter $H \in (0, \frac{1}{2})$. Below we explore the convergence of the Poisson fractional field to the MfBf with $H = \frac{1}{2}$.

The function $b_K(u) = \mathrm{V}_{d-1}(\mathrm{pr}_{u^\perp} K)$, $u \in \mathbb{S}^{d-1}$, is the support function of the projection body ΠK to K. The corresponding polar body is denoted by $\Pi^* K$ and called the *polar projection body*, see [31, p. 570], so that $\|u\|_{\Pi^* K} = b_K(u)$ for $u \in \mathbb{S}^{d-1}$. It is shown in [12] that the polar projection body $\Pi^* K$ can be obtained as the limit of $((p+1)\,\mathrm{V}_d(K))^{-1/p} R_p K$ as $p \downarrow -1$.

Theorem 5.1. *The finite-dimensional distributions of the random field $\sqrt{1-2H}\,\xi_{K,H}(z)$, $z \in \mathbb{R}^d$, converge as $H \uparrow \frac{1}{2}$ to the finite-dimensional distributions of the MfBf $X_F(z)$, $z \in \mathbb{R}^d$, with the Hurst parameter $H = \frac{1}{2}$ and the associated star body $F = \frac{1}{2}\Pi^* K$.*

Proof. Using (14), we have for $\|z\| = 1$

$$(1-2H)\mathbf{E}\xi_{K,H}(z)^2 = \frac{1}{H}\int_{z^\perp} \ell_{z,K}(y)^{-2H+1} dy\,.$$

For $H = \frac{1}{2}$, the integral equals $\|z\|_{\Pi^* K}$, see [12]. Similarly to the proof of Theorem 4.3, we obtain for arbitrary $z \in \mathbb{R}^d$ the convergence

$$(1-2H)\mathbf{E}\xi_{K,H}(z)^2 \to \|z\|_F \quad \text{as } H \uparrow \frac{1}{2}.$$

By (12),

$$\begin{aligned}
&\mathbf{E}\exp\left\{\sum_{j=1}^{k} \imath t_j \sqrt{1-2H}\,\xi_{K,H}(z_j)\right\} \\
&\qquad = \exp\left\{\int_{\mathbb{R}^d\times(0,\infty)} (\cos\theta - 1)\, r^{-d-1+2H} dx\, dr\right\} = \exp\{I_1 + I_2\},
\end{aligned}$$

where

$$\begin{aligned}
\theta &= \sqrt{1-2H}\sum_{j=1}^{k} t_j \left(\mathbb{1}_{z_j\in rK+x} - \mathbb{1}_{0\in rK+x}\right), \\
I_1 &= -\frac{1}{2}\int_{\mathbb{R}^d\times(0,\infty)} \theta^2 r^{-d-1+2H} dx\, dr, \\
I_2 &= \int_{\mathbb{R}^d\times(0,\infty)} \left(\cos\theta - 1 + \frac{\theta^2}{2}\right) r^{-d-1+2H} dx\, dr.
\end{aligned}$$

Then

$$\begin{aligned}
I_1 = &-\frac{1-2H}{2}\int_{\mathbb{R}^d\times(0,\infty)} \sum_{j,k} t_j t_k \left(\mathbb{1}_{z_j\in rK+x} - \mathbb{1}_{0\in rK+x}\right) \\
&\times \left(\mathbb{1}_{z_k\in rK+x} - \mathbb{1}_{0\in rK+x}\right) r^{-d-1+2H} dx\, dr \\
= &-\frac{1-2H}{2}\sum_{j,k} t_j t_k \mathbf{E}\left[\xi_{K,H}(z_j)\xi_{K,H}(z_k)\right] \\
\to &-\frac{1}{4}\sum_{j,k} t_j t_k \left(\|z_j\|_F + \|z_k\|_F - \|z_j - z_k\|_F\right) \qquad \text{as } H \uparrow \frac{1}{2}.
\end{aligned}$$

The elementary inequality $\left|\cos\theta - 1 + \frac{\theta^2}{2}\right| \leq \frac{|\theta|^3}{6}$ yields that

$$\begin{aligned}|I_2| \leq \frac{1}{6}(1-2H)^{3/2} \sum_{i,j,k} |t_i t_j t_k| \int_{\mathbb{R}^d\times(0,\infty)} \big|(\mathbb{1}_{z_i \in rK+x} - \mathbb{1}_{0\in rK+x}) \\ \times \left(\mathbb{1}_{z_j\in rK+x} - \mathbb{1}_{0\in rK+x}\right)(\mathbb{1}_{z_k\in rK+x} - \mathbb{1}_{0\in rK+x})\big| r^{-d-1+2H} dx\, dr.\end{aligned}$$

The Hölder inequality implies that

$$|I_2| \leq \frac{\sqrt{1-2H}}{6} \sum_{i,j,k} |t_i t_j t_k| \sqrt[3]{\mu_H(z_i)\mu_H(z_j)\mu_H(z_k)},$$

where

$$\begin{aligned}\mu_H(z) &= (1-2H) \int_{\mathbb{R}^d\times(0,\infty)} |\mathbb{1}_{z\in rK+x} - \mathbb{1}_{0\in rK+x}|^3\, r^{-d-1+2H} dx\, dr \\ &= (1-2H)\mathbf{E}\xi_{K,H}(z)^2 \to \|z\|_F \quad \text{as } H \uparrow \frac{1}{2}\end{aligned}$$

for all $z \in \mathbb{R}^d$. Thus, $|I_2| \to 0$ as $H \uparrow \frac{1}{2}$. □

The integral (10) fails to converge if $H \geq \frac{1}{2}$. It is possible to truncate it to ensure the convergence as follows. For $C > 0$ and $p > \frac{1}{2}$, define

$$\eta_{C,p}(z) = \int_{\mathbb{R}^d\times[0,C]} (\mathbb{1}_{z\in x+rK} - \mathbb{1}_{0\in x+rK})\, N_p(dx, dr)\,,$$

where N_p is the Poisson process with intensity ν_H from (9) for $H = p$. It is easy to see that $\eta_{C,p}$ is well defined. The following result shows that its normalised version converges to the MfBf with $H = \frac{1}{2}$ no matter what p is.

Theorem 5.2. *For any $p > \frac{1}{2}$, the finite-dimensional distributions of the random field $C^{1/2-p}\eta_{C,p}(z)$, $z \in \mathbb{R}^d$, converge as $C \to \infty$ to the finite-dimensional distributions of the MfBf with the Hurst parameter $H = \frac{1}{2}$ and the associated star body $F = \left(p - \frac{1}{2}\right)\Pi^* K$.*

Proof. First let us show that

$$C^{1-2p}\,\mathbf{E}\eta_{C,p}(z)^2 \to \|z\|_F\,. \tag{15}$$

Let $\|z\|=1$. Similarly to (14),

$$C^{1-2p}\,\mathbf{E}\eta_{C,p}(z)^2 = 2C^{1-2p}\int_{1/C}^{\infty}\int_{z^\perp}\min(s,\ell_{z,K}(y))s^{-1-2p}dyds.$$

Further,

$$\begin{aligned}
C^{1-2p}\,\mathbf{E}\eta_{C,p}(z)^2 &= 2C^{1-2p}\int_{z^\perp}\int_{1/C}^{\infty}\ell_{z,K}(y)s^{-1-2p}\mathbb{1}_{\ell_{z,K}(y)<\frac{1}{C}}dsdy\\
&\quad + 2C^{1-2p}\int_{z^\perp}\int_{1/C}^{\ell_{z,K}(y)}s^{-2p}\mathbb{1}_{\ell_{z,K}(y)\geq\frac{1}{C}}dsdy\\
&\quad + 2C^{1-2p}\int_{z^\perp}\int_{\ell_{z,K}(y)}^{\infty}\ell_{z,K}(y)s^{-1-2p}\mathbb{1}_{\ell_{z,K}(y)\geq\frac{1}{C}}dsdy\\
&= \frac{C}{p}\int_{z^\perp}\ell_{z,K}(y)\mathbb{1}_{\ell_{z,K}(y)<\frac{1}{C}}dy + \frac{2}{2p-1}\int_{z^\perp}\mathbb{1}_{\ell_{z,K}(y)\geq\frac{1}{C}}dy\\
&\quad + \frac{C^{1-2p}}{p(2p-1)}\int_{z^\perp}\ell_{z,K}(y)^{1-2p}\mathbb{1}_{\ell_{z,K}(y)\geq\frac{1}{C}}dy.
\end{aligned}$$

The first term in the right-hand side can be bounded as follows

$$\begin{aligned}
C\int_{z^\perp}\ell_{z,K}(y)\mathbb{1}_{\ell_{z,K}(y)<\frac{1}{C}}dy &\leq \int_{z^\perp}\mathbb{1}_{\ell_{z,K}(y)<\frac{1}{C}}dy\\
&= V_{d-1}\left(\left\{y\in \mathrm{pr}_{z^\perp}K : \ell_{z,K}(y)<\tfrac{1}{C}\right\}\right)\to 0 \quad \text{as } C\to\infty.
\end{aligned}$$

The second term converges to $\frac{2}{2p-1}\|z\|_{\Pi^*K}=\|z\|_F$. For the third term,

$$\begin{aligned}
&C^{1-2p}\int_{z^\perp}\ell_{z,K}(y)^{1-2p}\mathbb{1}_{\ell_{z,K}(y)\geq\frac{1}{C}}dy\\
&\quad = C^{1-2p}\int_{z^\perp}\ell_{z,K}(y)^{1-2p}\mathbb{1}_{\frac{1}{C}\leq\ell_{z,K}(y)<\frac{1}{\sqrt{C}}}dy\\
&\qquad + C^{1-2p}\int_{z^\perp}\ell_{z,K}(y)^{1-2p}\mathbb{1}_{\ell_{z,K}(y)\geq\frac{1}{\sqrt{C}}}dy\\
&\quad \leq \int_{z^\perp}\mathbb{1}_{\frac{1}{C}\leq\ell_{z,K}(y)<\frac{1}{\sqrt{C}}}dy + C^{1/2-p}\int_{z^\perp}\mathbb{1}_{\ell_{z,K}(y)\geq\frac{1}{\sqrt{C}}}dy\\
&\quad \leq \int_{z^\perp}\mathbb{1}_{\ell_{z,K}(y)<\frac{1}{\sqrt{C}}}dy + C^{1/2-p}\|z\|_{\Pi^*K}\to 0
\end{aligned}$$

as $C\to\infty$. Thus, (15) holds if $\|z\|=1$. Since, for $t>0$,

$$C^{1-2p}\,\mathbf{E}\eta_{C,p}(tz)^2 = C^{1-2p}t^{2p}\,\mathbf{E}\eta_{C/t,p}(z)^2\to t\|z\|_F \quad \text{as } C\to\infty,$$

(15) holds for arbitrary $z \in \mathbb{R}^d$.

The convergence of characteristic functions can be verified similarly to the proof of Theorem 5.1. $\square$

6. Other constructions of fractional Poisson fields

The section describes other constructions of Poisson random fields that share the same covariance structure with the MfBf.

6.1. Introducing the directional component to the Poisson process

Let F be an L_p-ball with $p = 2H$ for $H \in (0, \frac{1}{2})$, and let σ be the corresponding spectral measure defined by (2). Consider a Poisson point process N'_H on $\mathbb{R} \times (0, \infty) \times \mathbb{S}^{d-1}$ with the intensity measure

$$\nu'_H(dx, dr, dv) = dx\, r^{-d-1+2H} dr\, \sigma(dv).$$

Define the random field

$$\zeta(z) = \int_{\mathbb{R}\times(0,\infty)\times\mathbb{S}^{d-1}} \left(\mathbb{1}_{\langle z,v\rangle\in[x-r,x+r]} - \mathbb{1}_{0\in[x-r,x+r]}\right)\, N'_H(dx, dr, dv),$$

$z \in \mathbb{R}^d$. Let $\xi(y)$, $y \in \mathbb{R}$, be the univariate fractional Poisson field with Hurst index H and the shape parameter $K = [-1, 1]$. Then $\mathbf{E}\xi(y)^2 = c_H |y|^{2H}$, where $c_H = \frac{2^{1-2H}}{H(1-2H)}$, see [3]. Therefore,

$$\begin{aligned}\mathbf{E}\zeta(z)^2 &= \int_{\mathbb{R}^d\times(0,\infty)\times\mathbb{S}^{d-1}} \left(\mathbb{1}_{\langle z,v\rangle\in[x-r,x+r]} - \mathbb{1}_{0\in[x-r,x+r]}\right)^2\, \nu'_H(dx, dr, dv)\\ &= \int_{\mathbb{S}^{d-1}} \mathbf{E}\xi(\langle z, v\rangle)^2\, \sigma(dv) = c_H \int_{\mathbb{S}^{d-1}} |\langle z, v\rangle|^{2H}\, \sigma(dv) = c_H \|z\|_F^{2H}.\end{aligned}$$

Hence, the covariance function of $\zeta(z)$ is, up to a constant, the covariance function of MfBf with the associated star body F.

6.2. Poisson processes on Grassmannians

The affine Grassmannian $A(d, q)$ in $\mathbb{R}^d$ is the family of q-dimensional affine subspaces of $\mathbb{R}^d$, see [32, Sec. 13.2]. In particular, there is a unique invariant normalised Haar measure μ_q on $A(d, q)$, see [32, Th. 13.2.12]. Let $N_{H,q} = \{(L_i, r_i)\}$ be the Poisson process on $A(d, q) \times (0, \infty)$ with intensity being the product measure of μ_q and the measure with the density $r^{-d-1+q+2H}$ on $(0,\infty)$. Define

$$\xi(z) = \int_{A(d,q)\times(0,\infty)} (\mathbb{1}_{z\in r_i K+L_i} - \mathbb{1}_{0\in r_i K+L_i})\, N_{H,q}(dL,dr)\,.$$

By [32, Eq. (13.9)],

$$\begin{aligned}\int_{A(d,q)} &\left|\mathbb{1}_{z+r\check{K}\cap L\neq\emptyset} - \mathbb{1}_{r\check{K}\cap L\neq\emptyset}\right| \mu_q(dL)\\ &\qquad\qquad = \int_{G(d,q)} \mathrm{V}_{d-q}((\mathrm{pr}_{L^\perp}(rK)+z_{L^\perp}) \triangle \mathrm{pr}_{L^\perp}(rK))\nu_q(dL)\,,\end{aligned}$$

where $\check{K} = \{-x : x\in K\}$, ν_q is the Haar probability measure on the Grassmannian $G(d,q)$ (the family of all q-dimensional linear subspaces in $\mathbb{R}^d$), $\mathrm{pr}_{L^\perp}$ denotes the projection on the subspace $L^\perp$ orthogonal to L, and $z_{L^\perp}$ is the projection of z onto $L^\perp$. The integrand is bounded by a constant times $\min(r^{d-q}, r^{d-1-q})$, so that $\xi(z)$ is well defined for $H\in(0,\frac{1}{2})$.

The variance of $\xi(z)$ is given by

$$\mathbf{E}\xi(z)^2 = \int_{G(d,q)} \int_0^\infty \mathrm{V}_{d-q}((\mathrm{pr}_{L^\perp}(K)+sz_{L^\perp}) \triangle \mathrm{pr}_{L^\perp}(K))\nu_q(dL) s^{-1-2H} ds\,.$$

Denote the right-hand side of (13) by $F_H(K)$. Arguing as in the proof of Theorem 4.3, we obtain that

$$\begin{aligned}\mathbf{E}\xi(z)^2 &= \frac{1}{H(1-2H)} \int_{G(d,q)} \int_{z^\perp\cap L^\perp} \ell_{z_{L^\perp},\mathrm{pr}_{L^\perp} K}(y)^{-2H-1} dy \nu_q(dL)\\ &= \frac{1}{H}\int_{G(d,q)}\int_{\mathrm{pr}_{L^\perp} K} \rho(x, z_{L^\perp})^{-2H} dx \nu_q(dL)\\ &= \int_{G(d,q)} \|z_{L^\perp}\|^{2H}_{F_H(\mathrm{pr}_{L^\perp} K)} \nu_q(dL)\\ &= \int_{G(d,q)} \|z\|_{F_H(\mathrm{pr}_{L^\perp} K)} \nu_q(dL) = \|z\|^{2H}_{\tilde{F}}\,.\end{aligned}$$

The penultimate equation follows from the fact that $F_H(\mathrm{pr}_{L^\perp} K)$ is given by the sum of L and a subset of $L^\perp$. The associated star body $\tilde{F}$ is obtained as the limit of the p-sums (with $p = 2H$) of scaled radial pth mean bodies of the projections of K. Such MfBf can be viewed as the integral

$$\int_{G(d,q)} \eta_L(z_{L^\perp})\nu_q(dL)$$

of the independent MfBf's on $L^\perp$ indexed by $L \in G(d,q)$, each having the associated star body $F_H(\mathrm{pr}_{L^\perp} K)$ being the scaled radial pth mean body of $\mathrm{pr}_{L^\perp} K$.

Acknowledgements

IM was supported in part by Swiss National Science Foundation grant 200021-137527. KR was supported by the Swiss Government Excellence Scholarship. The authors are grateful to Richard Gardner for advise on the radial pth mean bodies and to Nikolay Leonenko for comments on the draft version. The comments of the referee have led to numerous improvements to the presentation.

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