

An Avron Rule for Fragments of R-mingle

George Metcalfe

Mathematical Institute, University of Bern
Sidlerstrasse 5, 3012 Bern, Switzerland
george.metcalfe@math.unibe.ch*

Abstract

Axiomatic bases of admissible rules are obtained for fragments of the substructural logic R-mingle. In particular, it is shown that a “modus-ponens-like” rule introduced by Arnon Avron forms a basis for the admissible rules of its implication and implication-fusion fragments, while a basis for the admissible rules of the full multiplicative fragment requires an additional countably infinite set of rules. Indeed, this latter case provides an example of a three-valued logic with a finitely axiomatizable consequence relation that has no finite basis for its admissible rules.

1 Introduction

In [5] Arnon Avron showed that the “modus-ponens-like” rule

$$(A) \quad \{p, (p \rightarrow (q \rightarrow q)) \rightarrow (p \rightarrow q)\} / q$$

is admissible but not derivable in the multiplicative fragment of the substructural logic R-mingle (RM for short) (see [1–5, 9, 13, 27]). Namely, $\{p, (p \rightarrow (q \rightarrow q)) \rightarrow (p \rightarrow q)\} \not\vdash_{\text{RM}} q$, but if formulas φ and $(\varphi \rightarrow (\psi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$ are theorems of this fragment, then ψ is also a theorem. In this paper, it is shown that (A) in fact provides an axiomatic basis for the admissible rules of the implication and implication-fusion fragments of RM; that is, supplementing axiomatizations of these fragments with (A) provides axiomatizations of their admissible rules. However, it is also shown that to obtain a basis for the admissible rules of the full multiplicative fragment of RM, an additional countably infinite set of rules is required. Indeed, this case provides an example of a three-valued logic (axiomatized as the multiplicative fragment of RM extended with (A)) that has a finitely axiomatized consequence relation but no finite basis for its admissible rules.

In the negation-free fragment of RM, admissibility and derivability coincide; that is, this fragment is structurally complete. A proof of this result, implicit in Meyer and Slaney’s proof of structural completeness for the implication-conjunction fragment of the relevant logic R [32], is extended here with some modifications to the implication-disjunction and implication-fusion-disjunction fragments of RM. On the other hand, the full logic RM is famously not structurally complete, being closed under the non-derivable disjunctive syllogism rule $\{p, \neg p \vee q\} / q$ (see, e.g., [1]). In other words, $\{p, \neg p \vee q\} \not\vdash_{\text{RM}} q$, but whenever φ and $\neg\varphi \vee \psi$ are theorems of RM, also ψ is a theorem.

Axiomatic bases for admissible rules have been provided for a range of non-classical logics, often making crucial use of earlier foundational work by Rybakov [30, 31] and Ghilardi [14, 15]. In particular, Iemhoff [16] and Rozière [29] demonstrated independently that an infinite set of “Visser rules” conjectured by De Jongh and Visser provide a basis for the admissible rules of intuitionistic logic. Bases and analytic Gentzen-style proof systems were subsequently obtained

*Supported by Swiss National Science Foundation grants 20002.129507 and 200021.146748 and Marie Curie Reintegration Grant PIRG06-GA-2009-256492.

for the admissible rules of certain intermediate logics [12, 17, 18], transitive modal logics [18, 19], and temporal logics [6, 7]. For substructural logics, however, much less is known. Structural completeness for such logics and their algebras has been investigated in [11, 25], but non-trivial bases have been obtained so far only for Łukasiewicz logics [20, 21] and certain classes of De Morgan and Kleene algebras [10, 22].

2 Preliminaries

Let us first recall some general notions. A (*propositional*) *language* \mathcal{L} is a set of connectives with specified finite arities, where the set $\text{Fm}_{\mathcal{L}}$ of \mathcal{L} -formulas over a countably infinite set of propositional variables is defined inductively in the usual way. Arbitrary \mathcal{L} -formulas will be denoted by φ, ψ and sets of \mathcal{L} -formulas by Γ . An \mathcal{L} -substitution σ is an endomorphism on the formula algebra $\mathbf{Fm}_{\mathcal{L}}$, writing $\sigma(\Gamma)$ for $\{\sigma(\varphi) \mid \varphi \in \Gamma\}$.

A *logic* $L \subseteq \mathcal{P}(\text{Fm}_{\mathcal{L}}) \times \text{Fm}_{\mathcal{L}}$ in the language \mathcal{L} is a finitary structural consequence relation on $\mathbf{Fm}_{\mathcal{L}}$; that is, writing $\Gamma \vdash_L \varphi$ for $(\Gamma, \varphi) \in L$:

- $\{\varphi\} \vdash_L \varphi$.
- If $\Gamma \vdash_L \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_L \varphi$.
- If $\Gamma \vdash_L \varphi$ and $\Gamma \cup \{\varphi\} \vdash_L \psi$, then $\Gamma \vdash_L \psi$.
- If $\Gamma \vdash_L \varphi$, then $\sigma(\Gamma) \vdash_L \sigma(\varphi)$ for each \mathcal{L} -substitution σ .
- If $\Gamma \vdash_L \varphi$, then there is a finite set $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_L \varphi$.

An *L-theorem* is a formula φ such that $\emptyset \vdash_L \varphi$ (abbreviated to $\vdash_L \varphi$).

Let L_1 be a logic in a language \mathcal{L}_1 . A logic L_2 in a language $\mathcal{L}_2 \supseteq \mathcal{L}_1$ is called an *expansion* of L_1 , and an *extension* if \mathcal{L}_1 and \mathcal{L}_2 coincide, if for all $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}_1}$: $\Gamma \vdash_{L_1} \varphi$ implies $\Gamma \vdash_{L_2} \varphi$. Such an expansion is called *conservative* if $\Gamma \vdash_{L_2} \varphi$ implies $\Gamma \vdash_{L_1} \varphi$ for each $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}_1}$. In this case, L_1 is the (unique) \mathcal{L}_1 -*fragment* of L_2 , denoted by $L_2 \upharpoonright \mathcal{L}_1$.

A logic L is called *structurally complete* if all of its extensions have new theorems, and *hereditarily structurally complete* if all of its extensions are structurally complete. Moreover, let us define a *rule* for a language \mathcal{L} to be an ordered pair Γ / φ , where $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ is finite. Γ / φ is called *L-derivable* if $\Gamma \vdash_L \varphi$, and *L-admissible* if for each \mathcal{L} -substitution σ , whenever $\vdash_L \sigma(\psi)$ for all $\psi \in \Gamma$, then $\vdash_L \sigma(\varphi)$. It follows that L is structurally complete if and only if (henceforth, iff) every L-admissible rule is L-derivable (see, e.g., [25, 31]).

Now define for $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$:

$$\Gamma \sim_L \varphi \quad \text{iff} \quad \Gamma' / \varphi \text{ is L-admissible for some finite } \Gamma' \subseteq \Gamma.$$

Then $\sim_L = \{(\Gamma, \varphi) \in \mathcal{P}(\text{Fm}_{\mathcal{L}}) \times \text{Fm}_{\mathcal{L}} \mid \Gamma \sim_L \varphi\}$ is also a logic. For a set of rules \mathcal{R} , let $L + \mathcal{R}$ denote the smallest logic containing $L \cup \mathcal{R}$, writing $L + r_1 + \dots + r_n$ if $\mathcal{R} = \{r_1, \dots, r_n\}$. The set \mathcal{R} is called a *basis for the admissible rules of L* if $L + \mathcal{R}$ is \sim_L .

The following simple lemma will play a key role in our proofs that a given set of rules forms a basis for the admissible rules of a logic.

Lemma 2.1. *Let L be a logic and let \mathcal{R} be a set of L-admissible rules such that $L + \mathcal{R}$ is structurally complete. Then \mathcal{R} is a basis for the admissible rules of L .*

Proof. Suppose that $L + \mathcal{R}$ is structurally complete. Then $\Gamma \vdash_{L+\mathcal{R}} \varphi$ implies $\Gamma \sim_L \varphi$ since \sim_L is a logic containing $L \cup \mathcal{R}$. Suppose now that $\Gamma \sim_L \varphi$. If $\vdash_{L+\mathcal{R}} \sigma(\psi)$ for all $\psi \in \Gamma$, then $\vdash_L \sigma(\psi)$ for all $\psi \in \Gamma$, so $\vdash_L \sigma(\varphi)$ and therefore also $\vdash_{L+\mathcal{R}} \sigma(\varphi)$. Hence $\Gamma \sim_{L+\mathcal{R}} \varphi$. But $L + \mathcal{R}$ is structurally complete, so also $\Gamma \vdash_{L+\mathcal{R}} \varphi$. \square

Let us now consider the algebraic setting. We fix \mathbf{K} to be a class of algebraic structures (algebras) of the same language \mathcal{L} . For a set of \mathcal{L} -identities $\Sigma \cup \{\varphi \approx \psi\}$, we write $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ if for each $\mathbf{A} \in \mathbf{K}$ and homomorphism $h: \mathbf{Fm}_{\mathcal{L}} \rightarrow \mathbf{A}$, whenever $h(\varphi') = h(\psi')$ for all $\varphi' \in \psi' \in \Sigma$, also $h(\varphi) = h(\psi)$. We say that \mathbf{K} is a (*quasi*)variety if it is axiomatizable by a set of (quasi-)identities. We also define the *variety* $\mathbb{V}(\mathbf{K})$ and *quasivariety* $\mathbb{Q}(\mathbf{K})$ generated by \mathbf{K} to be the smallest variety and quasivariety containing \mathbf{K} , respectively.

The logical and algebraic settings are connected via the theory of algebraizable logics (see [8]). A logic L is *algebraizable* if there is a quasivariety \mathbf{K} together with a set of identities in one variable $E(p)$ and a set of formulas in two variables $D(p, q)$ (called *translations*) satisfying

1. $\Gamma \vdash_L \varphi$ iff $\bigcup \{E(\psi) \mid \psi \in \Gamma\} \models_{\mathbf{K}} E(\varphi)$.
2. $\Sigma \models_{\mathbf{K}} \varphi \approx \psi$ iff $\bigcup \{D(\varphi', \psi') \mid \varphi' \approx \psi' \in \Sigma\} \vdash_L D(\varphi, \psi)$.
3. $\bigcup \{D(\varphi', \psi') \mid \varphi' \approx \psi' \in E(\varphi)\} \vdash_L \varphi$ and $\{\varphi\} \vdash_L D(\varphi', \psi')$ for each $\varphi' \approx \psi' \in E(\varphi)$.
4. $\bigcup \{E(\varphi') \mid \varphi' \in D(\varphi, \psi)\} \models_{\mathbf{K}} \varphi \approx \psi$ and $\{\varphi \approx \psi\} \models_{\mathbf{K}} E(\varphi')$ for each $\varphi' \in D(\varphi, \psi)$.

\mathbf{K} is then called an *equivalent quasivariety* of L .

The following theorem is very helpful for dealing with fragments of logics. Recall that an \mathcal{L} -*subreduct* of an algebra \mathbf{A} is a subalgebra of an \mathcal{L} -reduct of \mathbf{A} .

Theorem 2.2 ([8]). *Let L be an algebraizable logic for a language \mathcal{L} with an equivalent quasivariety \mathbf{Q} and translations in $\mathcal{L}' \subseteq \mathcal{L}$. Then $L \upharpoonright \mathcal{L}'$ is algebraizable with the same translations and has an equivalent quasivariety $\mathbf{Q} \upharpoonright \mathcal{L}'$ consisting of all \mathcal{L}' -subreducts of algebras from \mathbf{Q} .*

Given an algebraizable logic L with equivalent quasivariety \mathbf{Q} , the algebraic notion corresponding to an L -admissible rule is a quasi-identity that holds in the free algebra of \mathbf{Q} on countably infinitely many generators, denoted by $\mathbf{F}_{\mathbf{Q}}$ (see, e.g., [31]). Below, we recall a useful simple condition (a particular case of a theorem from [11]) for the logic of a quasivariety generated by a single algebra to be structurally complete. We make use of the *canonical morphism* $h_{\mathbf{Q}}: \mathbf{Fm}_{\mathcal{L}} \rightarrow \mathbf{F}_{\mathbf{Q}}$ which maps each formula φ to its equivalence class in the free algebra.

Lemma 2.3 ([11]). *Let L be a logic for a language \mathcal{L} with an equivalent quasivariety $\mathbb{Q}(\mathbf{A})$ such that $h_{\mathbb{Q}(\mathbf{A})} \circ g$ embeds \mathbf{A} into $\mathbf{F}_{\mathbb{Q}(\mathbf{A})}$ for some map $g: \mathbf{A} \rightarrow \mathbf{Fm}_{\mathcal{L}'}$ and $\mathcal{L}' \subseteq \mathcal{L}$. Then $L \upharpoonright \mathcal{L}'$ is structurally complete.*

3 R-mingle and its fragments

The logic R-mingle (for short, RM) is traditionally formulated in a language with the binary connectives \rightarrow (implication), \wedge (conjunction), and \vee (disjunction), and the unary connective \neg (negation). However, in the landscape of substructural logics, the “multiplicative” conjunction \cdot (fusion), definable in RM as $\varphi \cdot \psi =_{\text{def}} \neg(\varphi \rightarrow \neg\psi)$, plays an important role as the partner of its residuum \rightarrow . An axiomatization of RM in this extended language is provided in Figure 1. As usual, we identify axiomatizations with their associated consequence relations.

Consider the following binary functions on the set of integers \mathbb{Z} , where \wedge and \vee are min and max, respectively, \neg is the usual unary minus operation, and $|x|$ is the absolute value of x :

$$x \cdot y = \begin{cases} x \wedge y & \text{if } |x| = |y| \\ y & \text{if } |x| < |y| \\ x & \text{if } |y| < |x| \end{cases} \quad x \rightarrow y = \begin{cases} \neg x \vee y & \text{if } x \leq y \\ \neg x \wedge y & \text{otherwise.} \end{cases}$$

Let \mathbf{Z} be the algebra $\langle \mathbb{Z}, \rightarrow, \cdot, \wedge, \vee, \neg \rangle$. We denote by SA the quasivariety $\mathbb{Q}(\mathbf{Z})$ of *Sugihara algebras*, noting that this name is used in [9] for the same algebras without the (definable) fusion operation. Sugihara algebras may also be characterized as t -free subreducts of distributive idempotent involutive commutative residuated lattices (see, e.g., [27] for details).

<p>(B) $(q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$ (C) $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$ (I) $p \rightarrow p$ (\cdot1) $p \rightarrow (q \rightarrow (p \cdot q))$ (\cdot2) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \cdot q) \rightarrow r)$ (\neg1) $(p \rightarrow \neg q) \rightarrow (q \rightarrow \neg p)$ (\neg2) $\neg \neg p \rightarrow p$ (W) $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$ (mp) $\{p, p \rightarrow q\} / q$</p>	<p>(\wedge1) $(p \wedge q) \rightarrow p$ (\wedge2) $(p \wedge q) \rightarrow q$ (\wedge3) $((p \rightarrow q) \wedge (p \rightarrow r)) \rightarrow (p \rightarrow (q \wedge r))$ (\vee1) $p \rightarrow (p \vee q)$ (\vee2) $q \rightarrow (p \vee q)$ (\vee3) $((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$ (DIS) $(p \wedge (q \vee r)) \rightarrow ((p \wedge q) \vee (p \wedge r))$ (M) $p \rightarrow (p \rightarrow p)$ (adj) $\{p, q\} / p \wedge q$</p>
---	--

Figure 1: An axiomatization of RM.

For any Sugihara algebra \mathbf{A} and $x \in A$, let us fix $|x| =_{\text{def}} x \rightarrow x$. Then the lattice order of \mathbf{A} is equationally definable for $x, y \in A$ by

$$x \leq y \quad \text{iff} \quad x \rightarrow y = |x \rightarrow y|.$$

Note in particular that $0 \leq x$ in \mathbf{Z} iff $x = |x|$.

Taking advantage of this formulation, which covers all fragments considered in this paper, let us define for $\{\rightarrow\} \subseteq \mathcal{L} \subseteq \{\rightarrow, \cdot, \wedge, \vee, \neg\}$, a class of \mathcal{L} -subreducts of Sugihara algebras \mathbf{K} , and $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$:

$$\Gamma \models_{\mathbf{K}} \varphi \quad \text{iff} \quad \{\psi \approx |\psi| \mid \psi \in \Gamma\} \models_{\mathbf{K}} \varphi \approx |\varphi|$$

abbreviating $\Gamma \models_{\{\mathbf{A}\}} \varphi$ to $\Gamma \models_{\mathbf{A}} \varphi$. Then RM has the following strong completeness theorem:

Theorem 3.1 (Meyer, Dunn [1, 13]). *For $\{\rightarrow\} \subseteq \mathcal{L} \subseteq \{\rightarrow, \cdot, \wedge, \vee, \neg\}$ and $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$:*

$$\Gamma \vdash_{\text{RM}} \varphi \quad \text{iff} \quad \Gamma \models_{\text{SA}} \varphi.$$

Note also that for any set of identities $\Sigma \cup \{\varphi \approx \psi\}$ for \mathcal{L} :

$$\Sigma \models_{\text{SA}} \varphi \approx \psi \quad \text{iff} \quad \bigcup_{\varphi' \approx \psi' \in \Sigma} \{\varphi' \rightarrow \psi', \psi' \rightarrow \varphi'\} \vdash_{\text{RM}} \varphi \rightarrow \psi$$

$$\text{and } \bigcup_{\varphi' \approx \psi' \in \Sigma} \{\varphi' \rightarrow \psi', \psi' \rightarrow \varphi'\} \vdash_{\text{RM}} \psi \rightarrow \varphi.$$

That is, SA is an equivalent quasivariety for the logic RM with translations $E(p) = \{p \approx |p|\}$ and $D(p, q) = \{p \rightarrow q, q \rightarrow p\}$. Moreover, by Lemma 2.2, we obtain equivalent quasivarieties for all fragments of RM with implication.

To see now that RM and some of its fragments enjoy a deduction theorem, consider the following connective introduced by Avron in [2]:

$$\varphi \supset \psi =_{\text{def}} (\varphi \rightarrow \psi) \vee \psi.$$

Theorem 3.2 ([2]). $\Gamma \cup \{\varphi\} \vdash_{\text{RM}} \psi \quad \text{iff} \quad \Gamma \vdash_{\text{RM}} \varphi \supset \psi.$

It is easily checked that the following identities hold in all Sugihara algebras, either by calculation in the algebra \mathbf{Z} or by providing derivations in Avron's elegant hypersequent calculus for RM [3]:

$$\begin{aligned} x \wedge y &\approx \neg(\neg x \vee \neg y) \\ x \vee y &\approx \neg(\neg x \wedge \neg y) \\ x \cdot y &\approx \neg(x \rightarrow \neg y) \\ x \supset y &\approx (x \wedge |y|) \rightarrow y \\ x \vee y &\approx ((x \rightarrow y) \supset y) \wedge ((y \rightarrow x) \supset x) \\ x \cdot y &\approx ((x \rightarrow |y|) \rightarrow y) \wedge ((y \rightarrow |x|) \rightarrow x). \end{aligned}$$

The first three identities are well-known and hold in all involutive commutative residuated lattices, while the fourth and fifth have appeared already in [9]. The last identity, which shows that fusion is definable using implication and conjunction, seems to be new, however. Together, they allow us to distinguish the various *superimplicational* (i.e., containing implication) fragments of RM (see Figure 2).

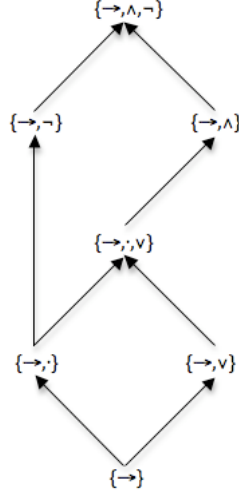


Figure 2: Superimplicational fragments of RM.

Theorem 3.3. *The distinct superimplicational fragments of RM consist of the following sets of connectives:*

$$\{\rightarrow, \wedge, \neg\} \quad \{\rightarrow, \wedge\} \quad \{\rightarrow, \cdot, \vee\} \quad \{\rightarrow, \vee\} \quad \{\rightarrow, \neg\} \quad \{\rightarrow, \cdot\} \quad \{\rightarrow\}.$$

Proof. It follows directly from the identities listed above that these sets of connectives exhaust the possibilities for the various superimplicational fragments of RM. In particular the $\{\rightarrow, \wedge, \neg\}$ -fragment coincides with the whole of RM and the $\{\rightarrow, \wedge\}$ -fragment is the positive fragment of RM. Hence the diagram in Figure 2 represents all inclusions between fragments of RM and it remains to check that these inclusions are proper.

Note first that it is shown in [9] that the $\{\rightarrow, \wedge, \neg\}$, $\{\rightarrow, \wedge\}$, $\{\rightarrow, \vee\}$, $\{\rightarrow, \neg\}$, and $\{\rightarrow\}$ fragments are distinct. Moreover, since classical logic is an axiomatic extension of RM where \wedge and \cdot collapse, and the implication, implication-negation, and implication-conjunction fragments of classical logic are distinct, it follows also that the $\{\rightarrow\}$, $\{\rightarrow, \cdot\}$, $\{\rightarrow, \vee\}$, $\{\rightarrow, \cdot, \vee\}$, and $\{\rightarrow, \neg\}$ -fragments of RM are distinct.

Finally, we show that the $\{\rightarrow, \cdot, \vee\}$ and $\{\rightarrow, \wedge\}$ fragments are distinct by considering the subalgebra of the $\{\rightarrow, \cdot, \vee\}$ -reduct of $\mathbf{Z} \times \mathbf{Z}$ generated by $(1, 0)$ and $(0, 1)$. This subalgebra contains the elements $(1, 1)$, $(1, 0)$, $(0, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$, but not the meet $(0, 0)$ of $(1, 0)$ and $(0, 1)$. Hence \wedge is not definable in the $\{\rightarrow, \cdot, \vee\}$ -fragment. \square

Following [9], we call superimplicational fragments of RM with at least one of \wedge and \vee , *upper fragments*, and the rest, *lower fragments*.

4 The Upper Fragments

Certain *positive* (lacking negation) upper fragments of RM are hereditarily structurally complete. Indeed, this was established implicitly for the $\{\rightarrow, \wedge\}$ -fragment of RM in [32] (see also [25]), although to the best of our knowledge, the $\{\rightarrow, \vee\}$ and $\{\rightarrow, \cdot, \vee\}$ fragments have not been considered in the literature. Here we adapt the approach of [32] to obtain a uniform hereditary structural completeness proof for all three of these fragments.

Theorem 4.1. *The $\{\rightarrow, \vee\}$, $\{\rightarrow, \cdot, \vee\}$, and $\{\rightarrow, \wedge\}$ fragments of RM are hereditarily structurally complete.*

Proof. Let $\{\varphi_1, \dots, \varphi_n\}$ be a finite set of formulas for one of these fragments. We write $\vec{\varphi} \supset \psi$ for the formula $\varphi_1 \supset (\varphi_2 \supset \dots \supset (\varphi_n \supset \psi) \dots)$, and define the substitution:

$$\sigma(p) = \vec{\varphi} \supset p.$$

We prove by induction on formula complexity that for all $\psi \in \text{Fm}_{\{\rightarrow, \cdot, \wedge, \vee\}}$:

$$\vdash_{\text{RM}} \sigma(\psi) \rightarrow (\vec{\varphi} \supset \psi) \quad \text{and} \quad \vdash_{\text{RM}} (\vec{\varphi} \supset \psi) \rightarrow \sigma(\psi).$$

The base case is immediate. For the inductive step, assume for $i = 1, 2$ that $\vdash_{\text{RM}} \sigma(\psi_i) \rightarrow (\vec{\varphi} \supset \psi_i)$ and $\vdash_{\text{RM}} (\vec{\varphi} \supset \psi_i) \rightarrow \sigma(\psi_i)$. Then for each connective $\# \in \{\rightarrow, \cdot, \wedge, \vee\}$:

$$\vdash_{\text{RM}} \sigma(\psi_1 \# \psi_2) \rightarrow ((\vec{\varphi} \supset \psi_1) \# (\vec{\varphi} \supset \psi_2)) \quad \text{and} \quad \vdash_{\text{RM}} ((\vec{\varphi} \supset \psi_1) \# (\vec{\varphi} \supset \psi_2)) \rightarrow \sigma(\psi_1 \# \psi_2).$$

But also (using, e.g., Avron's hypersequent calculus [3])

$$\begin{aligned} \vdash_{\text{RM}} ((\vec{\varphi} \supset \psi_1) \rightarrow (\vec{\varphi} \supset \psi_2)) &\rightarrow (\vec{\varphi} \supset (\psi_1 \rightarrow \psi_2)) \\ \vdash_{\text{RM}} (\vec{\varphi} \supset (\psi_1 \rightarrow \psi_2)) &\rightarrow ((\vec{\varphi} \supset \psi_1) \rightarrow (\vec{\varphi} \supset \psi_2)) \\ \vdash_{\text{RM}} ((\vec{\varphi} \supset \psi_1) \cdot (\vec{\varphi} \supset \psi_2)) &\rightarrow (\vec{\varphi} \supset (\psi_1 \cdot \psi_2)) \\ \vdash_{\text{RM}} (\vec{\varphi} \supset (\psi_1 \cdot \psi_2)) &\rightarrow ((\vec{\varphi} \supset \psi_1) \cdot (\vec{\varphi} \supset \psi_2)) \\ \vdash_{\text{RM}} ((\vec{\varphi} \supset \psi_1) \wedge (\vec{\varphi} \supset \psi_2)) &\rightarrow (\vec{\varphi} \supset (\psi_1 \wedge \psi_2)) \\ \vdash_{\text{RM}} (\vec{\varphi} \supset (\psi_1 \wedge \psi_2)) &\rightarrow ((\vec{\varphi} \supset \psi_1) \wedge (\vec{\varphi} \supset \psi_2)) \\ \vdash_{\text{RM}} ((\vec{\varphi} \supset \psi_1) \vee (\vec{\varphi} \supset \psi_2)) &\rightarrow (\vec{\varphi} \supset (\psi_1 \vee \psi_2)) \\ \vdash_{\text{RM}} (\vec{\varphi} \supset (\psi_1 \vee \psi_2)) &\rightarrow ((\vec{\varphi} \supset \psi_1) \vee (\vec{\varphi} \supset \psi_2)). \end{aligned}$$

Hence for each $\# \in \{\rightarrow, \cdot, \wedge, \vee\}$, as required:

$$\vdash_{\text{RM}} \sigma(\psi_1 \# \psi_2) \rightarrow (\vec{\varphi} \supset (\psi_1 \# \psi_2)) \quad \text{and} \quad \vdash_{\text{RM}} (\vec{\varphi} \supset (\psi_1 \# \psi_2)) \rightarrow \sigma(\psi_1 \# \psi_2).$$

Now for each $i = 1 \dots n$, since $\vdash_{\text{RM}} \vec{\varphi} \supset \varphi_i$, also, using the above claim, $\vdash_{\text{RM}} \sigma(\varphi_i)$. Moreover, since $\vdash_{\text{RM}} \sigma(\psi) \rightarrow (\vec{\varphi} \supset \psi)$, it follows that $\vdash_{\text{RM}} \vec{\varphi} \supset (\sigma(\psi) \rightarrow \psi)$ and hence, by the deduction theorem for \supset (Theorem 3.2), that $\{\varphi_1, \dots, \varphi_n\} \vdash_{\text{RM}} \sigma(\psi) \rightarrow \psi$. Suppose finally that L is an extension of the $\{\rightarrow, \vee\}$, $\{\rightarrow, \cdot, \vee\}$, or $\{\rightarrow, \wedge\}$ fragment of RM, and that $\{\varphi_1, \dots, \varphi_n\} / \psi$ is an L -admissible rule. Since $\vdash_L \sigma(\varphi_i)$ for $i = 1 \dots n$, also $\vdash_L \sigma(\psi)$. But $\{\varphi_1, \dots, \varphi_n\} \vdash_L \sigma(\psi) \rightarrow \psi$, so using modus ponens, $\{\varphi_1, \dots, \varphi_n\} \vdash_L \psi$ as required. That is, L is structurally complete. So the $\{\rightarrow, \vee\}$, $\{\rightarrow, \cdot, \vee\}$, and $\{\rightarrow, \wedge\}$ fragments of RM are hereditarily structurally complete. \square

The strategy described in the above proof was first used by Prucnal in [26] to prove hereditary structural completeness for the implication fragment of each intermediate logic, and is hence often referred to as ‘‘Prucnal’s trick’’. In fact, the proof establishes that each finite set of formulas Γ in the considered fragment L is L -projective, the substitution σ being a so-called L -projective unifier of Γ : namely $\vdash_L \sigma(\psi)$ for all $\psi \in \Gamma$ and for each variable p , $\Gamma \vdash_L p \rightarrow \sigma(p)$ and $\Gamma \vdash_L \sigma(p) \rightarrow p$ (see [14, 15] for further details). We note also that a more complicated proof of hereditary structural completeness has been provided by Olson and Raftery in [24] for the positive fragment of RM^t , axiomatized by extending an axiomatization for the positive fragment of RM with axioms $p \rightarrow (t \rightarrow p)$ and t for the constant t .

5 The Lower Fragments

We turn our attention now to the lower fragments of RM: the implication fragment $\text{RM}|\{\rightarrow\}$, the implication-fusion fragment $\text{RM}|\{\rightarrow, \cdot\}$, and the multiplicative fragment $\text{RM}|\{\rightarrow, \cdot, \neg\}$. In this setting, the following algebra (the $\{\rightarrow, \cdot, \neg\}$ -reduct of a three element Sugihara algebra) plays a pivotal role:

$$\mathbf{S} = \langle \{-1, 0, 1\}, \rightarrow, \cdot, \neg \rangle$$

where \rightarrow , \cdot , and \neg are the operations of \mathbf{Z} restricted to $\{-1, 0, 1\}$. Recall that (as in \mathbf{Z}) $x \geq 0$ in \mathbf{S} iff $x = |x|$. So in particular, $\Gamma \models_{\mathbf{S}} \varphi$ iff for every homomorphism $h: \mathbf{Fm}_{\{\rightarrow, \cdot, \neg\}} \rightarrow \mathbf{S}$, whenever $h(\psi) \geq 0$ for all $\psi \in \Gamma$, also $h(\varphi) \geq 0$.

For convenience, let us denote the multiplicative fragment $\text{RM}|\{\rightarrow, \cdot, \neg\}$ by RM_m and its equivalent quasivariety by \mathbf{SA}_m . The following key result for this fragment was established by Sobociński in [33] (see also [3] and [9] for proof-theoretic and algebraic proofs, respectively):

Theorem 5.1 ([33]). $\mathbb{V}(\mathbf{SA}_m) = \mathbb{V}(\mathbf{S})$.

However, consider now Avron's defined implication

$$\varphi \Rightarrow \psi =_{\text{def}} (\varphi \rightarrow |\psi|) \rightarrow (\varphi \rightarrow \psi)$$

and the modus ponens rule for this connective

$$(A) \quad \{p, p \Rightarrow q\} / q.$$

Then $\{p, p \Rightarrow q\} \not\models_{\mathbf{SA}_m} q$: just consider the $\{\rightarrow, \cdot, \neg\}$ -reduct of \mathbf{Z} and evaluate p as 2 and q as -1 . But $\{p, p \Rightarrow q\} \models_{\mathbf{S}} q$. So \mathbf{SA}_m is not generated by \mathbf{S} as a quasivariety. On the other hand, if $\Gamma \models_{\mathbf{S}} \varphi$ for some finite $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}$ where $\{\rightarrow\} \subseteq \mathcal{L} \subseteq \{\rightarrow, \cdot, \neg\}$, then, since $\mathbb{V}(\mathbf{SA}_m) = \mathbb{V}(\mathbf{S})$, the rule Γ / φ is $\text{RM}|\mathcal{L}$ -admissible. In particular:

Lemma 5.2. (A) is $\text{RM}|\mathcal{L}$ -admissible for $\{\rightarrow\} \subseteq \mathcal{L} \subseteq \{\rightarrow, \cdot, \neg\}$.

Our initial goal will be to show that $\{(A)\}$ in fact constitutes a basis for the admissible rules of the $\{\rightarrow\}$ and $\{\rightarrow, \cdot\}$ fragments of RM. First, we note that, as shown in a more general setting by Avron in [5], the connective \Rightarrow provides a deduction theorem with respect to \mathbf{S} .

Lemma 5.3 (see [5], Theorem III.2.2). For $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\{\rightarrow, \cdot, \neg\}}$:

$$\Gamma \cup \{\varphi\} \models_{\mathbf{S}} \psi \quad \text{iff} \quad \Gamma \models_{\mathbf{S}} \varphi \Rightarrow \psi.$$

Proof. (\Rightarrow) Suppose that $\Gamma \cup \{\varphi\} \models_{\mathbf{S}} \psi$ and that each formula in Γ is evaluated as 0 or 1 in \mathbf{S} . If φ is evaluated as 0 or 1, then ψ is evaluated as 0 or 1, and therefore so also is $\varphi \Rightarrow \psi$. (The only tricky case when φ is 1 and ψ is 0 follows from the fact that $|\psi|$ is then 0 and so $\varphi \rightarrow |\psi|$ and $\varphi \rightarrow \psi$ are both -1 .) If φ is evaluated as -1 , then $\varphi \Rightarrow \psi$ is clearly evaluated as 0 or 1.

(\Leftarrow) If $\Gamma \models_{\mathbf{S}} \varphi \Rightarrow \psi$, then $\Gamma \cup \{\varphi\} \models_{\mathbf{S}} \varphi \Rightarrow \psi$ and since also $\Gamma \cup \{\varphi\} \models_{\mathbf{S}} \varphi$ and $\{\varphi, \varphi \Rightarrow \psi\} \models_{\mathbf{S}} \psi$, it follows that $\Gamma \cup \{\varphi\} \models_{\mathbf{S}} \psi$. \square

Similarly to the connective \supset defined in Section 3, which provides a deduction theorem for fragments of RM containing both \rightarrow and \vee , the connective \Rightarrow may be understood as incorporating a form of implication with weakening. Note, however, that these two implications do not coincide; e.g., evaluating p as 2 and q as -1 in \mathbf{Z} , we have that $p \Rightarrow q$ is 2 and that $p \supset q$ is -1 .

Next, adapting similar proofs by Avron in [4, 5], we obtain an axiomatization of the logic with equivalent quasivariety $\mathbb{Q}(\mathbf{S})$.

Lemma 5.4. $\text{RM}_m + (A)$ has equivalent quasivariety $\mathbb{Q}(\mathbf{S})$.

Proof. It suffices to show that for any $\Gamma = \{\psi_1, \dots, \psi_n\} \subseteq \text{Fm}_{\{\rightarrow, \cdot, \neg\}}$ and $\varphi \in \text{Fm}_{\{\rightarrow, \cdot, \neg\}}$:

$$\Gamma \vdash_{\text{RM}_m+(A)} \varphi \quad \text{iff} \quad \Gamma \models_{\mathbf{S}} \varphi.$$

(\Rightarrow) If $\Gamma \vdash_{\text{RM}_m} \varphi$, then $\Gamma \models_{\mathbf{S}} \varphi$ since \mathbf{S} is the reduct of a Sugihara algebra. Also $\{p, p \Rightarrow q\} \models_{\mathbf{S}} q$. Hence the logic defined by $\models_{\mathbf{S}}$ contains $\text{RM}_m \cup \{(A)\}$. So $\Gamma \vdash_{\text{RM}_m+(A)} \varphi$ implies $\Gamma \models_{\mathbf{S}} \varphi$.

(\Leftarrow) Suppose that $\Gamma \models_{\mathbf{S}} \varphi$, and let $\vec{\psi} \Rightarrow \varphi$ denote the formula $\psi_1 \Rightarrow (\psi_2 \Rightarrow \dots \supset (\psi_n \Rightarrow \varphi) \dots)$. Then by Lemma 5.3, $\models_{\mathbf{S}} \vec{\psi} \Rightarrow \varphi$. So by Theorem 5.1, $\vdash_{\text{RM}_m} \vec{\psi} \Rightarrow \varphi$ and hence $\vdash_{\text{RM}_m+(A)} \vec{\psi} \Rightarrow \varphi$. But also $\Gamma \vdash_{\text{RM}_m+(A)} \psi_i$ for $i = 1 \dots n$. Hence, using the rule (A) n times, we obtain $\Gamma \vdash_{\text{RM}_m+(A)} \varphi$ as required. \square

Moreover, for the implication and implication-fusion fragments of RM, adding the rule (A) also secures structural completeness.

Lemma 5.5. $\text{RM}\{\rightarrow\} + (A)$ and $\text{RM}\{\rightarrow, \cdot\} + (A)$ are structurally complete.

Proof. By Lemma 2.3, it suffices to show that $h_{\mathbb{Q}(\mathbf{S}\{\rightarrow, \cdot\})} \circ g$ embeds $\mathbf{S}\{\rightarrow, \cdot\}$ into $\mathbf{F}_{\mathbb{Q}(\mathbf{S}\{\rightarrow, \cdot\})}$ where $g: \{-1, 0, 1\} \rightarrow \text{Fm}_{\{\rightarrow\}}$ is defined by:

$$\begin{aligned} g(1) &= (p \rightarrow q) \rightarrow (p \rightarrow q) \\ g(0) &= q \rightarrow q \\ g(-1) &= ((p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow (q \rightarrow q). \end{aligned}$$

First we check (e.g., using Avron's sequent calculus [3]) that the image of $\mathbf{S}\{\rightarrow, \cdot\}$ under $h_{\mathbb{Q}(\mathbf{S}\{\rightarrow, \cdot\})} \circ g$ behaves appropriately with respect to \cdot . For all formulas $\varphi, \psi \in \text{Fm}_{\{\rightarrow, \cdot, \neg\}}$, we have $\models_{\mathbf{S}} \varphi \approx \varphi \cdot \varphi$ and $\models_{\mathbf{S}} \varphi \cdot \psi \approx \psi \cdot \varphi$. Hence we need only check that $\models_{\mathbf{S}} g(1) \cdot g(0) \approx g(1)$, $\models_{\mathbf{S}} g(-1) \cdot g(0) \approx g(-1)$, and $\models_{\mathbf{S}} g(1) \cdot g(-1) \approx g(-1)$, i.e.

$$\begin{aligned} \models_{\mathbf{S}} ((p \rightarrow q) \rightarrow (p \rightarrow q)) \cdot (q \rightarrow q) &\approx (p \rightarrow q) \rightarrow (p \rightarrow q) \\ \models_{\mathbf{S}} (((p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow (q \rightarrow q)) \cdot (q \rightarrow q) &\approx ((p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow (q \rightarrow q) \\ \models_{\mathbf{S}} ((p \rightarrow q) \rightarrow (p \rightarrow q)) \cdot (((p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow (q \rightarrow q)) &\approx ((p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow (q \rightarrow q). \end{aligned}$$

Similarly, we may check (again using Avron's sequent calculus [3]) that the image of $\mathbf{S}\{\rightarrow, \cdot\}$ under $h_{\mathbb{Q}(\mathbf{S}\{\rightarrow, \cdot\})} \circ g$ behaves appropriately with respect to \rightarrow . That is, we check that for each $a, b \in \{-1, 0, 1\}$:

$$\models_{\mathbf{S}} g(a) \rightarrow g(b) \approx g(a \rightarrow b).$$

Finally, we observe that $h_{\mathbb{Q}(\mathbf{S}\{\rightarrow, \cdot\})} \circ g$ is 1-1: there are homomorphisms $h: \mathbf{Fm}_{\{\rightarrow, \cdot\}} \rightarrow \mathbf{S}$ that distinguish the formulas in the image of g . \square

Since (A) is admissible for the $\{\rightarrow\}$ and $\{\rightarrow, \cdot\}$ fragments of RM (Lemma 5.2), we obtain immediately from Lemma 2.1:

Theorem 5.6. $\{(A)\}$ is a basis for the admissible rules of the $\{\rightarrow\}$ and $\{\rightarrow, \cdot\}$ fragments of RM.

We turn our attention now to the multiplicative fragment RM_m . The following defined bi-implication will be useful:

$$\varphi \leftrightarrow \psi =_{\text{def}} (\varphi \rightarrow \psi) \cdot (\psi \rightarrow \varphi).$$

Observe that $\varphi \leftrightarrow \psi$ is -1 in \mathbf{S} if φ and ψ take different values, 0 if they both take value 0 , and 1 if they both take the same value -1 or 1 . We write $\varphi_1 \leftrightarrow \varphi_2 \leftrightarrow \varphi_3 \dots \leftrightarrow \varphi_n$ for $(\dots (\varphi_1 \leftrightarrow \varphi_2) \leftrightarrow \varphi_3 \dots) \leftrightarrow \varphi_n$ and consider the following rules for $n \in \mathbb{N} \setminus \{0\}$:

$$(R_n) \quad \{ \neg((p_1 \rightarrow p_1) \leftrightarrow \dots \leftrightarrow (p_n \rightarrow p_n)) \} / q.$$

Note that in \mathbf{S} , for all $a_1, \dots, a_n \in \{-1, 0, 1\}$:

$$\neg((a_1 \rightarrow a_1) \leftrightarrow \dots \leftrightarrow (a_n \rightarrow a_n)) = \begin{cases} 0 \text{ or } 1 & \text{if } 0 \in \{a_1, \dots, a_n\} \\ -1 & \text{otherwise.} \end{cases}$$

For convenience, let us call this property (*).

Lemma 5.7. (R_n) is RM_m -admissible for each $n \in \mathbb{N} \setminus \{0\}$.

Proof. No substitution makes the premise of (R_n) ($n \in \mathbb{N} \setminus \{0\}$) into a theorem of classical logic. Hence the same also holds for RM_m , and each (R_n) ($n \in \mathbb{N} \setminus \{0\}$) is RM_m -admissible. \square

Consider now the subalgebra $\mathbf{2} = \langle \{-1, 1\}, \rightarrow, \cdot, \neg \rangle$ of \mathbf{S} , term-equivalent to the two element Boolean algebra. Since \mathbf{S} is a homomorphic image of $\mathbf{2} \times \mathbf{S}$, we have $\mathbb{V}(\mathbf{2} \times \mathbf{S}) = \mathbb{V}(\mathbf{S})$. In what follows, we will show that a particular axiomatization of $\mathbb{Q}(\mathbf{2} \times \mathbf{S})$ is structurally complete and hence provides a basis for the admissible rules of RM_m .

Lemma 5.8. $\text{RM}_m + (A) + \{(R_n) \mid n \in \mathbb{N} \setminus \{0\}\}$ has equivalent quasivariety $\mathbb{Q}(\mathbf{2} \times \mathbf{S})$.

Proof. For convenience, let

$$\mathbf{L} = \text{RM}_m + (A) + \{(R_n) \mid n \in \mathbb{N} \setminus \{0\}\}.$$

Then it suffices to show that for any finite $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\{\rightarrow, \cdot, \neg\}}$:

$$\Gamma \vdash_{\mathbf{L}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathbf{2} \times \mathbf{S}} \varphi.$$

(\Rightarrow) If $\Gamma \vdash_{\text{RM}_m} \varphi$, then, since $\mathbf{2}$ and \mathbf{S} are both reducts of Sugihara algebras, $\Gamma \models_{\mathbf{2} \times \mathbf{S}} \varphi$. Also, $\{p, p \Rightarrow q\} \models_{\mathbf{2} \times \mathbf{S}} q$ as $\{p, p \Rightarrow q\} \models_{\mathbf{2}} q$ and $\{p, p \Rightarrow q\} \models_{\mathbf{S}} q$. Finally, $\{\neg((p_1 \rightarrow p_1) \leftrightarrow \dots \leftrightarrow (p_n \rightarrow p_n))\} \models_{\mathbf{2} \times \mathbf{S}} q$ for each $n \in \mathbb{N} \setminus \{0\}$ since $\neg((p_1 \rightarrow p_1) \leftrightarrow \dots \leftrightarrow (p_n \rightarrow p_n))$ is not satisfiable in $\mathbf{2}$. That is, the logic defined by $\models_{\mathbf{2} \times \mathbf{S}}$ contains $\text{RM}_m \cup \{(A)\} \cup \{(R_n) \mid n \in \mathbb{N} \setminus \{0\}\}$. So $\Gamma \vdash_{\mathbf{L}} \varphi$ implies $\Gamma \models_{\mathbf{2} \times \mathbf{S}} \varphi$.

(\Leftarrow) Suppose that $\Gamma \models_{\mathbf{2} \times \mathbf{S}} \varphi$, where Γ contains the variables p_1, \dots, p_n . If $\Gamma \models_{\mathbf{S}} \varphi$, then $\Gamma \vdash_{\mathbf{L}} \varphi$ since $\mathbb{Q}(\mathbf{S})$ is an equivalent quasivariety for $\text{RM}_m + (A)$ (Lemma 5.4). Suppose then that $\Gamma \not\models_{\mathbf{S}} \varphi$. I.e., there is a homomorphism $e: \mathbf{Fm}_{\{\rightarrow, \cdot, \neg\}} \rightarrow \mathbf{S}$ such that $e(\psi) \geq 0$ for all $\psi \in \Gamma$ and $e(\varphi) = -1$. By assumption, $\Gamma \models_{\mathbf{2} \times \mathbf{S}} \varphi$. Hence Γ is not satisfiable in $\mathbf{2}$. Otherwise, for some $k: \mathbf{Fm}_{\{\rightarrow, \cdot, \neg\}} \rightarrow \mathbf{2}$, we have $k(\psi) = 1$ for all $\psi \in \Gamma$ and we could define $j: \mathbf{Fm}_{\{\rightarrow, \cdot, \neg\}} \rightarrow \mathbf{2} \times \mathbf{S}$ by $j(p) = (k(p), e(p))$ such that $j(\psi) \geq (1, 0)$ for all $\psi \in \Gamma$ and $j(\varphi) \not\geq (1, 0)$.

Now consider any homomorphism $h: \mathbf{Fm}_{\{\rightarrow, \cdot, \neg\}} \rightarrow \mathbf{S}$ satisfying Γ : i.e., such that $h(\psi) \geq 0$ for all $\psi \in \Gamma$. Since Γ is not satisfiable in the subalgebra $\mathbf{2}$ of \mathbf{S} , we must have $h(p_i) = 0$ for some $i \in \{1, \dots, n\}$. But then, making use of property (*) above:

$$\Gamma \models_{\mathbf{S}} \neg((p_1 \rightarrow p_1) \leftrightarrow \dots \leftrightarrow (p_n \rightarrow p_n)).$$

Hence $\Gamma \vdash_{\text{RM}_m + (A)} \neg((p_1 \rightarrow p_1) \leftrightarrow \dots \leftrightarrow (p_n \rightarrow p_n))$ by Lemma 5.4, and clearly $\Gamma \vdash_{\mathbf{L}} \neg((p_1 \rightarrow p_1) \leftrightarrow \dots \leftrightarrow (p_n \rightarrow p_n))$. Therefore, finally, using (R_n) , $\Gamma \vdash_{\mathbf{L}} \varphi$, as required. \square

Lemma 5.9. $\text{RM}_m + (A) + \{(R_n) \mid n \in \mathbb{N} \setminus \{0\}\}$ is structurally complete.

Proof. By Lemmas 5.8 and 2.3, it suffices to show that $h_{\mathbb{Q}(\mathbf{2} \times \mathbf{S})} \circ g$ embeds $\mathbf{2} \times \mathbf{S}$ into $\mathbf{F}_{\mathbb{Q}(\mathbf{2} \times \mathbf{S})}$ where $g: \{-1, 1\} \times \{-1, 0, 1\} \rightarrow \text{Fm}_{\{\rightarrow, \cdot, \neg\}}$ is defined by:¹

$$\begin{aligned} g((-1, -1)) &= \neg((p \rightarrow q) \rightarrow (p \rightarrow q)) \\ g((-1, 0)) &= \neg(p \rightarrow p) \\ g((-1, 1)) &= \neg((q \rightarrow q) \rightarrow (p \rightarrow p)) \\ g((1, -1)) &= (q \rightarrow q) \rightarrow (p \rightarrow p) \\ g((1, 0)) &= p \rightarrow p \\ g((1, 1)) &= (p \rightarrow q) \rightarrow (p \rightarrow q). \end{aligned}$$

Recall that $\mathbb{V}(\mathbf{2} \times \mathbf{S}) = \mathbb{V}(\mathbf{S})$, so in particular $\mathbf{F}_{\mathbb{Q}(\mathbf{2} \times \mathbf{S})} = \mathbf{F}_{\mathbf{S}A_m}$.

It is almost immediate to see that $h_{\mathbb{Q}(\mathbf{2} \times \mathbf{S})} \circ g$ respects the negation of $\mathbf{2} \times \mathbf{S}$. To show that $h_{\mathbb{Q}(\mathbf{2} \times \mathbf{S})} \circ g$ preserves \cdot , we can use the commutativity and idempotence of \cdot in Sugihara

¹This embedding was in fact discovered automatically using an algorithm described in [23].

algebras; hence, it suffices to check (using, e.g., Avron's sequent calculus [3]) that for all $x \in \{-1, 1\} \times \{-1, 0, 1\}$:

$$\begin{aligned} \models_{\mathbf{S}} g((-1, -1)) \cdot g(x) &\approx g((-1, -1)) \\ \models_{\mathbf{S}} g((1, 0)) \cdot g(x) &\approx g(x) \end{aligned}$$

and also

$$\begin{aligned} \models_{\mathbf{S}} g((-1, 0)) \cdot g((-1, 1)) &\approx g((-1, 1)) \\ \models_{\mathbf{S}} g((-1, 0)) \cdot g((1, -1)) &\approx g((-1, -1)) \\ \models_{\mathbf{S}} g((-1, 0)) \cdot g((1, 1)) &\approx g((-1, 1)) \\ \models_{\mathbf{S}} g((-1, 1)) \cdot g((1, -1)) &\approx g((-1, -1)) \\ \models_{\mathbf{S}} g((-1, 1)) \cdot g((1, 1)) &\approx g((-1, 1)) \\ \models_{\mathbf{S}} g((1, -1)) \cdot g((1, 1)) &\approx g((1, -1)). \end{aligned}$$

Since $x \rightarrow y \approx \neg(x \cdot \neg y)$ holds in all Sugihara algebras, $h_{\mathbb{Q}(\mathbf{2} \times \mathbf{S})} \circ g$ also preserves \rightarrow . Finally, $h_{\mathbb{Q}(\mathbf{2} \times \mathbf{S})} \circ g$ is 1-1: we can find homomorphisms $h: \mathbf{Fm}_{\{\rightarrow, \cdot, \neg\}} \rightarrow \mathbf{S}$ that distinguish the formulas in the image of g . \square

Since (A) and (R_n) ($n \in \mathbb{N} \setminus \{0\}$) are \mathbf{RM}_m -admissible (Lemmas 5.2 and 5.7), we obtain immediately from Lemma 2.1:

Theorem 5.10. *The rules (A) and $\{(R_n) \mid n \in \mathbb{N} \setminus \{0\}\}$ form a basis for the admissible rules of the multiplicative fragment of \mathbf{RM} .*

The natural question to ask at this point is whether a *finite* set of rules would suffice. This is answered negatively as follows:

Theorem 5.11. *The admissible rules of the multiplicative fragment of \mathbf{RM} have no finite basis.*

Proof. By compactness, any finite set of rules derivable in $\mathbf{RM}_m + (\text{A}) + \{(R_n) \mid n \in \mathbb{N} \setminus \{0\}\}$ is derivable in $\mathbf{RM}_m + (\text{A}) + \{(R_n) \mid 1 \leq n \leq k\}$ for some $k \in \mathbb{N} \setminus \{0\}$. Hence it suffices to show that for each $k \in \mathbb{N} \setminus \{0\}$, the rules (A) and $\{(R_n) \mid 1 \leq n < k\}$ do not form a basis for the admissible rules of \mathbf{RM}_m . Consider the following subalgebra of \mathbf{S}^k :

$$\mathbf{A} = \{(\beta_1, \dots, \beta_k) \in \{-1, 0, 1\}^k \mid \text{at most one of the } \beta_i \text{'s is } 0\}.$$

We will prove that:

$$(\star) \quad \{\neg((p_1 \rightarrow p_1) \leftrightarrow \dots \leftrightarrow (p_n \rightarrow p_n))\} \models_{\mathbf{A}} q \quad \text{iff} \quad n < k.$$

It follows from (\star) that \mathbf{A} belongs to the equivalent quasivariety for $\mathbf{RM}_m + (\text{A}) + \{(R_n) \mid 1 \leq n < k\}$, but not to $\mathbb{Q}(\mathbf{2} \times \mathbf{S})$, the equivalent quasivariety for $\mathbf{RM}_m + (\text{A}) + \{(R_n) \mid n \in \mathbb{N} \setminus \{0\}\}$. But then the rules (A) and $\{(R_n) \mid 1 \leq n < k\}$ cannot form a basis for the admissible rules of \mathbf{RM}_m .

To prove (\star) , note first that for each $n \in \mathbb{N} \setminus \{0\}$:

$$\{\neg((p_1 \rightarrow p_1) \leftrightarrow \dots \leftrightarrow (p_n \rightarrow p_n))\} \not\models_{\mathbf{A}} q$$

iff for some homomorphism $h: \mathbf{Fm}_{\{\rightarrow, \cdot, \neg\}} \rightarrow \mathbf{A}$,

$$h(\neg((p_1 \rightarrow p_1) \leftrightarrow \dots \leftrightarrow (p_n \rightarrow p_n))) \geq (0, \dots, 0),$$

i.e., for some $a_1, \dots, a_n \in A$ and $(\beta_1, \dots, \beta_k) = \neg((a_1 \rightarrow a_1) \leftrightarrow \dots \leftrightarrow (a_n \rightarrow a_n))$, we have $\beta_i \geq 0$ for $i = 1 \dots k$. Equivalently, by property (\star) , there should exist $a_1, \dots, a_n \in A$ such that for each $i = 1 \dots k$, we have $a_j = (\alpha_1, \dots, \alpha_{i-1}, 0, \alpha_{i+1}, \dots, \alpha_k)$ for some $j \in \{1, \dots, n\}$. But A consists of all those members of S^k with at most one 0. So the latter holds iff $n \geq k$, and we obtain (\star) . \square

Moreover, since the admissible rules of $\mathbf{RM}_m + (\text{A})$ coincide with the admissible rules of \mathbf{RM}_m , we also obtain:

Corollary 5.12. *The admissible rules of the finitely axiomatizable three-valued logic $\mathbf{RM}_m + (\text{A})$ have no finite basis.*

Table 1: Admissibility in proper fragments of RM.

Fragment	Admissible Rules	Reference
$\{\rightarrow, \wedge\}$	Hereditarily structurally complete	Theorem 4.1
$\{\rightarrow, \cdot, \vee\}$	Hereditarily structurally complete	Theorem 4.1
$\{\rightarrow, \vee\}$	Hereditarily structurally complete	Theorem 4.1
$\{\rightarrow, \cdot\}$	Basis: $\{(A)\}$	Theorem 5.6
$\{\rightarrow\}$	Basis: $\{(A)\}$	Theorem 5.6
$\{\rightarrow, \neg\}$	Basis: $\{(A)\} \cup \{(R_n) \mid n \in \mathbb{N} \setminus \{0\}\}$	Theorem 5.10

This contrasts with the fact, established by Rautenberg in [28], that every two-valued logic is both finitely axiomatizable and hereditarily structurally complete. Examples of tabular intermediate and modal logics with more than three values whose admissible rules are not finitely axiomatizable are given by Rybakov in [31], but to the best of our knowledge, $\text{RM}_m + (A)$ is the first example of a three-valued logic having this property.

Alternative bases for the admissible rules of the multiplicative and implication-fusion fragments are provided by swapping (A) for $\{p \cdot q\} / q$, since as shown by Avron in [4], $\mathbb{Q}(\mathbf{S})$ is an equivalent quasivariety for RM_m extended with this rule. Finally, the situation for proper fragments of R-mingle is summarized in Table 1, recalling that the full logic is not structurally complete but that no basis for its admissible rules has yet been found.

References

- [1] A. R. Anderson and N. D. Belnap, *Entailment*, Vol. 1, Princeton Univ. Press, 1975.
- [2] A. Avron, *On an implication connective of RM*, Notre Dame J. Form. Log. **27** (1986), 201–209.
- [3] ———, *A constructive analysis of RM*, J. Symbolic Logic **52** (1987), no. 4, 939–951.
- [4] ———, *Multiplicative conjunction as an extensional conjunction*, Log. J. IGPL **5** (1997), 181–208.
- [5] ———, *Implicational f-structures and implicational relevance logics*, J. Symbolic Logic **65** (2000), no. 2, 788–802.
- [6] S. Babenyshev and V. Rybakov, *Linear temporal logic LTL: Basis for admissible rules*, J. Log. Comput. **21** (2011), no. 2, 157–177.
- [7] ———, *Unification in linear temporal logic LTL*, Ann. Pure Appl. Logic **162** (2011), no. 12, 991–1000.
- [8] W. J. Blok and D. Pigozzi, *Algebraizable logics*, Mem. Amer. Math. Soc., vol. 77, Amer. Math. Soc., 1989.
- [9] W. J. Blok and J. G. Raftery, *Fragments of R-mingle*, Studia Logica **78** (2004), no. 1-2, 59–106.
- [10] L. M. Cabrer and G. Metcalfe, *Admissibility via natural dualities*. Submitted.
- [11] P. Cintula and G. Metcalfe, *Structural completeness in fuzzy logics*, Notre Dame J. Form. Log. **50** (2009), no. 2, 153–183.
- [12] ———, *Admissible rules in the implication-negation fragment of intuitionistic logic*, Ann. Pure Appl. Logic **162** (2010), no. 10, 162–171.
- [13] J. M. Dunn, *Algebraic completeness for R-mingle and its extensions*, J. Symbolic Logic **35** (1970), 1–13.
- [14] S. Ghilardi, *Unification in intuitionistic logic*, J. Symbolic Logic **64** (1999), no. 2, 859–880.
- [15] ———, *Best solving modal equations*, Ann. Pure Appl. Logic **102** (2000), no. 3, 184–198.
- [16] R. Iemhoff, *On the admissible rules of intuitionistic propositional logic*, J. Symbolic Logic **66** (2001), no. 1, 281–294.
- [17] ———, *Intermediate logics and Visser’s rules*, Notre Dame J. Form. Log. **46** (2005), no. 1, 65–81.
- [18] R. Iemhoff and G. Metcalfe, *Proof theory for admissible rules*, Ann. Pure Appl. Logic **159** (2009), no. 1–2, 171–186.
- [19] E. Jeřábek, *Admissible rules of modal logics*, J. Logic Comput. **15** (2005), 411–431.
- [20] ———, *Admissible rules of Łukasiewicz logic*, J. Logic Comput. **20** (2010), no. 2, 425–447.
- [21] ———, *Bases of admissible rules of Łukasiewicz logic*, J. Logic Comput. **20** (2010), no. 6, 1149–1163.

- [22] G. Metcalfe and C. Röthlisberger, *Admissibility in De Morgan algebras*, *Soft Comput.* **16** (2012), no. 11, 1875–1882.
- [23] ———, *Admissibility in finitely generated quasivarieties*, *Log. Methods Comput. Sci.* **9** (2013), no. 2, 1–19.
- [24] J. S. Olson and J. G. Raftery, *Positive Sugihara monoids*, *Algebra Universalis* **57** (2007), 75–99.
- [25] J. S. Olson, J. G. Raftery, and C. J. Van Alten, *Structural completeness in substructural logics*, *Log. J. IGPL* **16** (2008), no. 5, 453–495.
- [26] T. Prucnal, *On the structural completeness of some pure implicational propositional calculi*, *Studia Logica* **32** (1973), no. 1, 45–50.
- [27] J. G. Raftery, *Representable idempotent commutative residuated lattices*, *Trans. Amer. Math. Soc.* **359** (2007), 4405–4427.
- [28] W. Rautenberg, *2-element matrices*, *Studia Logica* **40** (1981), no. 4, 315–353 (1982).
- [29] P. Rozière, *Regles admissibles en calcul propositionnel intuitionniste*, Ph.D. Thesis, 1992.
- [30] V. Rybakov, *A criterion for admissibility of rules in the modal system S_4 and the intuitionistic logic*, *Algebra Logic* **23** (1984), 369–384.
- [31] ———, *Admissibility of logical inference rules*, *Studies in Logic and the Foundations of Mathematics*, vol. 136, Elsevier, Amsterdam, 1997.
- [32] J. K. Slaney and R. K. Meyer, *A structurally complete fragment of relevant logic*, *Notre Dame J. Form. Log.* **33** (1992), 561–566.
- [33] B. Sobociński, *Axiomatization of a partial system of three-valued calculus of propositions*, *The Journal of Computing Systems* **1** (1952), 23–55.