Saddlepoint approximations to the probability of ruin in finite time for the compound Poisson risk process perturbed by diffusion

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Abstract

A large deviations type approximation to the probability of ruin within a finite time for the compound Poisson risk process perturbed by diffusion is derived. This approximation is based on the saddlepoint method and generalizes the approximation for the non-perturbed risk process by Barndorff-Nielsen and Schmidli (1995). An importance sampling approximation to this probability of ruin is also provided. Numerical illustrations assess the accuracy of the saddlepoint approximation using importance sampling as a benchmark. The relative deviations between saddlepoint approximation and importance sampling are very small, even for extremely small probabilities of ruin. The saddlepoint approximation is however substantially faster to compute.

Key words

Conditional distribution, cumulant generating function, Gerber-Shiu function, importance sampling, Laplace transform, large deviations techniques, Monte Carlo simulation, relative error.

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1 Introduction

This article considers the risk process perturbed by diffusion as a stochastic model for the fluctuations of the insurer reserve over the time. Let $X_1, X_2, \ldots > 0$ be independent individual claim amounts with absolutely continuous distribution function F_X, let $\{N_t\}_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ and let $\{W_t\}_{t\geq 0}$ be a Wiener process. The individual claim amounts, the Poisson process and the Wiener process are assumed independent. All these random elements are defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The Cramér-Lundberg perturbed risk process is then defined by

$$Y_t = x + ct - \sum_{i=0}^{N_t} X_i + \sigma W_t, \tag{1}$$

$\forall\, t \geq 0$, where $X_0 \stackrel{\text{def}}{=} 0$, $x \geq 0$ is the initial capital, $c > 0$ is the premium rate and $\sigma > 0$ is a scale parameter for the perturbation. Thus $Y_0 = x$. The diffusion term takes the uncertainties related with claim amounts or with premium incomes into consideration. Model (1) is a perturbation of the standard risk process, which can be obtained by setting $\sigma = 0$, and its origins go back to Gerber (1970) and Dufresne and Gerber (1991). There is a considerable literature on perturbed risk processes and a review is provided by Schmidli (1999). Two extensive references on risk processes are Asmussen and Albrecher (2010) and Rolski et al. (1999). Note that (1) is an instance of a spectrally negative Lévy process.

Let us define the time of ruin as

$$T = \begin{cases} \inf\{t \in (0,\infty) | Y_t \leq 0\}, & \text{if the infimum exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

The probability of ruin within the finite time horizon $(0, t)$, for any $t \in (0,\infty)$, is defined by $\psi(x,t) = \mathbb{P}(T < t \mid Y_0 = x)$. It is the probability that $\{Y_t\}_{t\geq 0}$ falls below the zero line prior to time t. The probability of ruin within the infinite time horizon is defined by $\psi(x) = \mathbb{P}(T < \infty \mid Y_0 = x)$. It is the probability that $\{Y_t\}_{t\geq 0}$ ever falls below the zero line. In the following, unless the time horizon is stated explicitly, the probability of ruin refers to the infinite time horizon. The quantity of interest in this article is the probability of ruin within a finite time horizon and the aim is to provide an accurate and efficient computational technique for this probability, by generalizing the saddlepoint approximation of Barndorff-Nielsen and Schmidli (1995) to the perturbed risk process (1). This approximation is then compared with importance sampling based on exponential change of measure. For this purpose, a generalization of the importance sampling algorithm for the unperturbed risk process, given by Asmussen (1985) and Asmussen and Albrecher (2010, Section XV.4), is provided. Although computer intensive, importance sampling is typically very accurate, see e. g. Gatto and Mosimann (2012) for the infinite time probability of ruin. So it is here considered as a benchmark for the numerical performance of the saddlepoint approximation. Another Monte Carlo method obtained from the generation of a sample path of a dual process is also considered. We show that the saddlepoint approximation and importance sampling, which are large deviations techniques, are numerically very close, in the sense that their relative deviations are very small. The Monte Carlo method with the dual process is generally distant from importance sampling (and so from the saddlepoint approximation).

Some literature on computational methods for probabilities of ruin is the following. Dufresne and Gerber (1989) propose two methods for computing the probability of ruin of the risk process without perturbation: one based on recursive evaluations of upper and lower bounds and another one based on the simulation of a dual process. Gatto and Mosimann (2012) extend both methods to the perturbed risk process (1). They also propose saddlepoint approximations to the probability of ruin $\psi(x)$ and to $\psi^{(1)}(x) \stackrel{\text{def}}{=} \mathbb{P}(T < \infty, Y_T = 0 \mid Y_0 = x)$, namely the probability that the zero line is first crossed by an oscillation of the path of the risk process, i. e. by creeping. With these approximations one clearly obtains $\psi^{(2)}(x) \stackrel{\text{def}}{=} \mathbb{P}(T < \infty, Y_T < 0 \mid Y_0 = x) = \psi(x) - \psi^{(1)}(x)$ as well, which is the probability that the zero line is first crossed by a jump or a claim of the path risk process. Gatto and Baumgartner (2014) define the value at ruin (VaRu) as the initial capital required to reach a fixed probability of ruin and the tail value at ruin (TVaRu) as an extension of it. They provide saddlepoint approximations to these two measures of risk. For phase type individual claim amounts, Asmussen and Rolski (1991) derive an exact formula for the probability of ruin in risk process without perturbation, see also Rolski et al. (1999, Theorem 8.3.1). Phase type distributions are characterized as matrix exponential distributions and include hypo-exponential distributions (i. e. convolutions of exponential distributions), see e.g. Rolski et al. (1999, Section 8.2). In this context, Stanford et al. (2011) suggest an algorithm for computing finite time probabilities of ruin.

The remaining part of this article is as follows. Section 2 presents the saddlepoint approximation. In Section 2.1 we summarize saddlepoint approximations to distribution functions and to conditional distribution functions. We show to apply them in presence of defective probability measures, following Barndorff-Nielsen and Schmidli (1995, Section 2). In Section 2.2 we provide the double Laplace transform of the time of ruin and the initial capital, when represented as an exponential random variable, by generalizing results by Barndorff-Nielsen and Schmidli (1995, Section 4). This Laplace transform with the saddlepoint approximations of Section 2.1 yield the approximation to the finite time probability of ruin. Practical computational aspects are presented in Section 2.3. Section 3 shows the generalization of the importance sampling algorithm for unperturbed risk process by Asmussen (1985) to the perturbed process. In Section 4 we show the results of an intensive numerical study. A short conclusion is given in Section 5. Finally, the Appendix presents: a proof of the double Laplace transform of Section 2.2, some proofs of the importance sampling algorithms of Section 3 and a summary of the Monte Carlo method based on the dual process.

2 Saddlepoint approximations

2.1 Saddlepoint approximations to conditional distribution functions and application to defective distributions

The saddlepoint technique of asymptotic analysis provides very accurate large deviations approximations to densities or distribution functions. The saddlepoint approximation to the density of the sample mean of n i. i. d. random variables was proposed by Daniels (1954). Its relative error is $\mathrm{O}(n^{-1})$, as $n \to \infty$, uniformly over any bounded set, i. e. over large deviation regions. Asymptotic errors of the Edgeworth expansion hold over sets which converge towards the population mean at the rate $n^{-1/2}$, as $n \to \infty$, i. e. over normal deviation regions. Lugannani and Rice (1980) proposed a saddlepoint approximation to distribution functions and Skovgaard (1987) to conditional distribution functions.

Let $X, X_1, \ldots, X_n$ be independent, identically distributed and absolutely continuous random variables with cumulant generating function (CGF) K, let $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$ and denote by ϕ and Φ the standard normal density and distribution function. Lugannani and Rice's saddlepoint approximation to $\mathbb{P}(\bar{X} \geq x)$ is given by

$$1 - \Phi(r_x) + \phi(r_x)\left(\frac{1}{s_x} - \frac{1}{r_x}\right), \tag{2}$$

where

$$r_x = \mathrm{sgn}(\alpha_x)\Big\{2n\big[x\alpha_x - K(\alpha_x)\big]\Big\}^{\frac{1}{2}}, \;\; s_x = \alpha_x\big\{nK''(\alpha_x)\big\}^{\frac{1}{2}}$$

and α_x is the solution in v of the saddlepoint equation

$$K'(v) = x,$$

for x in the range of X. The relative error of (2) is $\mathrm{O}(n^{-1})$, as $n \to \infty$, uniformly over bounded sets. Both $1/r_x$ and $1/s_x$ become arbitrarily large as x approaches $\mathbb{E}[X]$ and for this reason Lugannani and Rice's approximation (2) can be numerically misleading for x over small neighborhoods of $\mathbb{E}[X]$. Let

$$r_x^* = r_x + \frac{1}{r_x}\log\frac{s_x}{r_x},$$

then Jensen (1992, Lemma 2.1) shows that $1-\Phi(r_x^*)$ is an approximation to (2) with relative error $\mathrm{O}(n^{-1})$, as $n\to\infty$, uniformly over bounded sets. The quantity r_x has the interpretation of a modified signed likelihood ratio statistic and r_x^* is an adjustment due to Barndorff-Nielsen (1986, 1990*a*,*b*, 1991), yielding asymptotic normality with absolute error of the order $\mathrm{O}(n^{-3/2})$.

Now let $(T,X),(T_1,X_1),\ldots,(T_n,X_n)$ be i. i. d. absolutely continuous bivariate random vectors with CGF K. Let

$$K''(v_1,v_2)=\left(\frac{\partial^2}{\partial v_i\,\partial v_j}K(v_i,v_j)\right)_{i,j=1,2} \tag{3}$$

be the Hessian of K and

$$K''_{22}(0,v)=\frac{\partial^2}{\partial v^2}K(0,v). \tag{4}$$

Let us denote $(\bar{T},\bar{X})=n^{-1}\sum_{i=1}^n(T_i,X_i)$. Skovgaard's approximation to the conditional survival function $\mathbb{P}(\bar{T}\geq t\mid\bar{X}=x)$ is given by

$$1-\Phi(r_{t,x})+\phi(r_{t,x})\left(\frac{1}{s_{t,x}}-\frac{1}{r_{t,x}}\right), \tag{5}$$

where

$$r_{t,x}=\mathrm{sgn}(\beta_{t,x})\Big\{2n\big[(\alpha_{t,x},\beta_{t,x})(t,x)^\top-K(\alpha_{t,x},\beta_{t,x})-\gamma_x x+K(0,\gamma_x)\big]\Big\}^{\frac{1}{2}}, \tag{6}$$

$$s_{t,x}=\beta_{t,x}\left(n\frac{\det\big(K''(\alpha_{t,x},\beta_{t,x})\big)}{K''_{22}(0,\gamma_x)}\right)^{\frac{1}{2}}, \tag{7}$$

$(\alpha_{t,x},\beta_{t,x})$ is the solution in (v_1,v_2) of the bivariate saddlepoint equation

$$\left(\frac{\partial}{\partial v_1}K(v_1,v_2),\frac{\partial}{\partial v_2}K(v_1,v_2)\right)=(t,x) \tag{8}$$

and γ_x is the solution in v of the marginal saddlepoint equation

$$\frac{\partial}{\partial v}K(0,v)=x, \tag{9}$$

for t in the range of T and for x in the range of X. The relative error of (5) is $\mathrm{O}(n^{-1})$, as $n\to\infty$, uniformly over bounded sets. As with Lugannani and Rice's formula, both $1/r_{t,x}$ and $1/s_{t,x}$ become arbitrarily large as (t,x) approaches $(\mathbb{E}[T],\mathbb{E}[X])$, so (5) can be numerically misleading for (t,x) over small neighborhoods of $(\mathbb{E}[T],\mathbb{E}[X])$.

Let

$$r_{t,x}^*=r_{t,x}+\frac{1}{r_{t,x}}\log\frac{s_{t,x}}{r_{t,x}},$$

then

$$1-\Phi(r_{t,x}^*)$$

is an approximation to (5) with relative error $\mathrm{O}(n^{-1})$, as $n\to\infty$, uniformly over bounded sets. The generalization of (5) to the conditional distribution of an M-estimator given another M-estimator is provided by Gatto and Jammalamadaka (1999). General references on saddlepoint approximations are e. g. Field and Ronchetti (1990) and Jensen (1995).

The following developments are from Barndorff-Nielsen and Schmidli (1995). The random variable T has range $(0,\infty]$, the random variable X has range $(0,\infty)$ and we assume $\mathbb{P}(T=\infty \mid X=x)>0$, $\forall x>0$. In this sense T has a defective distribution under the probability measure $\mathbb{P}$. Thus, if f denotes the absolutely continuous part of the joint density of (T,X) under $\mathbb{P}$, then $\int_0^\infty\int_0^\infty f(t,x)\,\mathrm{d}t\,\mathrm{d}x<1$. Let us define

$$K(\upsilon_1,\upsilon_2)=\log\int_0^\infty\int_0^\infty \mathrm{e}^{\upsilon_1 t+\upsilon_2 x} f(t,x)\,\mathrm{d}t\,\mathrm{d}x, \tag{10}$$

and $\operatorname{dom} K=\{(\upsilon_1,\upsilon_2)\in\mathbb{R}^2 | K(\upsilon_1,\upsilon_2)<\infty\}$. We assume the following steepness property of K,

$$\lim_{\varepsilon\downarrow 0}\frac{\partial}{\partial\varepsilon}K\big((1-\varepsilon)(\upsilon_1,\upsilon_2)+\varepsilon(\bar\upsilon_1,\bar\upsilon_2)\big)=\infty,$$

$\forall(\upsilon_1,\upsilon_2)\in\operatorname{int}\operatorname{dom}K$ and $(\bar\upsilon_1,\bar\upsilon_2)\in\partial\operatorname{dom}K$. We now approximate $\mathbb{P}(T<t\mid X=x,T<\infty)$, $\forall x>0$, as follows. Let $\bar{\mathbb{P}}$ be the renormalized measure such that

$$\bar{\mathbb{P}}\big((T,X)\in B\big)=\mathrm{e}^{-K(0,0)}\mathbb{P}\big((T,X)\in B\big),$$

$\forall B\in\mathscr{B}\big((0,\infty)^2\big)$. Note that $\mathrm{e}^{K(0,0)}=\mathbb{P}(T<\infty)>0$. From $\bar{\mathbb{P}}\big(T\in(0,\infty)\big)=1$ we have $\bar{\mathbb{P}}\big((T,X)\in(0,\infty)^2\big)=1$ and, in this sense, $\bar{\mathbb{P}}$ can be interpreted as a proper probability measure. Let $\bar f$ be the induced proper density of (T,X) under $\bar{\mathbb{P}}$. Then we have

$$\mathbb{P}(T<t\mid X=x,T<\infty)=\frac{\int_0^t f(u,x)\,\mathrm{d}u}{\int_0^\infty f(u,x)\,\mathrm{d}u}=\frac{\int_0^t f(u,x)\,\mathrm{e}^{-K(0,0)}\,\mathrm{d}u}{\int_0^\infty f(u,x)\,\mathrm{e}^{-K(0,0)}\,\mathrm{d}u}=\frac{\int_0^t \bar f(u,x)\,\mathrm{d}u}{\int_0^\infty \bar f(u,x)\,\mathrm{d}u}=\bar{\mathbb{P}}(T<t\mid X=x),$$

$\forall t,x>0$. Thus approximating $\bar{\mathbb{P}}(T<t\mid X=x)$ formally by Skovgaard's formula with $n=1$ yields

$$\Phi(r_{t,x})-\phi(r_{t,x})\left(\frac{1}{s_{t,x}}-\frac{1}{r_{t,x}}\right), \tag{11}$$

where $r_{t,x}$ and $s_{t,x}$ are defined by (6) and (7) as before but with K now given by (10), as saddlepoint approximation to $\mathbb{P}(T<t\mid X=x,T<\infty)$.

2.2 Application to probabilities of ruin within finite time

Theorem 2.1 gives the joint or double Laplace transform of the time of ruin T and the initial capital Y_0, considered random, which is essential for obtaining the saddlepoint approximation. It generalizes generalizes Barndorff-Nielsen and Schmidli (1995, Lemma 1), which is included as a special case if $\sigma=0$.

Theorem 2.1. *Let $\{Y_t\}_{t\geq 0}$ be the risk process with diffusion* (1) *and T its time of ruin. Let $M_X(\beta)=\mathbb{E}\left[\mathrm{e}^{\beta X_1}\right]$ be the moment generating function* (MGF) *of the absolutely continuous individual claim amounts, let $\bar\beta=\sup\{\beta\in\mathbb{R} | M_X(\beta)<\infty\}$ and assume $\lim_{\beta\uparrow\bar\beta}M_X'(\beta)=\infty$. Further, let us denote the* CGF *of L_1 (i. e. of the Lévy loss process at time $t=1$) by*

$$\kappa(\beta)\stackrel{\text{def}}{=}\log\mathbb{E}\left[\mathrm{e}^{\beta L_1}\right]=\tfrac{1}{2}\beta^2\sigma^2-c\beta+\lambda\left\{M_X(\beta)-1\right\},$$

$\forall\beta\in(-\infty,\bar\beta)$, *and let $\upsilon(\alpha)\leq\tilde\upsilon(\alpha)$ denote the two solutions in β of*

$$\alpha+\kappa(\beta)=0,$$

assuming their existence. In the case where the solution is unique, we denote it $v(\alpha)$ *and we set* $\bar{v}(\alpha) = \bar{\beta}$. *Denote by* $(\mathring{\beta}, -\mathring{\alpha})$ *the coordinates of the minimum of* κ. *Then,* $\forall(\alpha,\beta) \in D \overset{\text{def}}{=} \{(\alpha,\beta) \in \mathbb{R}^2 | \alpha \leq \mathring{\alpha}, \beta < \bar{v}(\alpha)\}$, *the double Laplace transform of the time of ruin* T *and the initial capital* Y_0 *is given by*

$$\int_0^\infty \mathrm{e}^{\beta x} f_\alpha(x)\,\mathrm{d}x = -\frac{\frac{\alpha}{v(\alpha)} + \frac{\kappa(\beta)}{\beta}}{\alpha + \kappa(\beta)}, \tag{12}$$

where

$$f_\alpha(x) \overset{\text{def}}{=} \mathbb{E}\left[\mathrm{e}^{\alpha T} \mathbb{1}\{T < \infty\} \mid Y_0 = x\right]. \tag{13}$$

Remark 2.2. The adjustment coefficient (or Lundberg exponent) of the risk process $\{Y_t\}_{t\geq 0}$, defined as the positive solution of $k(\beta) = 0$ in β, if it exists, is given by $\bar{v}(0)$, if it exists within $(0, \bar{\beta})$.

Remark 2.3. The function $f_\alpha(x)$ given in (13) is in fact a special case of the so-called expected discounted penalty function or Gerber-Shiu function, introduced by Gerber and Shiu (1998) for the compound Poisson risk process and generalized to the perturbed risk process (1) by Gerber and Landry (1998). In the form suggested by Tsai and Willmot (2002), the penalty function of the risk process (1) takes the form

$$f_{-\delta}(x; w_0, w) \overset{\text{def}}{=} w_0 \mathbb{E}\left[\mathrm{e}^{-\delta T} \mathbb{1}\{T < \infty, Y_T = 0\} \mid Y_0 = x\right] + \mathbb{E}\left[\mathrm{e}^{-\delta T} w(Y_{T_-}, |Y_T|) \mathbb{1}\{T < \infty, Y_T < 0\} \mid Y_0 = x\right], \tag{14}$$

where $\delta \geq 0$ is a constant discounting factor and where the constant $w_0 \geq 0$ and $w : (\mathbb{R}_- \times \mathbb{R}_+, \mathscr{B}(\mathbb{R}_- \times \mathbb{R}_+)) \to (\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$ determine the penalty scheme to be inflicted the insurer in case of ruin. Thus, w_0 is the inflicted penalty when the insurer is ruined by creeping and w is the penalty when ruin is due to a claim. Several generalizations of this penalty function can be found in recent actuarial literature and, in fact, the analysis of Gerber-Shiu functions has become a major research topic in insurance and finance (in the context of option pricing). We mention a few articles on this topic. Li and Garrido (2005) consider the renewal (or Sparre Andersen) risk process with Wiener perturbation, where the inter-claim times are distributed as a sum of n independent exponentials. They show that if the individual claim amount density and the penalty function w are twice continuously differentiable, then $f_{-\delta}(x; w_0, w)$ is $2n$ continuously differentiable with respect to x. Then they provide a renewal equation to (14) under these smoothness assumptions. Biffis and Morales (2010) suggest a more sophisticated penalty scheme which considers the surplus at the last minimum before ruin. Moreover, they consider the more general Lévy risk process corresponding to a perturbed subordinator. Feng (2011) considers processes defined by a stochastic integral, which are Lévy processes and include (1). For these processes, he provides a generalization of the Gerber-Shiu function together with a method of solution based on the theory of operators. This generalized Gerber-Shiu function is called expected present value of total operating costs up to default and it is given by

$$H_{d,\delta}(x) = \mathbb{E}\left[\int_0^{T_d} \mathrm{e}^{-\delta t} l(Y_t)\,\mathrm{d}t \,\middle|\, Y_0 = x\right], \tag{15}$$

where

$$T_d = \begin{cases} \inf\{t \in (0,\infty) | Y_t \leq d\}, & \text{if the infimum exists,} \\ \infty, & \text{otherwise,} \end{cases}$$

$d \geq 0$ is a prescribed default level and $l : (\mathbb{R}, \mathscr{B}(\mathbb{R})) \to (\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$ is the operating cost depending on the surplus level. Feng and Shimizu (2013) extends the analysis of this new class of functions to the general

spectrally negative Lévy risk process. In particular, Feng and Shimizu (2013, Proposition 5.1) tells that the Gerber-Shiu function $f_{-\delta}(x; w_0, w)$ in (14) is indeed a special case of $H_{0,\delta}(x)$ given in (15), for an appropriate choice of l depending on w_0 and w. From the statistical side, a nonparametric estimator of the Gerber-Shiu function for the risk process (1) is suggested by Shimizu (2012). Note finally that an introductory survey on Gerber-Shiu functions can be found in Asmussen and Albrecher (2010, Chapter XII)

With Remark 2.3 we see that taking $w_0 = 1$ and w always equal to one in (14) yields (13), i.e. $f_\alpha(x) = f_\alpha(x; 1, 1)$. From this fact, the double Laplace transform (12) of Theorem 2.1 can be obtained from known results on Gerber-Shiu functions. We refer to the Appendix for the related details.

We now have the two main components for computing the probability of ruin in finite time with the risk process perturbed by diffusion. The first component is the joint CGF K of the time of ruin T and the initial capital Y_0 on $\{T < \infty\}$, when Y_0 is assumed a standard exponential random variable on $(\Omega, \mathscr{F}, \mathbb{P})$. In this situation we find

$$K(\alpha, \beta) \stackrel{\text{def}}{=} \log \mathbb{E}\left[\mathrm{e}^{\alpha T + (\beta+1) Y_0} \mathbb{1}\{T < \infty\} \right] = \log \frac{-\frac{\alpha}{\nu(\alpha)} - \frac{\kappa(\beta)}{\beta}}{\alpha + \kappa(\beta)}, \tag{16}$$

$\forall (\alpha, \beta) \in D$, from (12) in Theorem 2.1. It follows that $K(\alpha, \beta - 1)$ is the joint CGF of (T, Y_0) on $\{T < \infty\}$, which is defined over $\{(\alpha, \beta) \in \mathbb{R}^2 | \alpha \leq \mathring{\alpha}, \beta < \tilde{\nu}(\alpha) + 1\}$. The second main component is the saddlepoint approximations to the survival function by Lugannani and Rice (1980) and to the conditional survival function by Skovgaard (1987) together with the extensions for defective distributions of Section 2.1. Following Barndorff-Nielsen and Schmidli (1995, Section 5), the finite time probability of ruin can be written as

$$\psi(x, t) = \mathbb{P}(T \leq t \mid Y_0 = x) = \mathbb{P}(T \leq t, T < \infty \mid Y_0 = x) = \bar{\psi}(x, t)\psi(x), \tag{17}$$

where $\bar{\psi}(x, t) = \mathbb{P}(T \leq t \mid Y_0 = x, T < \infty)$ and $\psi(x) = \mathbb{P}(T < \infty \mid Y_0 = x)$, $\forall x, t > 0$. Hence, the finite time probability of ruin can be computed according to the two following situations.

$c \leq \lambda\mu$. In this case, ruin is almost sure, i. e. $\psi(x) = \mathbb{P}(T < \infty \mid Y_0 = x) = 1$. Here (17) yields $\psi(x, t) = \bar{\psi}(x, t)$, which can be computed by the conditional saddlepoint approximation (11).

$c > \lambda\mu$. In this case ruin is uncertain, i. e. $\mathbb{P}(T = \infty \mid Y_0 = x) > 0$. From (17), $\psi(x, t)$ can be computed by the conditional saddlepoint approximation (11) applied to $\bar{\psi}(x, t)$, multiplied the saddlepoint approximation to $\psi(x)$ given by Gatto and Mosimann (2012, Section 2), which is obtained by Lugannani and Rice (1980) formula (2) applied to the CGF

$$u \mapsto \begin{cases} \log \frac{(c-\lambda\mu)u}{cu - \frac{1}{2}\sigma^2 u^2 + \lambda\left(1 - M_X(u)\right)}, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

Note that we can apply the saddlepoint formulae directly to $K(\alpha, \beta)$, instead of the CGF $K(\alpha, \beta - 1)$. By doing so, we should however remember that the correct saddlepoint at (t, x) is given by $(\alpha_{t,x}, \beta_{t,x} + 1)$, even though this fact has no influence on numerical implementation.

2.3 Computational aspects

Let $t, x \geq 0$ and consider the function

$$\begin{aligned} \varphi_{t,x} : \quad & D \to \mathbb{R} \\ & (\alpha, \beta) \mapsto K(\alpha, \beta) - (\alpha, \beta)(t, x)^\top, \end{aligned} \tag{18}$$

where K is given by (16). Solving the saddlepoint equations (8) and (9) is equivalent to minimizing $\varphi_{t,x}$, which can be easily performed by standard optimization algorithms. Therefore we define

$$(\alpha_{t,x}, \beta_{t,x}) = \operatorname*{arg\,min}_{(\alpha,\beta)\in D} \varphi_{t,x}(\alpha, \beta)$$

and

$$\gamma_x = \operatorname*{arg\,min}_{(0,\beta)\in D} \varphi_{t,x}(0, \beta) = \operatorname*{arg\,min}_{-\infty<\beta<\tilde{v}(0)} \varphi_{t,x}(0, \beta).$$

Optimization programs often return the position and the value of the minimum together with an accurate approximation to the Hessian at the minimum. These are the values required by $r_{t,x}$ and $s_{t,x}$, see (6) and (7).

It is important to note that the function v, implicitly defined in Theorem 2.1, is generally not available in closed form and needs to be computed by numerical inversion of the function $-\kappa$. Because this numerical inversion would be required at every optimization step, the search for the saddlepoint would become too computationally intensive. In order to overcome this important numerical issue, we can exploit the following result.

Lemma 2.4. *Consider the functions κ and v defined in Theorem 2.1 and the function $\varphi_{t,x}$ given by (18), for $t, x \geq 0$. Let $(\alpha_{t,x}, \beta_{t,x})$ denote the bivariate saddlepoint, i.e. the unique solution to* (8)*. Consider the transformation*

$$\begin{aligned}\Theta\colon \operatorname{dom}\Theta &\to D\\ (\eta,\beta) &\mapsto (-\kappa(\eta),\beta),\end{aligned}$$

where $\operatorname{dom}\Theta \stackrel{\text{def}}{=} \{(\eta,\beta)\in\mathbb{R}^2 | \eta \leq \mathring{\alpha}, (v(\eta),\beta)\in D\}$*. Then*

$$(\alpha_{t,x}, \beta_{t,x}) = \Theta(\eta_{t,x}, \beta_{t,x}),$$

where

$$(\eta_{t,x}, \beta_{t,x}) = \operatorname*{arg\,min}_{\Theta(\eta,\beta)\in D} (\varphi_{t,x}\circ\Theta)(\eta,\beta).$$

Furthermore

$$\mathbb{D}^2 K(\alpha_{t,x}, \beta_{t,x}) = \begin{pmatrix} H_{11}\big(\kappa'(\eta_{t,x})\big)^2 & -H_{12}\kappa'(\eta_{t,x}) \\ -H_{12}\ \kappa'(\eta_{t,x}) & H_{22}\end{pmatrix},$$

where $H = \mathbb{D}^2(\varphi_{t,x}\circ\Theta)(\eta_{t,x}, \beta_{t,x})$ is the Hessian of $\varphi_{t,x}\circ\Theta$ at the minimum.

Proof. Let $t, x > 0$. As K is convex, $\varphi_{t,x}$ is also convex. Since the coordinate transformation Θ acts on both arguments independently through the functions $-\kappa$ and the identity, which are both continuous and strictly increasing over $\operatorname{dom}\Theta$, then Θ is a continuous bijection, with inverse given by $\Theta^{-1}(\eta,\beta) = (v(\eta),\beta)$. (We apply $-\kappa$ only to values η such that $-\kappa(\eta) \leq \mathring{\alpha}$.) Therefore,

$$\operatorname*{arg\,min}_{D} \varphi_{t,x} = \Theta\left(\operatorname*{arg\,min}_{\Theta^{-1}(D)} \big(\varphi_{t,x}\circ\Theta\big)\right),$$

which yields the first assertion.

The Hessian of K at the saddlepoint is obtained by $\mathbb{D}^2 K = \mathbb{D}^2\varphi_{t,x}$ and $\mathbb{D}K(\alpha_{t,x}, \beta_{t,x}) - (t, x) = 0$. The Hessian of K at $(\alpha_{t,x}, \beta_{t,x})$, as given in the lemma, can be directly obtained. □

Remark 2.5. The Hessian of K at the saddlepoint is useful in the frequent situations where the available optimization routine returns an approximation to the Hessian H of the objective function $\varphi_{t,x} \circ \Theta$ at the optimum.

Remark 2.6. By applying l'Hospital's rule, we can obtain the following limits:

$$\lim_{\beta \to \eta} (K \circ \Theta)(\eta, \beta) = \log \frac{\kappa(\eta) - \eta\kappa'(\eta)}{\eta^2 \kappa'(\eta)}, \quad \lim_{\eta \to 0} (K \circ \Theta)(\eta, \beta) = \log \frac{\frac{\kappa(\beta)}{\beta} - \kappa'(0)}{k(\beta)}$$

and

$$\lim_{(\eta,\beta) \to (0,0)} (K \circ \Theta)(\eta, \beta) = \log \frac{\kappa''(0)}{2\kappa'(0)}.$$

The cases $\eta \to \beta$ and $\beta \to 0$ can be obtained by using the symmetry of $K \circ \Theta$ in its arguments, i. e. $K \circ \Theta(\eta, \beta) = K \circ \Theta(\beta, \eta)$, $\forall (\eta, \beta) \in \Theta^{-1}(D)$. These values are useful for the computation of the saddlepoint approximation.

3 Importance sampling approximations

This section provides importance sampling estimators for the probabilities of ruin within finite and infinite time horizons of the perturbed risk process (1). Because ruin is often a rare event, approximating its probability by the rate of simulated processes reaching ruin over the total number of simulated processes entails a large variability. Therefore, selecting an appropriate sampling distribution is essential for obtaining approximations to finite and infinite time horizon probabilities of ruin, which are accurate is the sense of having bounded relative error or, at least, logarithmic efficiency[1]. The central idea of the importance sampling algorithm presented here originates from Siegmund (1976). Other references are e. g. Asmussen (1985), Asmussen and Albrecher (2010, Sections X.3–4), and Gatto and Mosimann (2012, Section 5). The idea is to sample after an exponential tilt of the probability measure, instead of directly from the original measure $\mathbb{P}$.

Let us define the Lévy loss process $L_t = x - Y_t$, $\forall\, t \geq 0$, on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$ and let $\mathscr{F}_T = \{A \in \mathscr{F} | \forall\, t \geq 0, A \cap \{T \leq t\} \in \mathscr{F}_t\}$. Let $u \in \mathbb{R}$ such that $k(u) < \infty$ and let $A \in \mathscr{F}_T$ such that $A \subset \{T < \infty\}$, then

$$\mathbb{P}(A) = \mathbb{P}_0(A) = \mathbb{E}_u \left[\mathbb{1}_A \frac{\mathrm{d}\mathbb{P}_0}{\mathrm{d}\mathbb{P}_u} \right], \tag{19}$$

where $\mathbb{E}_u$ is the expectation under $\mathbb{P}_u$ and $\mathbb{P}_u$ is the equivalent probability measure on the restriction to $\mathscr{F}_T$, determined by $\mathrm{d}\mathbb{P}_0 / \mathrm{d}\mathbb{P}_u = \mathrm{e}^{-uL_T + T\kappa(u)}$, see e. g. Asmussen and Albrecher (2010, Section IV.4, Theorem 4.3). We define the deficit or overshoot at ruin as $D_x = -Y_T = L_T - x \geq 0$, on $\{T < \infty\}$. Setting $A = \{T \leq t\}$ in (19) yields $\psi(x, t) = \mathbb{E}_u [Z(x, t, u)]$, where

$$Z(x, t, u) \stackrel{\text{def}}{=} \mathbb{1}\{T \leq t\} \mathrm{e}^{-uL_T + T\kappa(u)} = \mathrm{e}^{-ux} \mathbb{1}\{T \leq t\} \mathrm{e}^{-uD_x + T\kappa(u)}$$

[1] Two optimality criteria of rare event simulation are the following. Assume that $z(\xi) \stackrel{\text{def}}{=} \mathbb{P}\big(A(\xi)\big) \to 0$, as $\xi \uparrow \infty$, then the unbiased estimator $Z(\xi) = \mathbb{1}_{A(\xi)}$ of $z(\xi)$, $\forall \xi > 0$, is logarithmic efficient if

$$\liminf_{\xi \to \infty} \left| \frac{\log \mathrm{var}\big(Z(\xi)\big)}{\log z^2(\xi)} \right| \geq 1$$

and it has bounded relative error if

$$\limsup_{\xi \to \infty} \frac{\mathrm{var}\big(Z(\xi)\big)}{z^2(\xi)} < \infty.$$

is the Monte Carlo estimator of $\psi(x,t)$ under the probability measure $\mathbb{P}_u$. The corresponding Monte Carlo approximation is then $n^{-1}\sum_{k=1}^{n} Z_k(x,t,u)$, where $Z_1(x,t,u),\dots,Z_n(x,t,u)$ are independent generations of $Z(x,t,u)$ under $\mathbb{P}_u$. Setting $A=\{T<\infty\}$ in (19) yields $\psi(x)=\mathbb{E}_u[Z(x,u)]$, where

$$Z(x,u)=\mathrm{e}^{-ux}\mathbb{1}\{T<\infty\}\,\mathrm{e}^{-uD_x+T\kappa(u)}$$

is the Monte Carlo estimator of $\psi(x)$ under $\mathbb{P}_u$. The Monte Carlo approximation is $n^{-1}\sum_{i=1}^{n} Z_k(x,u)$, where $Z_1(x,u),\dots,Z_n(x,u)$ are independent generations of $Z(x,u)$ under $\mathbb{P}_u$. We note that the form of the Lévy process $\{L_t\}_{t\geq 0}$ is invariant under exponential tilt. Precisely, under $\mathbb{P}_u$, $\{L_t\}_{t\geq 0}$ is again a compound Poisson process perturbed by a Wiener process, with: independent individual claims having distribution function $F_u(x)=\int_0^x \mathrm{e}^{uy}\,\mathrm{d}F(y)/M_X(u)$, $\forall x>0$; Poisson counting process $\{N_t\}_{t\geq 0}$ with intensity $\lambda_u=\lambda M_X(u)$; premium rate $c_u=c-\sigma^2 u$; and Wiener perturbation $\{\sigma W_t\}_{t\geq 0}$, where $\{W_t\}_{t\geq 0}$ is a standard Wiener Process.

The next result gives the optimal choices of u yielding either logarithmic efficiency or bounded relative error.

Result 3.1. *Assume $c>\lambda\mu$ and factorize the finite time horizon as $t=xy$, for some $y>0$, where $x>0$ is the initial capital. Let u_y be the solution in u of*

$$\kappa'(u)=\frac{1}{y}. \tag{20}$$

Let $y_0=1/\kappa'(\tilde{\nu}(0))$, where $\tilde{\nu}(0)$ is the adjustment coefficient, defined in Remark 2.2. In the finite time horizon, $t<\infty$ and we distinguish the short time horizon, where $t<x/\kappa'(\tilde{\nu}(0))\Leftrightarrow y<y_0$, and the long time horizon, where $t>x/\kappa'(\tilde{\nu}(0))\Leftrightarrow y>y_0$.

In the short time horizon,

$$Z(x,t,u_y)=\mathrm{e}^{-u_y x}\mathbb{1}\{T\leq t\}\,\mathrm{e}^{-u_y D_x+T\kappa(u_y)}$$

is a logarithmic efficient estimator of $\psi(x,t)$, as $x\to\infty$, under $\mathbb{P}_{u_y}$.

In the long time horizon,

$$Z(x,t,\tilde{\nu}(0))=\mathrm{e}^{-\tilde{\nu}(0)x}\mathbb{1}\{T\leq t\}\,\mathrm{e}^{-\tilde{\nu}(0)D_x}$$

is an estimator with bounded relative error of $\psi(x,t)$, as $x\to\infty$, under $\mathbb{P}_{\tilde{\nu}(0)}$.

In the infinite time horizon,

$$Z(x,\tilde{\nu}(0))=\mathrm{e}^{-\tilde{\nu}(0)(x+D_x)}$$

is an estimator with bounded relative error of $\psi(x)$, as $x\to\infty$, under $\mathbb{P}_{\tilde{\nu}(0)}$.

A partial proof of these statements, which are known for the risk process without perturbation, see e.g. Asmussen and Albrecher (2010, Sections XV.3-4), is reported to the Appendix.

Remark 3.2. Some intuitive justifications of 3.1 are the following. We can first note that the sampling measure of the long time horizon case is the same as the one of the infinite time horizon case, namely Lundberg's conjugated measure $\mathbb{P}_{\tilde{\nu}(0)}$, which makes ruin almost sure in the infinite time horizon. The variability of $\exp\{uD_x+T\kappa(u)\}$ is reduced when $u=\tilde{\nu}(0)$, because $T\kappa(\tilde{\nu}(0))=0$ and the overshoot D_x vanishes as $x\to\infty$. Regarding the short time horizon case, it can be shown that $\mathbb{E}_{u_y}[T]\to t$, as $x\to\infty$, and thus $\mathbb{P}_{u_y}$ is a re-centering of $\mathbb{P}$ at the asymptotic mean of T, which is in some sense the optimal shift to consider before approximating the distribution of T by simulation (or by normal approximation). Note that, just like $\mathbb{P}_{\tilde{\nu}(0)}$, $\mathbb{P}_{u_y}$ does also make ruin almost sure in the infinite time horizon, although it does not remove the time of ruin T from the exponent of the estimator.

Remark 3.3. These importance sampling estimators require simulating discretized sample paths of the risk process (1). The claim occurrence times are obtained by independent and exponential inter-arrival times, which are then assigned to their closest points of the selected discretization mesh. The Wiener perturbation can be simulated with the method of circulant embedding, which is an efficient method based on the Fast Fourier transform, see e. g. Dietrich and Newsam (1997) or Asmussen and Glynn (2007, Section XI.3).

4 Numerical examples and comparison

The aim of this section is to provide numerical comparisons between the saddlepoint approximation proposed in Section 2 and the importance sampling method of Section 3. In order to emphasize the importance of large deviations techniques (which are the saddlepoint approximation and importance sampling), a comparison with a Monte Carlo method based on the simulation of a dual process, explained in the Appendix, is also shown. The probability of ruin with infinite time horizon can be computed exactly (by using partial fractions decomposition) if the individual claim amounts have a hypo-exponential distribution, see Dufresne and Gerber (1991, Section 6) for details. This yields an upper bound to our approximations (because $\psi(x,t) \leq \psi(x)$, $\forall\, t \in (0,\infty)$) and a limiting value (because $\lim_{t\to\infty}\psi(x,t) = \psi(x)$). Let $V_1,\ldots,V_n$ be independent random variables having exponential distribution with parameters $\nu_1,\ldots,\nu_n > 0$, respectively, i. e. $\mathbb{P}(V_i > u) = \mathrm{e}^{-\nu_i u}$, $\forall\, u > 0$ and $i \in \{1,\ldots,n\}$. Then $X = \sum_{i=1}^n V_i$ is said to have a hypo-exponential (or generalized Erlang) distribution with parameter $(\nu_1,\ldots,\nu_n)$. We assume $\nu_1 < \ldots < \nu_n$. Thus $\mathbb{E}[X] = \sum_{i=1}^n \nu_i^{-1}$ and the MGF of X is given by $\mathbb{E}[\mathrm{e}^{\beta X}] = \prod_{i=1}^n (1-\beta/\nu_i)^{-1}$, $\forall\, \beta < \min\{\nu_1,\ldots,\nu_n\}$. We consider the risk process with diffusion $\{Y_t\}_{t\geq 0}$ given in (1) with the following parameters. The claim frequency is $\lambda = 1$, the premium rate is $c = 2$ and the infinitesimal variance of the Wiener process is $\sigma^2 = 0.4$. The individual claim amounts $X_1, X_2, \ldots$ are hypo-exponentially distributed with vector parameter $(\nu_1,\nu_2) = (1,10)$. Thus $\mu = \mathbb{E}[X_1] = 1.1$ and the MGF of X_1 is given by $M_X(\beta) = 10/\{(1-\beta)(10-\beta)\}$, $\forall\, \beta < \tilde{\beta} = 1$.

The saddlepoint approximation, importance sampling and simulation based on the dual process are applied to this example, $\forall (x,t) \in \{0, 0.04, \ldots, 10\} \times \{0, 0.08, \ldots, 14.96\}$. For importance sampling of Section 3, 40 000 paths of the process have been simulated at each argument pair (x,t). Because the importance sampling measure depends on the actual choice of (x,t) (see Section 3), the simulated processes cannot be re-used at other arguments (x,t). The Monte Carlo method using the dual process $\{Q_t\}_{t\geq 0}$ given in the Appendix is based on a total of $2^{20} = 1\,048\,576$ simulated paths. Now each one of the simulated paths can be used for all pairs (x,t). In the absence of exact values for the finite time probabilities of ruin, the errors of the saddlepoint and of the dual Monte Carlo approximations are defined as deviations from the importance sampling approximation.

The graph in Figure 1 shows the saddlepoint approximation to the probability of ruin within finite time described in Section 2, using the formulae of Lugannani and Rice (1980). If the initial reserve lies very close to the expected value of the maximal aggregate loss or if the denominator of the MGF e^K, see (16), lies close to zero, then the saddlepoint approximation has an erratic behavior. In the graph of Figure 1, these two facts lead to irregularities along a straight line parallel to the time axis and along a curve emanating from the origin, respectively. In order to reduce these irregularities, a moving median filter (with range given by 3×3 sub-grid) has been applied to the approximation values. After this filtering, the first irregularity becomes almost invisible and the second irregularity is reduced. So excepting over the line of the second irregularity, Figure 1 shows that the saddlepoint approximation yields a smooth

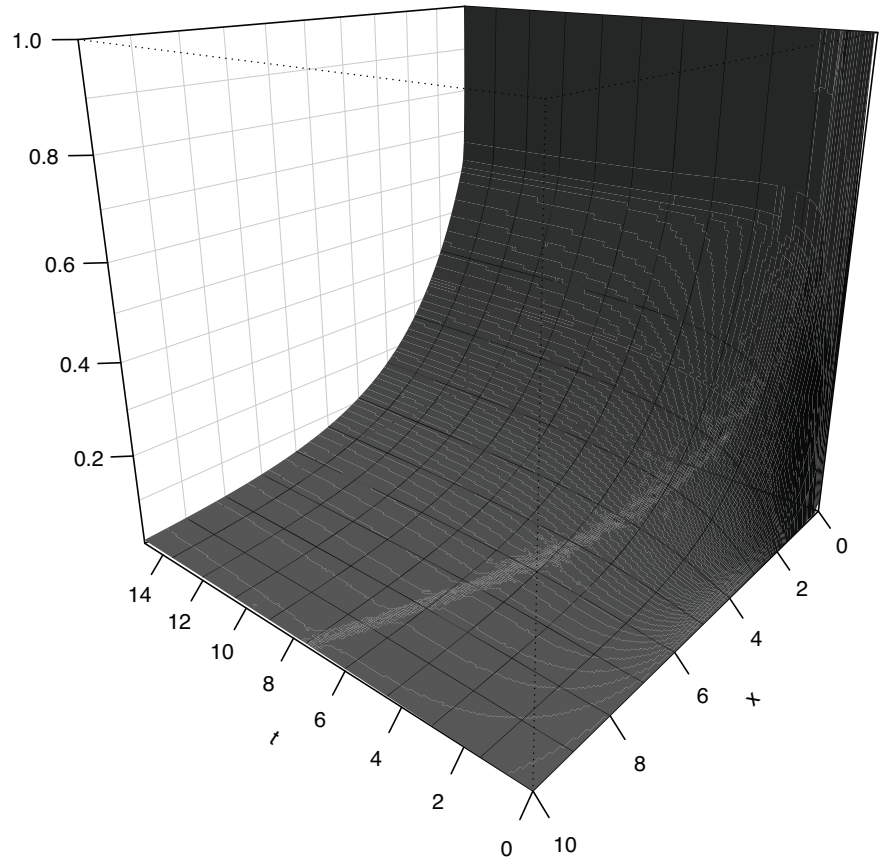

Figure 1. Saddlepoint approximation to the probability of ruin within finite time t and with initial capital x.

surface.

Figure 2 depicts the relative error of the saddlepoint approximation shown in Figure 1 using importance sampling as a reference, i. e. $|1-\psi_S(x,t)/\psi_I(x,t)|$, where $\psi_S(x,t)$ and $\psi_I(x,t)$ are the saddlepoint and importance sampling approximations to $\psi(x,t)$. The two variants of importance sampling, referred to as long and short time horizon cases in Section 3, can be distinguished by lighter and darker shading in Figure 2, respectively. In this example, these two variants produce quite close values. Because the importance sampling approximation is not smooth (in x and t), a moving median filter (acting on 5×5 sub-grids) has been applied to the data in order to provide a smoother graph. It should be noted that the important peaks, which lie over a line passing trough the point $(x,t)=(9,6)$, are caused by the erratic behavior of the saddlepoint approximation mentioned above. Otherwise, with the exception of very small initial reserves x or very small time horizons t, the relative errors are very small, typically below 0.05. This graph provides good evidence, that the saddlepoint approximation is very accurate.

Figure 3 shows a graph with relative errors analogous to Figure 2, with the simulation of the the dual process replacing the saddlepoint approximation, again using importance sampling as a reference. As in Figure 2, the long and short time horizon cases are distinguished by lighter and darker shading, and and a moving median filter (acting on 7×7 sub-grids) has been applied to the data. With the exception of small values of x and large values of t, the relative errors are substantial. For small t and large x, the relative errors can reach levels as high as 0.6. This provides good evidence that the Monte Carlo method using the dual process is inaccurate and thus emphasizes the importance of large deviations techniques.

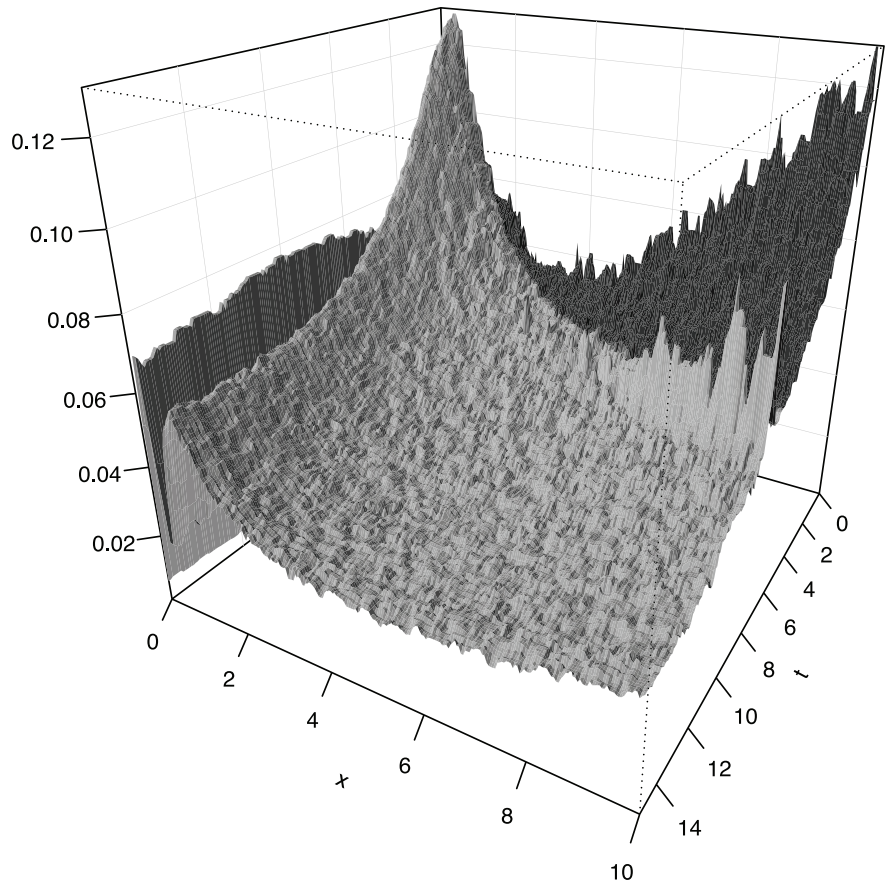

Figure 2. Relative error of the saddlepoint approximation to the probability of ruin within finite time t and with initial capital x, with importance sampling as reference.

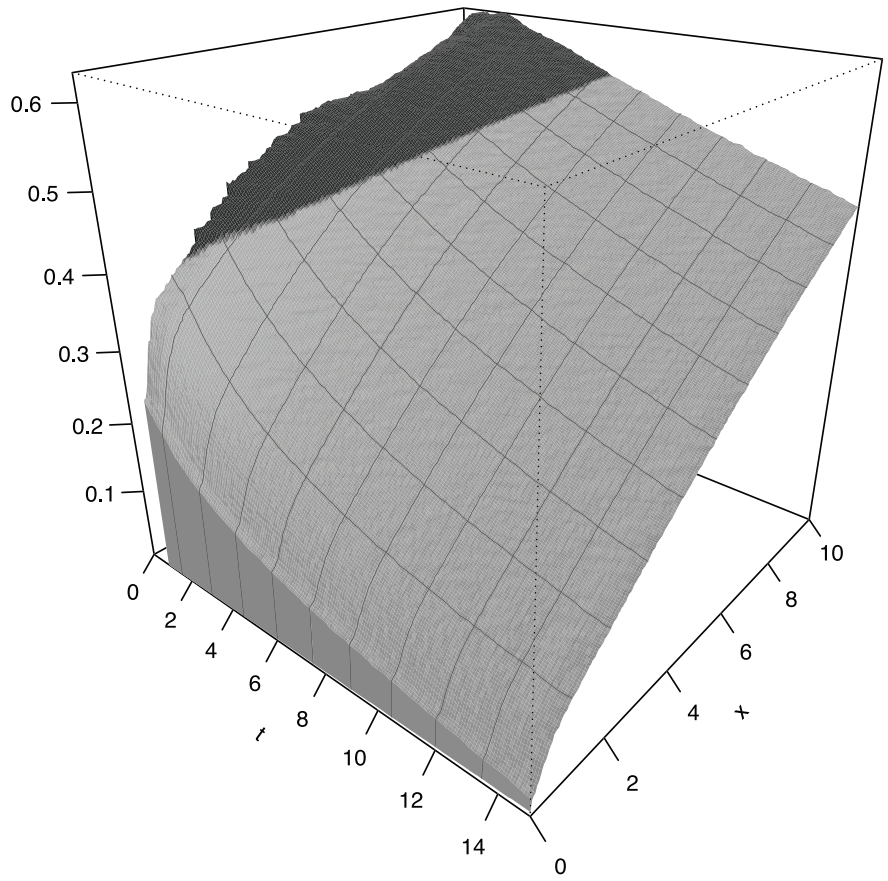

Figure 3. Relative error of the approximation based on the simulation of the dual process to the probability of ruin within finite time t and with initial capital x, with importance sampling as reference.

5 Conclusions and remarks

This article provides a saddlepoint approximation to the probability of ruin within finite time horizon for the perturbed compound Poisson process and it also addresses the related computational problems. This saddlepoint approximation is compared with two different Monte Carlo approximations to finite time probabilities of ruin. The first one is importance sampling and the second one is based on a dual process. Numerical results show that the saddlepoint approximation is very close to importance sampling. In contrast to this fact, the Monte Carlo approximation based on the dual process differs substantially from the two previous methods. These results suggest that the saddlepoint approximation and importance sampling are very accurate. However, the saddlepoint approximation is substantially faster to compute than importance sampling. Therefore this saddlepoint approximation offers a practical and accurate solution for computing the probability of ruin within finite time. Gatto and Mosimann (2012) obtained similar conclusions with these three types of methods, when applied to the computation of the infinite time horizon probability of ruin.

We now mention some related problems. As mentioned by a Referee, the methods presented in this article could be adapted to obtain the probability that the perturbed risk process first crosses the null level by creeping and this prior to a time $t \in (0,\infty)$, i.e. to obtain $\psi^{(1)}(x,t) \stackrel{\text{def}}{=} \mathbb{P}(T < t, Y_T = 0 \mid Y_0 = x)$. Gatto and Mosimann (2012) give a saddlepoint approximation to this probability in the infinite time horizon. Essentially, $\psi^{(1)}(x,t)$ can be obtained by setting $w_0 = 1$ and w equal to zero in (21) in the Appendix, which yields the double Laplace transform of the time ruin due to creeping

$$\hat{f}_\alpha(\beta;1,0) = \frac{\frac{\sigma^2}{2}\{\nu(\alpha)-\beta\}}{\alpha+\kappa(\beta)}.$$

Gatto and Baumgartner (2014) defined the VaRu of the perturbed risk process (1) as the minimal initial capital required to obtain an infinite time probability of ruin smaller than or equal to a small threshold. Following this, we now define the finite time VaRu of the perturbed risk process at level $\varepsilon \in (0,1)$ by $\text{VaRu}(\varepsilon,t) = \inf\{x \geq 0 | \psi(x,t) \leq \varepsilon\}$, $\forall\, t \in (0,\infty)$. They also defined the TVaRu by considering the expected maximal aggregate loss given that the maximal aggregate loss exceeds a fixed VaRu. Following this, we now define the finite time TVaRu of the perturbed risk process at level $\varepsilon \in (0,1)$ by $\text{TVaRu}(\varepsilon,t) = \mathbb{E}[M_t \mid M_t > \text{VaRu}(\varepsilon,t)]$, where M_t is defined in (22), $\forall\, t \in (0,\infty)$. Thus, a practical open problem would be the evaluations of the risk measures $\text{VaRu}(\varepsilon,t)$ and $\text{TVaRu}(\varepsilon,t)$ with the saddlepoint approximation suggested in this article.

All computer programs used for this article are written in `R` and can be found at `cran.r-project.org`.

Appendix

Complements to Section 2

A justification of the double Laplace transform of the time of ruin T and the initial capital Y_0 provided by (12) in Theorem 2.1 is the following.

Proof of Theorem 2.1. The starting point is the Laplace transform of the Gerber-Shiu function (14), given e.g. in Asmussen and Albrecher (2010, Equation 4.3, p. 387). Re-expressed with reversed sign and with our notation (i.e. with β, $-\delta$ and $-\rho_\delta$ replaced by λ, α and $\nu(\alpha)$, respectively), it is given by

$$\hat{f}_\alpha(\beta;w_0,w) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{e}^{\beta x} f_\alpha(x;w_0,w)\,\mathrm{d}x = \frac{\frac{\sigma^2}{2}w_0\{\nu(\alpha)-\beta\} + \lambda\{\hat{w}(\nu(\alpha)) - \hat{w}(\beta)\}}{\alpha+\kappa(\beta)}, \tag{21}$$

$\forall \alpha, \beta \leq 0$, where

$$\hat{w}(\beta) \stackrel{\text{def}}{=} \int_0^\infty \mathrm{e}^{\beta y} \int_y^\infty w(y, x-y)\, \mathrm{d}F_X(x)\, \mathrm{d}y.$$

Taking w equal to one yields $\hat{w}(\beta) = \beta^{-1}\{M_X(\beta) - 1\}$ and taking further $w_0 = 1$ in (21) yields

$$\hat{f}_\alpha(\beta; 1, 1) = \frac{\frac{\sigma^2}{2}\{\nu(\alpha) - \beta\} + \frac{\lambda}{\nu(\alpha)}\{M_X(\nu(\alpha)) - 1\} - \frac{\lambda}{\beta}\{M_X(\beta) - 1\}}{\alpha + \kappa(\beta)},$$

From the definition of $\nu(\alpha)$ follows directly

$$\frac{\alpha}{\nu(\alpha)} = c - \frac{1}{2}\nu(\alpha)\sigma^2 - \frac{\lambda}{\nu(\alpha)}\{M_X(\nu(\alpha)) - 1\}$$

and so we formally obtain (12).

Because D is a connected subset of $\mathbb{R}^2$ and because the right-hand side of (12) is an analytical function $\forall \alpha, \beta \in D$, follows that the double Laplace transform formula (12) holds over the entire set D. □

We can note, as indicated by a Referee, that $f_\alpha(x)$ is smooth in the sense that it has a bounded second derivative. This is established by Feng (2011, Lemma C.1), because the $f_\alpha(x)$ is a special case of the more general functional of T given in (15).

A proof of (12), which is indepedent from results on Gerber-Shiu functions, is given under `cran.r-project.org`.

Complements to Section 3

We now give a partial proof of Result 3.1, which is a direct generalization of the proof for the risk process without perturbation, see e.g. Asmussen and Albrecher (2010, Sections XV.3-4). We first mention two results from this last reference, which are here generalized to the perturbed risk process (1).

Lemma 5.1. *Assume $c > \lambda\mu$, then*

$$\frac{\psi(x, xy)}{\psi(x)} \xrightarrow{x\to\infty} \begin{cases} 0, \text{ if } y < y_0, \\ 1, \text{ if } y > y_0. \end{cases}$$

Lemma 5.2. *Assume $c > \lambda\mu$, $y < y_0$ and define $l_y = u_y - \kappa(u_y)y$, where κ is given by Theorem 2.1 and u_y is defined by (20). Then*

$$-\frac{\log\psi(x, xy)}{x} \xrightarrow{x\to\infty} l_y.$$

Proof of Result 3.1. In the short time horizon $y < y_0$, $\mathbb{E}_{u_y}[Z^2(x, xy, u_y)] \leq \mathrm{e}^{-2u_y x}\mathbb{E}_{u_y}[\exp\{2[xy\kappa(u_y) - u_y D_x]\}; T \leq xy] = \mathrm{e}^{-2u_y x}\mathbb{E}_{u_y}[\exp\{2[x(u_y - l_y) - u_y D_x]\}] \leq \mathrm{e}^{-2l_y x}$. With Lemma 5.2 we thus obtain

$$\liminf_{x\to\infty} \frac{-\log \mathrm{var}_{u_y}\left(Z^2(x, xy, u_y)\right)}{-\log\psi(x, xy)} \geq \liminf_{x\to\infty} \frac{-\log \mathbb{E}_{u_y}\left[Z^2(x, xy, u_y)\right]}{l_y x} \geq \liminf_{x\to\infty} \frac{2l_y x}{l_y x} = 2.$$

In the long time horizon $y > y_0$, Lemma 5.1 and the Cramér-Lundberg approximation, given by Dufresne and Gerber (1991), yield $\psi(x, xy) \sim \psi(x) \sim \gamma \mathrm{e}^{-\tilde{\nu}(0)x}$, as $x \to \infty$, for some $\gamma > 0$. Moreover, $\mathbb{E}_{\tilde{\nu}(0)}[Z^2(x, xy, \tilde{\nu}(0))] \leq \mathrm{e}^{-2\tilde{\nu}(0)x}\mathbb{E}_{\tilde{\nu}(0)}[\mathrm{e}^{-2\tilde{\nu}(0)D_x}] \leq \mathrm{e}^{-2\tilde{\nu}(0)x}$. Thus

$$\limsup_{x\to\infty} \frac{\mathbb{E}_{\tilde{\nu}(0)}\left[Z^2(x, xy, \tilde{\nu}(0))\right]}{\psi^2(x, xy)} \leq \limsup_{x\to\infty} \frac{\mathrm{e}^{-2\tilde{\nu}(0)x}}{(\gamma \mathrm{e}^{-\tilde{\nu}(0)x})^2} = \gamma^{-2}.$$

In the infinite time horizon, $\mathbb{E}_{\tilde{\nu}(0)}[Z^2(x, \tilde{\nu}(0))] \leq \mathrm{e}^{-2\tilde{\nu}(0)x}$ and the justification is similar. □

Complements to Section 4

An alternative Monte Carlo technique for approximating both finite and infinite time probabilities of ruin was suggested by Dufresne and Gerber (1989, Section 4), for the compound Poisson risk process, and adapted to the perturbed risk process by Gatto and Mosimann (2012, Section 5).

The main result is the following. Let $t \in (0,\infty)$,

$$M_t = \sup_{s\le t}\{L_s\} \text{ and } Q_t = L_t - \inf_{s\le t}\{L_s\}, \tag{22}$$

then

$$M_t \sim Q_t. \tag{23}$$

Let $L = \sup_{t\ge 0}\{L_t\}$ denote the maximal aggregate loss. We have $\psi(x,t) = \mathbb{P}(M_t > x)$ and $\psi(x) = \mathbb{P}(L > x)$. From monotone convergence, $\psi(x) = \lim_{t\to\infty}\mathbb{P}(M_t > x)$. Define now $G(x,t) = \mathbb{P}(Q_t \le x)$ and $G(x) = \lim_{t\to\infty} G(x,t)$, the stationary distribution function of $\{Q_t\}_{t\ge 0}$. Result (23) yields $1-\psi(x,t) = G(x,t)$, which in the limit as $t\to\infty$ yields $\psi(x) = 1-G(x)$. With these facts, we can approximate probabilities of ruin by generating a single path of the dual process $\{Q_t\}_{t\ge 0}$. More precisely, let $D(x,t)$ denote the duration in which $\{Q_t\}_{t\ge 0}$ lies below level x before time t, i. e. the Lebesgue measure of $\{s\in[0,t)|Q_s < x\}$, and let $\hat{G}(x,t) = D(x,t)/t$, then $\lim_{t\to\infty}\hat{G}(x,t) = G(x)$ a. s. and thus for t large, $\hat{G}(x,t)$ is an approximation to $G(x,t) = 1-\psi(x,t)$. Hence the simulation of a single path of $\{Q_s\}_{s\in[0,t]}$ suffices to approximate $\psi(x,t)$. Selecting t very large yields an approximation to $\psi(x)$.

In practice, computing $D(x,t)$ requires discretizing the simulated path with a small mesh size, because of the self-similarity of the Wiener process. Without Wiener perturbation, it would be sufficient to observe the dual process $\{Q_t\}_{t\ge 0}$ at times of claims only, yielding a substantially simpler algorithm.

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