Sharp comparison and maximum principles via horizontal normal mapping in the Heisenberg group

Zoltán M. Balogh[1], Andrea Calogero, and Alexandru Kristály [2]

Abstract

In this paper we solve a problem raised by Gutiérrez and Montanari about comparison principles for $H-$convex functions on subdomains of Heisenberg groups. Our approach is based on the notion of the sub-Riemannian horizontal normal mapping and uses degree theory for set-valued maps. The statement of the comparison principle combined with a Harnack inequality is applied to prove the Aleksandrov-type maximum principle, describing the correct boundary behavior of continuous $H-$convex functions vanishing at the boundary of horizontally bounded subdomains of Heisenberg groups. This result answers a question by Garofalo and Tournier. The sharpness of our results are illustrated by examples.

Keywords: Heisenberg group; $H-$convex functions; comparison principle; Aleksandrov-type maximum principle.

MSC: 35R03, 26B25.

Contents

[1]Z. M. Balogh was supported by the Swiss National Science Foundation, and the FP7 EU Commission Project CG-DICE.

[2]A. Kristály was supported by a CNCS-UEFISCDI grant no. PN-II-ID-PCE-2011-3-0241, and János Bolyai Research Scholarship.

1 Introduction

1.1 Motivation

It is well known that convex functions defined on subdomains of $\mathbb{R}^n$ are locally Lipschitz continuous and almost everywhere twice differentiable. Moreover, the celebrated maximum principle due to Aleksandrov provides a global regularity result for convex functions that are continuous on the closure and are vanishing on the boundary of the domain. More precisely, if $\Omega \subset \mathbb{R}^n$ is a bounded open and convex domain, and $u \in C(\overline{\Omega})$ is convex with $u = 0$ on $\partial\Omega$, then

$$|u(\xi_0)|^n \leq C_n \text{dist}(\xi_0, \partial\Omega) \text{diam}(\Omega)^{n-1} \mathcal{L}^n(\partial u(\Omega)), \ \forall \xi_0 \in \Omega, \tag{1.1}$$

where $C_n > 0$ is a constant depending only on the dimension n. In the above expression the notation $\mathcal{L}^n(\partial u(\Omega))$ stands for the measure of the range of the so-called normal mapping of u. To define this concept we need first the subdifferential $\partial u(\xi_0)$ of u at ξ_0, given by

$$\partial u(\xi_0) = \{p \in \mathbb{R}^n : u(\xi) \geq u(\xi_0) + p \cdot (\xi - \xi_0), \ \forall \xi \in \Omega\},$$

where $'\cdot'$ is the usual inner product in $\mathbb{R}^n$. The range of the normal mapping of u is defined by

$$\partial u(\Omega) = \bigcup_{\xi \in \Omega} \partial u(\xi).$$

A convenient way to deduce the Aleksandrov estimate (1.1) is to compare the ranges of normal mappings of the convex function u and the cone function $v : \overline{\Omega} \to \mathbb{R}$ with base on $\partial\Omega$ and vertex $(\xi_0, u(\xi_0))$ (see e.g. Gutiérrez [17, Theorem 1.4.2]).

It is well-known, that for any convex function $u \in C^2(\Omega)$,

$$\mathcal{L}^n(\partial u(\Omega)) = \int_\Omega \det[\text{Hess}(u)(x)] dx, \tag{1.2}$$

which implies by (1.1) the estimate:

$$|u(\xi_0)|^n \leq C_n \text{dist}(\xi_0, \partial\Omega) \text{diam}(\Omega)^{n-1} \int_\Omega \det[\text{Hess}(u)(x)] dx, \ \forall \xi_0 \in \Omega. \tag{1.3}$$

In recent years, the notion of convexity has been considered in the setting of Heisenberg groups by Lu, Manfredi and Stroffolini [23], and in more general Carnot groups by Danielli, Garofalo and Nhieu [14] and also Juutinen, Lu, Manfredi and Stroffolini [21]. The main idea behind this approach is to develop a concept of convexity that is adapted to the sub-Riemannian, or Carnot-Carathéodory geometry of the Carnot groups. In this way convexity is assumed only along *trajectories of left-invariant horizontal vector-fields* which are in the first layer of the Lie algebra of the group and generate the sub-Riemannian metric. This notion is called by many authors as *$H-$convexity*. This approach makes sense also in case of more general Carnot-Carathéodory spaces even in the absence of a groups structure, see Bardi and Dragoni [5].

Various results on local regularity properties such as local Lipschitz continuity or second differentiability a.e. in terms of the horizontal vector-fields have been already proven in this context. We refer to the paper of Balogh and Rickly [4] for the proof of the local Lipschitz continuity of $H-$convex functions on the Heisenberg group and Rickly [26] for Carnot groups.

It was pointed out to us by one of the referees, that the generalization of Aleksandrov's second order differentiability theorem of $H-$convex functions to the case of Carnot groups is a rather delicate issue. Magnani [24] proved second horizontal differentiability a.e. in the general Carnot setting of a $H-$convex function u, but only under the assumption that all entries of the symmetrized horizontal Hessian $u_{i,j}$ as well as the horizontal commutators $[X_i, X_j]u$ are Radon measures. The first condition was proved by Danielli, Garofalo and Nhieu in [14]. The second condition is more difficult, it was proven by Danielli, Garofalo, Nhieu and Tournier in [15] for the case of Carnot groups of step 2. The property that $[X_i, X_j]u$ are Radon measures is still open for general Carnot groups.

In this paper we will be concerned with first order regularity properties of $H-$convex functions on the Heisenberg group. We note first, that the behavior of $H-$convex functions in non-horizontal directions can still be pretty wild. Indeed, examples of $H-$convex functions are constructed by Balogh and Rickly in [4] which coincide with the Weierstrass function on a thick Cantor set of vertical lines. This fact indicates the intricate nature of $H-$convex functions as well as possible differences with respect to their Euclidean counterpart. In particular, the validity of an Aleksandrov-type estimate, similar to (1.1) becomes questionable.

The main goal of this paper is to prove *global regularity results* akin to (1.1) in the setting of general Heisenberg groups $\mathbb{H}^n$. This problem has been first considered by Gutiérrez and Montanari [18] in the setting of the first Heisenberg group $\mathbb{H}^1$ and by Garofalo and Tournier [16] for the second Heisenberg group $\mathbb{H}^2$ and the Engel group. In these papers, the methods of Trudinger and Wang [28, 29, 30] have been applied to obtain comparison estimates for integrals involving Hessians and related expressions in second order derivatives. Trudinger and Zhang [31] obtained recently a generalization of these results for integrals of k-th order Hessian measures of k-convex functions defined on $\mathbb{H}^n$. Such comparison estimates can be used to deduce weaker versions of Aleksandrov-type maximum principle (1.3). For instance, in [18] it is shown that if $u : B_H \to \mathbb{R}$ is a C^2-smooth, $H-$convex function defined on the unit Korányi-Cygan ball in the first Heisenberg group $\mathbb{H}^1$ which vanishes on the boundary, then

$$|u(\xi_0)|^2 \leq c_1(\xi_0) \int_{B_H} \left(\det[\mathrm{Hess}_H(u)(\xi)]^* + 12(Tu(\xi))^2\right) d\xi, \ \forall \xi_0 \in B_H, \tag{1.4}$$

where $[\mathrm{Hess}_H(u)(\xi)]^*$ denotes the symmetrized horizontal Hessian and Tu is the vertical derivative of u.

The main drawback of the estimate (1.4) is that the expression $c_1(\xi_0) > 0$ in front of the integral behaves like $\mathrm{dist}_H(\xi_0, \partial B_H)^{-\alpha}$ for some $\alpha > 0$, which is far to be optimal taking into account that $u = 0$ on ∂B_H. A similar result was obtained also in [16], where Garofalo and Tournier [16, p. 2013] formulated the question about existence of a suitable pointwise estimate that behaves like a positive power of the distance to the boundary.

1.2 Statements of main results

The primary goal of our paper is to provide a positive answer to the above question by proving an Aleksandrov-type estimate in the spirit of (1.1). More precisely, we shall prove the estimate

$$|u(\xi_0)|^{2n} \leq C_n \mathrm{dist}_H(\xi_0, \partial\Omega)\ \mathrm{diam}_{HS}(\Omega)^{2n-1} \mathcal{L}^{2n}_{HS}(\partial_H u(\Omega)), \ \forall \xi_0 \in \Omega, \tag{1.5}$$

where $\Omega \subset \mathbb{H}^n$ is any open horizontally bounded and convex domain, $u : \overline{\Omega} \to \mathbb{R}$ is a continuous $H-$convex function which vanishes at the boundary $\partial\Omega$, and $C_n > 0$ depends only on n. (The concept of horizontal boundedness will be introduced in the sequel.)

In the above estimate dist_H stands for the sub-Riemannian distance of the Heisenberg group. The quantities $\mathrm{diam}_{HS}(\Omega)$ and $\mathcal{L}^{2n}_{HS}(\partial_H u(\Omega))$ denote the *horizontal slicing diameter* of the horizontally bounded set Ω, resp. the *horizontal slicing measure* of the set $\partial_H u(\Omega)$. These notions are introduced in Definition 2.1 as the appropriate substitutes for their Euclidean counterparts $\mathrm{diam}(\Omega)$ and $\mathcal{L}^n(\partial u(\Omega))$, respectively.

We recall that $\partial_H u$ is the *horizontal normal mapping* of u introduced by Danielli, Garofalo and Nhieu [14] and studied by Calogero and Pini [9]. The concept of horizontal normal mapping turns out to be the right analogue to the normal mapping in the Euclidean space which made the estimate (1.5) possible. Roughly speaking, the horizontal normal mapping $\partial_H u$ includes all subdifferentials of u taken in the directions of the left-invariant horizontal directions on the Heisenberg group.

Until now, there was a major obstacle in applying the method of normal mapping due to the lack of good comparison principles for $H-$convex functions. Our first result overcomes this obstacle, and at the same time answers a question of Calogero and Pini [9] and Gutiérrez and Montanari [18]:

Theorem 1.1 (Comparison principle for the horizontal normal mapping) *Let $\Omega \subset \mathbb{H}^n$ be an open, horizontally bounded and $H-$convex set, and $u, v : \Omega \to \mathbb{R}$ be $H-$convex functions. Let $\Omega_0 \subset \mathbb{H}^n$ be open such that $\overline{\Omega_0} \subset \Omega$ and assume that $u < v$ in Ω_0 and $u = v$ on $\partial\Omega_0$. Then*

$$\partial_H v(\Omega_0) \subset \partial_H u(\Omega_0).$$

In fact, Theorem 1.1 is a consequence of a more general comparison result, see Theorem 3.1, where the novelty of our approach is shown by the application of a degree theoretical argument for upper semicontinuous set-valued maps, developed by Hu and Papageorgiou [20]. Due to the $H-$convexity of the functions u and v, the upper semicontinuous set-valued maps $\partial_H u$ and $\partial_H v$ show certain monotonicity properties, allowing to relate the set-valued degree of these maps via a suitable homotopy flow. A similar comparison principle to the previous one can be stated by requiring $u \leq v$ in Ω_0 but adding the strict $H-$ convexity of v, see Theorem 3.2.

We emphasize that the $H-$convexity of the functions u and v is indispensable in order to obtain comparison principles. Indeed, in the absence of convexity we construct an example for which the comparison principle fails on the first Heisenberg group $\mathbb{H}^1$, see Section 5.

Using Theorem 1.1 we can prove the following:

Theorem 1.2 (Horizontal comparison principle) *Let $\Omega \subset \mathbb{H}^n$ be an open, bounded and $H-$convex set, and $u, v : \overline{\Omega} \to \mathbb{R}$ be continuous $H-$convex functions. If for every Borel set $E \subset \Omega$ we have*

$$\mathcal{L}^{2n}(\partial_H v(E)) \leq \mathcal{L}^{2n}(\partial_H u(E)),$$

then

$$\min_{\xi \in \overline{\Omega}}(v(\xi) - u(\xi)) = \min_{\xi \in \partial\Omega}(v(\xi) - u(\xi)).$$

A consequence of Theorem 1.2 is the fact that the horizontal normal mapping characterizes uniquely the $H-$convex functions with prescribed boundary values.

Corollary 1.1 *Let $\Omega \subset \mathbb{H}^n$ be an open, bounded and $H-$convex set, and let $u, v : \overline{\Omega} \to \mathbb{R}$ be continuous $H-$convex functions. If for every Borel set $E \subset \Omega$ we have*

$$\mathcal{L}^{2n}(\partial_H u(E)) = \mathcal{L}^{2n}(\partial_H v(E))$$

and $u = v$ in $\partial\Omega$, then $u = v$ in Ω.

The main result of the paper is the following maximum principle.

Theorem 1.3 (Aleksandrov-type maximum principle) *Let $\Omega \subset \mathbb{H}^n$ be an open, horizontally bounded and convex set. If $u : \overline{\Omega} \to \mathbb{R}$ is a continuous $H-$convex function which verifies $u = 0$ on $\partial\Omega$, then*

$$|u(\xi_0)|^{2n} \leq C_n \mathrm{dist}_H(\xi_0, \partial\Omega)\ \mathrm{diam}_{HS}(\Omega)^{2n-1} \mathcal{L}^{2n}_{HS}(\partial_H u(\Omega)), \qquad \forall \xi_0 \in \Omega, \tag{1.6}$$

where $C_n > 0$ depends only on n.

The proof of Theorem 1.3 is a puzzle which is assembled by several pieces: basic comparison principle, maximum principle on horizontal planes, horizontal normal mapping of cone functions, Harnack-type inequality, and quantitative description of the twirling effect of horizontal planes. Some of the pieces in this puzzle are readily available in the current literature: in particular the Harnack-type inequality for $H-$convex functions has been proven by Gutierrez and Montanari in [18], in the same paper the authors apply this result to obtain estimates on the boundary behavior of $H-$convex functions.

Theorem 1.3 is *sharp* which is shown as follows: for a given $\varepsilon \in (0, 1)$ we construct an open, bounded and convex set $\Omega \subset \mathbb{H}^1$ and a continuous $H-$convex function $u : \overline{\Omega} \to \mathbb{R}$ which verifies $u = 0$ on $\partial\Omega$ and $u < 0$ in Ω such that $\mathcal{L}^2_{HS}(\partial_H u(\Omega)) < \infty$, and

$$\sup_{\xi \in \Omega} \frac{|u(\xi)|^2}{\mathrm{dist}_H(\xi, \partial\Omega)^{1+\varepsilon}} = +\infty. \tag{1.7}$$

Some comments concerning further perspectives are in order. Since the arguments in the proof of the comparison principles (see Theorems 1.1 and 3.2) are topological, it is clear that such results can be also extended to general Carnot groups. However, in this general setting certain technical difficulties will arise in the proof of the Aleksandrov-type maximum principle, e.g. the construction of specific cone functions; these issues will be considered in the forthcoming paper [3]. Furthermore, we expect that the approach presented in this paper can be successfully applied to establish interior $\Gamma^{1+\alpha}-$, or $W^{2,p}-$regularity of $H-$convex functions in the spirit of Caffarelli [7, 8] and Gutiérrez [17]. In the setting of Carnot groups a first step in this direction has been done by Capogna and Maldonado [11].

The paper is organized as follows. In Section 2 we fix notations and recall preliminary results on $H-$convex functions in the Heisenberg group. Section 3 is devoted to comparison principles; in particular we prove Theorems 1.1 and 1.2. In Section 4 we give the proof of our main result Theorem 1.3. Section 5 is devoted to the discussions related to sharpness of our results. First, we provide an example showing that comparison principles do not hold in the absence of the convexity assumption, see §5.1. Then, the above example (see (1.7)) is presented in detail, showing the sharpness of the Aleksandrov-type estimate, see §5.2. We also discuss the relationship between the horizontal Monge-Ampère operator and the horizontal normal mapping, see §5.3. To make the paper self-contained we add an Appendix containing two parts. In the first part we recall those results of Hu and Papageorgiu [20] on the degree theory for set-valued maps from which we need in our proof in Section 3. In the second part of the Appendix we give a detailed proof of the quantitative Harnack inequality following Gutierrez and Montanari [18] that we use in Section 4.

Acknowledgment. A. Calogero and A. Kristály are grateful to the Mathematisches Institute, Universität Bern for the warm hospitality where this work has been initiated. We thank the referees for the careful reading of the manuscript and for their comments.

2 Preliminaries

The Heisenberg group $\mathbb{H}^n$ is the simplest Carnot group of step 2 which serves as prototype of Carnot groups. For a comprehensive introduction to analysis on Carnot groups we refer to [6]. Here we recall just the necessary notation and background results used in the sequel. The Lie algebra $\mathfrak{h}$ of $\mathbb{H}^n$ admits a stratification $\mathfrak{h} = V_1 \oplus V_2$ with $V_1 = \text{span}\{X_i,\, Y_i;\ 1 \le i \le n\}$ being the first layer, and $V_2 = \text{span}\{T\}$ being the second layer which is one-dimensional. We assume $[X_i, Y_i] = -4T$ and the rest of commutators of basis vectors all vanish. The exponential map $\exp : \mathfrak{h} \to \mathbb{H}^n$ is defined in the usual way. By these commutator rules we obtain, using the Baker-Campbell-Hausdorff formula, that $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is endowed with the non-commutative group law given by

$$(z,t) \circ (z',t') = (z + z', t + t' + 2\text{Im}\langle z, z'\rangle), \tag{2.1}$$

where $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $t \in \mathbb{R}$, and $\langle z, z'\rangle = \sum_{j=1}^n z_j \overline{z'_j}$ is the Hermitian inner product. Denoting by $z_j = x_j + iy_j$, then $(x_1, ..., x_n, y_1, ..., y_n, t)$ form a real coordinate system for $\mathbb{H}^n$. Transporting the basis vectors of V_1 from the origin to an arbitrary point of the group by left-translations, we obtain a system of left-invariant vector fields written as first order differential operators as follows

$$\begin{aligned} X_j &= \partial_{x_j} + 2y_j \partial_t, \qquad j = 1, ..., n; \\ Y_j &= \partial_{y_j} - 2x_j \partial_t, \qquad j = 1, ..., n. \end{aligned} \tag{2.2}$$

These vector fields are called by an abuse of language *horizontal*. The *horizontal plane* in $\xi_0 \in \mathbb{H}^n$ is given by $H_{\xi_0} = \xi_0 \circ \exp(V_1 \times \{0\})$. It is easy to check that for $\xi_0 = (z_0, t_0) = (x_0, y_0, t_0) \in \mathbb{H}^n$ the equation of the horizontal plane is given by

$$H_{\xi_0} = \{(z,t) \in \mathbb{H}^n : t = t_0 + 2\text{Im}\langle z_0, z\rangle\} = \{(x,y,t) \in \mathbb{H}^n : t = t_0 + 2(x \cdot y_0 - x_0 \cdot y)\}.$$

The sub-Riemannian, or Carnot-Carathéodory metric on $\mathbb{H}^n$ is defined in terms of the above vector fields. Instead of the Carnot-Carathéodory metric, in this paper we shall work with the bi-Lipschitz equivalent *Korányi-Cygan metric* that is more suitable for concrete calculations and is defined explicitly as follows.

Let $N(z,t) = (|z|^4 + t^2)^{\frac{1}{4}}$ be the gauge norm on $\mathbb{H}^n$. It is an interesting exercise to check that the expression

$$d_H((z,t),(z',t')) = N((z',t')^{-1} \circ (z,t)),$$

satisfies the triangle inequality defining a metric on $\mathbb{H}^n$ (see [13]). This metric is the so-called Korányi-Cygan metric which is by left- translation and dilation invariance bi-Lipschitz equivalent to the Carnot-Carathéodory metric. Here, the non-isotropic Heisenberg dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ for $\lambda > 0$ are defined by $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$. If $A \subset \mathbb{H}^n$ and $\xi \in \mathbb{H}^n$, then $\text{dist}_H(\xi, A) = \inf_{\zeta\in A} d_H(\xi,\zeta)$. The Korányi-Cygan ball of center $(z_0,t_0) \in \mathbb{H}^n$ and radius $r > 0$ is given by $B_H((z_0,t_0),r) = \{(z,t) \in \mathbb{H}^n : d_H((z,t),(z_0,t_0)) < r\}$.

Let $\Omega \subset \mathbb{H}^n$ be an open set. The main idea of the analysis on the Heisenberg group is that general regularity properties of functions defined on the Heisenberg group should be expressed only in terms of horizontal vector fields (2.2). In particular, the appropriate gradient notion for a function is the so-called *horizontal gradient*, which is defined as the $2n$–vector $\nabla_H u(\xi) = (X_1u(\xi), ..., X_nu(\xi), Y_1u(\xi), ..., Y_nu(\xi))$ for a function $u \in \Gamma^1(\Omega)$. Here, the class $\Gamma^k(\Omega)$ is the Folland–Stein space of functions having continuous derivatives up to order k with respect to the vector fields X_i and Y_i, $i \in \{1, ..., n\}$. For general non-smooth functions $u : \Omega \to \mathbb{R}$ one defines the *horizontal subdifferential* $\partial_H u(\xi_0)$ of u at $\xi_0 \in \Omega$ given by

$$\partial_H u(\xi_0) = \left\{p \in \mathbb{R}^{2n} : u(\xi) \geq u(\xi_0) + p\cdot(\text{Pr}_1(\xi) - \text{Pr}_1(\xi_0)),\ \forall \xi \in \Omega \cap H_{\xi_0}\right\},$$

where $\text{Pr}_1 : \mathbb{H}^n \to \mathbb{R}^{2n}$ is the projection defined by $\text{Pr}_1(\xi) = \text{Pr}_1(x,y,t) = (x,y)$. (The same notation '$\cdot$' will be used for the inner products in $\mathbb{R}^n$ and $\mathbb{R}^{2n}$.) It is easy to see that if $u \in \Gamma^1(\Omega)$ and $\partial_H u(\xi) \neq \emptyset$, then $\partial_H u(\xi) = \{\nabla_H u(\xi)\}$.

The range of the *horizontal normal mapping* of the function u is defined by

$$\partial_H u(\Omega) = \bigcup_{\xi\in\Omega} \partial_H u(\xi).$$

A function $u : \Omega \to \mathbb{R}$ is called *$H-$subdifferentiable on* Ω if $\partial_H u(\xi) \neq \emptyset$ for every $\xi \in \Omega$. Let $\mathcal{S}_H(\Omega)$ be the set of all $H-$subdifferentiable functions on Ω, and $\mathcal{S}^0_H(\Omega)$ be set of all continuous $H-$subdifferentiable functions on Ω.

The main objects of study in this paper are $H-$ convex functions. There are several equivalent ways to define the concept of $H-$ convexity. The most intuitive property is to require the convexity of the restriction of the function on the trajectories of left invariant vector fields spanned by (2.2). Another definition using the group operation is as follows. A set $\tilde{\Omega} \subset \mathbb{H}^n$ is called *$H-$convex* if for every $\xi_1, \xi_2 \in \tilde{\Omega}$ with $\xi_1 \in H_{\xi_2}$ and $\lambda \in [0,1]$, we have $\xi_1 \circ \delta_\lambda(\xi_1^{-1} \circ \xi_2) \in \tilde{\Omega}$. It is clear that if $\tilde{\Omega}$ is convex (i.e. it is convex in $\mathbb{R}^{2n+1}-$sense), then it is also $H-$convex. If $\tilde{\Omega}$ is $H-$convex, a function $u : \tilde{\Omega} \to \mathbb{R}$ is called *$H-$convex* if for every $\xi_1, \xi_2 \in \tilde{\Omega}$ with $\xi_1 \in H_{\xi_2}$ and $\lambda \in [0,1]$, we have

$$u(\xi_1 \circ \delta_\lambda(\xi_1^{-1} \circ \xi_2)) \leq (1-\lambda)u(\xi_1) + \lambda u(\xi_2). \tag{2.3}$$

If the strict inequality holds in (2.3) for every $\xi_1 \neq \xi_2$, $\xi_1 \in H_{\xi_2}$ then u is called *strictly $H-$convex*. We denote by $\mathcal{C}_H(\tilde{\Omega})$ the set of all $H-$convex functions on $\tilde{\Omega}$.

We will now present some basic properties of $H-$convex functions which will be used through the paper. First, for various equivalent characterizations of $H-$convex functions and their regularity properties we refer to [4, 9, 14, 10] which can be summarized as follows:

Theorem 2.1 *Let $\Omega \subset \mathbb{H}^n$ be an open set. If $u : \Omega \to \mathbb{R}$ is a function, then $\partial_H u(\xi)$ is a convex and compact set of $\mathbb{R}^{2n}$ for every $\xi \in \Omega$. If Ω is $H-$convex, then $\mathcal{S}_H(\Omega) = \mathcal{S}^0_H(\Omega) = \mathcal{C}_H(\Omega)$.*

Now, we are dealing with the regularity of the set-valued map $\xi \mapsto \partial_H u(\xi)$. Let us recall that if X and Y are metric spaces, a set-valued map $F : X \to 2^Y \setminus \{\emptyset\}$ with compact values is upper semicontinuous at $x \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x' \in B_X(x,\delta)$ one has $F(x') \subset B_Y(F(x),\varepsilon)$. F is upper semicontinuous on $Z \subset X$ if it is upper semicontinuous at every point $x \in Z$. Here, $B_X(x,\delta)$ and $B_Y(y,\delta)$ denote the balls of radii δ and center x and y, respectively, in X and Y.

Proposition 2.1 *Let $\Omega \subset \mathbb{H}^n$ be an open set. If $u \in \mathcal{S}_H^0(\Omega)$ then $\partial_H u : \Omega \to 2^{\mathbb{R}^{2n}}$ is upper semicontinuous on Ω. Moreover, for every compact set $K \subset \Omega$, the set $\partial_H u(K)$ is compact.*

Proof. Let $\xi_0 \in \Omega$ be fixed and assume that $\partial_H u$ is not upper semicontinuous at ξ_0. On account of the upper semicontinuity and Theorem 2.1 this implies the existence of a sequence $\{\xi_k\} \subset \Omega$ such that $\xi_k \to \xi_0$ and $p_k \in \partial_H u(\xi_k)$ with $p_k \to p_0$ and $p_0 \notin \partial_H u(\xi_0)$. Note that $p_k \in \partial_H u(\xi_k)$ is equivalent to

$$u(\zeta) - u(\xi_k) \geq p_k \cdot (\mathrm{Pr}_1(\zeta) - \mathrm{Pr}_1(\xi_k)), \qquad \forall \zeta \in \Omega \cap H_{\xi_k}.$$

Let $\zeta \in \Omega \cap H_{\xi_0}$ be a given point and take a sequence $\zeta_k \in \Omega \cap H_{\xi_k}$ with $\zeta_k \to \zeta$. Then

$$u(\zeta_k) - u(\xi_k) \geq p_k \cdot (\mathrm{Pr}_1(\zeta_k) - \mathrm{Pr}_1(\xi_k)).$$

Since u is continuous, taking the limit in the above inequality, we have

$$u(\zeta) - u(\xi_0) \geq p_0 \cdot (\mathrm{Pr}_1(\zeta) - \mathrm{Pr}_1(\xi_0)).$$

Since $\zeta \in \Omega \cap H_{\xi_0}$ was arbitrary we obtain that $p_0 \in \partial_H u(\xi_0)$, a contradiction. The second statement follows (see [1, Proposition 1.1.3]) from the upper semicontinuity of the map $\partial_H u$. □

In the statement of our main result Theorem 1.3 the notions of horizontal slicing diameter $\mathrm{diam}_{HS}(\Omega)$ and horizontal slicing measure have been used. Roughly speaking, $\mathrm{diam}_{HS}(\Omega)$ stands for the supremum of diameters of horizontal slices of Ω and $\mathcal{L}^{2n}_{HS}(\partial_H u(\Omega))$ is the supremum of measures for the ranges of horizontal slices under the normal map. The precise definition is as follows:

Definition 2.1 *An open set $\Omega \subset \mathbb{H}^n$ is called* horizontally bounded *if*

$$\mathrm{diam}_{HS}(\Omega) = \sup\{\mathrm{diam}_H(\Omega \cap H_\xi)) : \xi \in \Omega\} < +\infty. \tag{2.4}$$

The quantity $\mathrm{diam}_{HS}(\Omega)$ *is called the* horizontal slicing diameter *of Ω. For a function $u : \Omega \to \mathbb{R}$ we define the horizontal slicing measure by*

$$\mathcal{L}^{2n}_{HS}(\partial_H u(\Omega)) = \sup_{\xi \in \Omega} \mathcal{L}^{2n}(\partial_H u(\Omega \cap H_\xi)).$$

It is clear that the quantity $\mathrm{diam}_{HS}(\Omega)$ is smaller than the Heisenberg diameter of Ω and that $\mathcal{L}^{2n}_{HS}(\partial_H u(\Omega)) \leq \mathcal{L}^{2n}(\partial_H u(\Omega))$. Theorem 1.3 implies therefore the weaker estimate

$$|u(\xi_0)|^{2n} \leq C_n \mathrm{dist}_H(\xi_0, \partial\Omega)\ \mathrm{diam}_H(\Omega)^{2n-1} \mathcal{L}^{2n}(\partial_H u(\Omega)), \qquad \forall \xi_0 \in \Omega. \tag{2.5}$$

Notice also that $\mathrm{diam}_{HS}(\Omega)$ could be finite for certain unbounded domains $\Omega \subset \mathbb{H}^n$, e.g., a cylinder around the vertical axis. Moreover, one can easily check that we have a natural scaling invariance property of Theorem 1.3 with respect to Heisenberg dilations δ_λ; see Remark 4.1.

We conclude this section by stating some properties of $H-$convex functions which are vanishing at the boundary.

Proposition 2.2 *Let $\Omega \subset \mathbb{H}^n$ be an open, horizontally bounded and $H-$convex set. If $u : \overline{\Omega} \to \mathbb{R}$ is an $H-$convex function which verifies $u = 0$ on $\partial\Omega$, then $u \leq 0$. Moreover, if Ω is (Euclidean) convex, either $u \equiv 0$ on $\overline{\Omega}$, or $u < 0$ in Ω.*

Proof. Let $\xi_0 \in \Omega$ be fixed. Let us consider arbitrarily a point $\xi \in \partial\Omega \cap H_{\xi_0}$. Since Ω is horizontally bounded and $H-$convex, there exists a unique point $\xi' \in (\partial\Omega \cap H_{\xi_0} \cap H_\xi) \setminus \{\xi\}$ such that $\xi_0 = \xi \circ \delta_\lambda(\xi^{-1} \circ \xi')$ for some $\lambda \in (0, 1)$. The $H-$convexity of $u : \overline{\Omega} \to \mathbb{R}$ implies that

$$u(\xi_0) \leq (1 - \lambda)u(\xi) + \lambda u(\xi') = 0,$$

which proves that $u \leq 0$ in Ω. For the proof of the second statement we show that any two points can be connected by a certain chain of balls where we can apply a Harnack-type inequality; we postpone this construction to the Appendix (see Subsection 6.2). $\square$

3 Comparison principles in Heisenberg groups

Let us recall that in order to prove the Aleksandrov-type estimate (1.1) in the Euclidean case, the following result is applied (see Gutiérrez [17, Lemma 1.4.1]):

Lemma 3.1 (Comparison lemma in Euclidean case) *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. If $u, v \in C(\overline{\Omega})$ with $u = v$ on $\partial\Omega$ and $u \leq v$ in Ω, then $\partial v(\Omega) \subset \partial u(\Omega)$.*

It is natural to ask whether a similar property holds in the setting of Heisenberg groups:
Question: *Let $\Omega \subset \mathbb{H}^n$ be an open and bounded set, $u, v \in C(\overline{\Omega})$ with $u = v$ on $\partial\Omega$ and $u \leq v$ in Ω. Does the inclusion $\partial_H v(\Omega) \subset \partial_H u(\Omega)$ hold?*

The answer to this question is *negative* in general; we postpone our counterexample to Section 5. However, we can give a *positive* answer to the Question formulated above, under the assumption of *$H-$convexity.*

3.1 Comparison lemma for the horizontal normal mapping

The main result of this section is a Heisenberg version of Lemma 3.1. While in the Euclidean case the proof of this comparison principle is rather trivial, the geometric structure of the Heisenberg group $\mathbb{H}^n$ causes serious difficulties in the proof of such a comparison result. Various authors including Gutiérrez and Montanari expressed their doubts about this method and used another approach to obtain Aleksandrov-type estimates [18]. Here we overcome the difficulties by using degree-theoretical arguments of set valued maps [20]; the results needed in the proof are collected in the Appendix. Our first result is the following:

Theorem 3.1 (Comparison lemma for horizontal normal mapping) *Let Ω_0 and $\Omega \subset \mathbb{H}^n$ be open, horizontally bounded sets such that Ω is $H-$convex, $\overline{\Omega_0} \subset \Omega$ and $u, v : \Omega \to \mathbb{R}$ are $H-$convex functions. Let $\xi_0 \in \Omega_0$ be fixed such that $u(\xi_0) \leq v(\xi_0)$ and $u \geq v$ on $\partial\Omega_0 \cap H_{\xi_0}$. If $p_0 \in \partial_H v(\xi_0)$ satisfies*

$$v(\xi) > v(\xi_0) + p_0 \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)), \qquad \forall \xi \in \partial\Omega_0 \cap H_{\xi_0}, \tag{3.1}$$

then $p_0 \in \partial_H u(\Omega_0 \cap H_{\xi_0})$.

Proof. The proof is divided into four steps.

Step 1. We consider the restriction of the standard projection Pr_1 to a horizontal plane: more precisely, consider $\mathrm{Pr}_1 : H_{\xi_0} \to \mathbb{R}^{2n}$ which gives a linear isomorphism between the horizontal plane H_{ξ_0} and $\mathbb{R}^{2n}$. Accordingly, we introduce the following notations, $\tilde{\xi_0} := \mathrm{Pr}_1(\xi_0)$, $\tilde{\xi} := \mathrm{Pr}_1(\xi)$, $\widetilde{\partial_H v} := \partial_H v \circ \mathrm{Pr}_1^{-1} : \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0}) \to 2^{\mathbb{R}^{2n}}$ and $\widetilde{\partial_H u} := \partial_H u \circ \mathrm{Pr}_1^{-1} : \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0}) \to 2^{\mathbb{R}^{2n}}$. In these notations the condition (3.1) reads as

$$v(\xi) > v(\xi_0) + p_0 \cdot (\tilde{\xi} - \tilde{\xi_0}), \qquad \forall \xi \in \partial\Omega_0 \cap H_{\xi_0}. \tag{3.2}$$

By Proposition 2.1 and Theorem 2.1, the set-valued maps $\widetilde{\partial_H u}$ and $\widetilde{\partial_H v}$ are upper semicontinuous on the compact set $\mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})$ with compact and convex values.

Step 2. Let $p_0 \in \partial_H v(\xi_0)$. We prove that

$$\deg_{SV}\left(\widetilde{\partial_H v}(\cdot) - p_0, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0\right) = 1, \tag{3.3}$$

where $\deg_{SV}$ denotes the degree function for set-valued maps, see Theorem 6.2 from the Appendix.

To verify (3.3), we first claim that

$$(p^v - p_0) \cdot (\tilde{\xi} - \tilde{\xi_0}) > 0, \ \forall \xi \in \partial\Omega_0 \cap H_{\xi_0}, \qquad \forall p^v \in \widetilde{\partial_H v}(\tilde{\xi}). \tag{3.4}$$

Let us fix $\xi \in \partial\Omega_0 \cap H_{\xi_0}$ and $p^v \in \widetilde{\partial_H v}(\tilde{\xi})$. Since $\xi \in \Omega$ and v is $H-$convex on Ω, one has that

$$v(\zeta) - v(\xi) \geq p^v \cdot (\tilde{\zeta} - \tilde{\xi}), \qquad \forall \zeta \in \Omega \cap H_\xi.$$

In particular, choosing $\zeta = \xi_0 \in \Omega_0 \cap H_{\xi_0}$ in the latter inequality, we obtain that

$$v(\xi_0) - v(\xi) \geq p^v \cdot (\tilde{\xi_0} - \tilde{\xi}). \tag{3.5}$$

Combining this inequality with (3.2), it yields precisely relation (3.4).

Now, we consider the parametric set-valued map $\mathcal{F}_\lambda : \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0}) \to 2^{\mathbb{R}^{2n}}$, $\lambda \in [0,1]$, defined by

$$\mathcal{F}_\lambda(\tilde{\xi}) = (1-\lambda)(\tilde{\xi} - \tilde{\xi_0}) + \lambda(\widetilde{\partial_H v}(\tilde{\xi}) - p_0).$$

It follows from Proposition 2.1 and Theorem 2.1 that the following properties hold:

- $\overline{\{\cup \mathcal{F}_\lambda(\tilde{\xi}) : (\lambda, \tilde{\xi}) \in [0,1] \times \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})\}}$ is compact in $\mathbb{R}^{2n}$;
- for every $(\lambda, \tilde{\xi}) \in [0,1] \times \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})$, the set $\mathcal{F}_\lambda(\tilde{\xi})$ is compact and convex in $\mathbb{R}^{2n}$;
- $(\lambda, \tilde{\xi}) \mapsto \mathcal{F}_\lambda(\tilde{\xi})$ is upper semicontinuous from $[0,1] \times \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})$ into $2^{\mathbb{R}^{2n}} \setminus \{\emptyset\}$.

According to Definition 6.2 from the Appendix, $\mathcal{F}_\lambda$ is of homotopy of class (P).

We now claim that for the constant curve $\gamma : [0,1] \to \mathbb{R}^{2n}$, $\gamma(\lambda) = 0$, we have $\gamma(\lambda) \notin \mathcal{F}_\lambda(\mathrm{Pr}_1(\partial\Omega_0 \cap H_{\xi_0}))$ for every $\lambda \in [0,1]$. By contrary, we assume that there exists $\lambda_0 \in [0,1]$ and $\xi \in \partial\Omega_0 \cap H_{\xi_0}$ such that $0 \in \mathcal{F}_{\lambda_0}(\tilde{\xi})$, i.e.,

$$0 \in (1-\lambda_0)(\tilde{\xi} - \tilde{\xi_0}) + \lambda_0(\widetilde{\partial_H v}(\tilde{\xi}) - p_0).$$

In particular, there exists $p^v \in \widetilde{\partial_H v}(\tilde{\xi})$ such that $0 = (1-\lambda_0)(\tilde{\xi} - \tilde{\xi}_0) + \lambda_0(p^v - p_0)$. Multiplying the latter relation by $(\tilde{\xi} - \tilde{\xi}_0) \neq 0$, on account of (3.4) we obtain the contradiction

$$0 = (1-\lambda_0)|\tilde{\xi} - \tilde{\xi}_0|^2 + \lambda_0(p^v - p_0) \cdot (\tilde{\xi} - \tilde{\xi}_0) > 0.$$

Therefore, by the homotopy invariance (see Theorem 6.2 from the Appendix), we have that $\lambda \mapsto \deg_{SV}(\mathcal{F}_\lambda, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0)$ is constant. In particular, by exploiting the basic properties of the set-valued and Brouwer degrees (see Appendix), it yields that

$$\begin{aligned}\deg_{SV}\left(\widetilde{\partial_H v} - p_0, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0\right) &= \deg_{SV}(\mathcal{F}_1, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0) = \\ &= \deg_{SV}(\mathcal{F}_0, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0) = \deg_{SV}(Id - \tilde{\xi}_0, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0) = \\ &= \deg_B(Id - \tilde{\xi}_0, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0) = \deg_B(Id, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), \tilde{\xi}_0) = 1,\end{aligned}$$

which shows (3.3).

Step 3. We prove that

$$\deg_{SV}\left(\widetilde{\partial_H u} - p_0, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0\right) = 1.$$

First of all, a similar reason as in (3.5) shows that

$$u(\xi_0) - u(\xi) \geq p^u \cdot (\tilde{\xi}_0 - \tilde{\xi}), \ \forall \xi \in \partial\Omega_0 \cap H_{\xi_0}, \qquad \forall p^u \in \widetilde{\partial_H u}(\tilde{\xi}). \tag{3.6}$$

We introduce the parametric set-valued map $\mathcal{G}_\lambda : \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0}) \to 2^{\mathbb{R}^{2n}}$, $\lambda \in [0,1]$, defined by

$$\mathcal{G}_\lambda(\tilde{\xi}) = (1-\lambda)(\widetilde{\partial_H v}(\tilde{\xi}) - p_0) + \lambda(\widetilde{\partial_H u}(\tilde{\xi}) - p_0).$$

We observe, again from Proposition 2.1 and Theorem 2.1 that

- $\overline{\{\cup\mathcal{G}_\lambda(\tilde{\xi}) : (\lambda, \tilde{\xi}) \in [0,1] \times \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})\}}$ is compact in $\mathbb{R}^{2n}$;
- for every $(\lambda, \tilde{\xi}) \in [0,1] \times \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})$, $\mathcal{G}_\lambda(\tilde{\xi})$ is compact and convex in $\mathbb{R}^{2n}$ (as the sum of two compact and convex sets);
- $(\lambda, \tilde{\xi}) \mapsto \mathcal{G}_\lambda(\tilde{\xi})$ is upper semicontinuous from $[0,1] \times \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})$ into $2^{\mathbb{R}^{2n}} \setminus \{\emptyset\}$.

Therefore, $\mathcal{G}_\lambda$ is a homotopy of class (P).

We prove that

$$0 \notin \mathcal{G}_\lambda(\mathrm{Pr}_1(\partial\Omega_0 \cap H_{\xi_0})), \qquad \forall \lambda \in [0,1]. \tag{3.7}$$

Assume the contrary, i.e., there exists $\lambda_0 \in [0,1]$ and $\xi \in \partial\Omega_0 \cap H_{\xi_0}$ such that $0 \in \mathcal{G}_{\lambda_0}(\tilde{\xi})$. It follows that

$$0 = (1-\lambda_0)(p^v - p_0) + \lambda_0(p^u - p_0) \tag{3.8}$$

for some $p^u \in \widetilde{\partial_H u}(\tilde{\xi})$ and $p^v \in \widetilde{\partial_H v}(\tilde{\xi})$. Combining (3.5), (3.6) and (3.8) respectively, we obtain that

$$(1-\lambda_0)v(\xi_0) + \lambda_0 u(\xi_0) - [(1-\lambda_0)v(\xi) + \lambda_0 u(\xi)] \geq p_0 \cdot (\tilde{\xi}_0 - \tilde{\xi}).$$

On the other hand, by adding the latter inequality to (3.2) applied for $\tilde{\xi}$, it yields

$$\lambda_0(-v(\xi_0) + u(\xi_0)) + \lambda_0(v(\xi) - u(\xi)) > 0.$$

Note that $u \geq v$ on $\partial\Omega_0 \cap H_{\xi_0}$; thus it follows that

$$\lambda_0(u(\xi_0) - v(\xi_0)) > \lambda_0(u(\xi) - v(\xi)) \geq 0.$$

Clearly, $\lambda_0 \neq 0$; thus, it yields that $u(\xi_0) > v(\xi_0)$ which contradicts the assumption that $v(\xi_0) \geq u(\xi_0)$. Therefore, (3.7) holds true.

Again, by the homotopy invariance (see Theorem 6.2 from the Appendix), we have that $\lambda \mapsto \deg_{SV}(\mathcal{G}_\lambda, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0)$ is constant, i.e., according to Step 2,

$$\deg_{SV}\left(\widetilde{\partial_H u} - p_0, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0\right) = \deg_{SV}\left(\widetilde{\partial_H v} - p_0, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0\right) = 1,$$

which concludes the proof of Step 3.

Step 4. By Step 3 and the definition of $\deg_{SV}$, for small $\varepsilon > 0$, one has that

$$\deg_B(f_\varepsilon^u - p_0, \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0}), 0) = 1, \tag{3.9}$$

where $f_\varepsilon^u : \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0}) \to \mathbb{R}^{2n}$ is a continuous approximate selector of the upper semicontinuous set-valued map $\widetilde{\partial_H u}$ such that

$$f_\varepsilon^u(\tilde{\xi}) \in \widetilde{\partial_H u}\left(B_{\mathbb{R}^{2n}}(\tilde{\xi}, \varepsilon) \cap \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})\right) + B_{\mathbb{R}^{2n}}(0, \varepsilon), \quad \forall \tilde{\xi} \in \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0}), \tag{3.10}$$

see Proposition 6.1 from the Appendix. Let $\varepsilon = \frac{1}{k}$ and let $\phi_k^u := f_{1/k}^u$, $k \in \mathbb{N}$. First of all, from (3.9) and the properties of the Brouwer degree d_B (see Theorem 6.1 from the Appendix), we have that for every $k \in \mathbb{N}$ there exists $\tilde{\xi}_k \in \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0})$ such that $p_0 = \phi_k^u(\tilde{\xi}_k)$. Up to a subsequence, we may assume that $\tilde{\xi}_k \to \tilde{\xi} \in \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})$. On the other hand, by relation (3.10), we have that

$$p_0 = \phi_k^u(\tilde{\xi}_k) \in \widetilde{\partial_H u}\left(B_{\mathbb{R}^{2n}}\left(\tilde{\xi}_k, \frac{1}{k}\right) \cap \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})\right) + B_{\mathbb{R}^{2n}}\left(0, \frac{1}{k}\right),$$

i.e., there exists $\tilde{\zeta}_k \in B_{\mathbb{R}^{2n}}(\tilde{\xi}_k, \frac{1}{k}) \cap \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0})$ and $p_k \in B_{\mathbb{R}^{2n}}(0, \frac{1}{k})$ such that $p_0 \in \widetilde{\partial_H u}(\tilde{\zeta}_k) + p_k$. Clearly, $\tilde{\zeta}_k \to \tilde{\xi}$ as $k \to \infty$. In the following, we shall show that $p_0 \in \widetilde{\partial_H u}(\tilde{\xi})$.

We assume by contradiction, that $p_0 \notin \widetilde{\partial_H u}(\tilde{\xi})$. Since $\widetilde{\partial_H u}(\tilde{\xi})$ is compact, it follows that $d_0 := \mathrm{dist}(p_0, \widetilde{\partial_H u}(\tilde{\xi})) > 0$. On account of the upper semicontinuity of $\widetilde{\partial_H u}$ at $\tilde{\xi}$, there exists $\delta > 0$ such that

$$\widetilde{\partial_H u}(\xi') \subset \widetilde{\partial_H u}(\tilde{\xi}) + B_{\mathbb{R}^{2n}}(0, d_0/4), \ \forall \xi' \in B_{\mathbb{R}^{2n}}(\tilde{\xi}, \delta) \cap \mathrm{Pr}_1(\overline{\Omega_0} \cap H_{\xi_0}).$$

Applying the latter relation for $\xi' = \tilde{\zeta}_k$, and taking into account that $p_k \to 0$, we obtain that for k large enough,

$$p_0 \in \widetilde{\partial_H u}(\tilde{\zeta}_k) + p_k \subset \widetilde{\partial_H u}(\tilde{\xi}) + B_{\mathbb{R}^{2n}}(0, d_0/2),$$

which contradicts the definition of d_0. Therefore, $p_0 \in \widetilde{\partial_H u}(\tilde{\xi})$.

We claim that $\tilde{\xi} \in \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0})$. To see this, we assume by contradiction that $\tilde{\xi} \in \mathrm{Pr}_1(\partial\Omega_0 \cap H_{\xi_0})$. Then, $p_0 \in \widetilde{\partial_H u}(\tilde{\xi})$ is equivalent to $0 \in \mathcal{G}_1(\tilde{\xi})$, which contradicts relation (3.7). Consequently, $\tilde{\xi} \in \mathrm{Pr}_1(\Omega_0 \cap H_{\xi_0})$; therefore,

$$p_0 \in \widetilde{\partial_H u}(\tilde{\xi}) = \partial_H u(\mathrm{Pr}_1^{-1}(\tilde{\xi})) = \partial_H u(\xi),$$

where $\xi = \mathrm{Pr}_1^{-1}(\tilde{\xi}) \in \Omega_0 \cap H_{\xi_0}$, which concludes the proof. □

3.2 Comparison principles for $H-$convex functions

In this subsection we apply Theorem 3.1 to prove Theorem 1.1 and Theorem 1.2. To do this, we shall compare $H-$convex functions with specific cone functions, that we will call slicing cones. Some properties on the horizontal normal mapping of such cones will be presented in the sequel.

We present in the sequel the construction of this specific cone function, taking into account that we are in a domain that is horizontally bounded (but it could be in general, unbounded).

Let $G_0 \subset \mathbb{H}^n$ be an open and horizontally bounded set and $\xi_0 \in G_0$ such that $G_0 \cap H_{\xi_0}$ is (Euclidean) convex. Let $c_v < c_b \leq 0$.

For every $\xi \in H_{\xi_0}$ with $\xi \neq \xi_0$, we define $\xi^\partial = \xi^\partial(\xi)$ the unique point in $\partial G_0 \cap H_{\xi_0}$ such that ξ belongs to the horizontal segment (that is exactly the geodesic in the Carnot-Carathéodory metric) from ξ_0 to ξ^∂. Moreover, for every such $\xi \in H_{\xi_0}$ with $\xi \neq \xi_0$, we define λ^ξ as the unique positive value such that

$$\xi = \xi_0 \circ \delta_{\lambda^\xi}(\xi_0^{-1} \circ \xi^\partial). \tag{3.11}$$

For $\xi = \xi_0$ we set $\lambda^{\xi_0} = 0$, we also define ξ_0^∂ to be an arbitrary point in $\partial G_0 \cap H_{\xi_0}$.

Now, for every $\xi \in \mathbb{H}^n$, we define $\xi^\perp \in H_{\xi_0}$ to be the Euclidean orthogonal projection of ξ on the plane H_{ξ_0}. Finally we define the *slicing cone* $V : \mathbb{R}^{2n+1} \to \mathbb{R}$ with vertex (ξ_0, c_v) and base $G_0 \cap H_{\xi_0}$ with the value c_b on $\partial G_0 \cap H_{\xi_0}$ by

$$V(\xi) = c_v \left(1 - \left(1 - \frac{c_b}{c_v}\right) \frac{N(\xi_0^{-1} \circ \xi^\perp)}{N(\xi_0^{-1} \circ (\xi^\perp)^\partial)}\right), \qquad \xi \in \mathbb{H}^n = \mathbb{R}^{2n+1}. \tag{3.12}$$

An easy computation shows that

$$V(\xi) = c_v \left(1 - \left(1 - \frac{c_b}{c_v}\right) \lambda^{\xi^\perp}\right), \qquad \xi \in \mathbb{H}^n. \tag{3.13}$$

Since $\lambda^{\xi^\perp} = \lambda^\xi = 1$, for every $\xi \in \partial G_0 \cap H_{\xi_0}$, we have $V(\xi) = c_b$.

By its definition, the function $V\big|_{H_{\xi_0}}$ is Euclidean convex which implies that V is Euclidean convex and hence $H-$convex.

Proposition 3.1 *Let $\Omega \subset \mathbb{H}^n$ be an open, horizontally bounded set, $G_0 \subset \Omega$ be an open (Euclidean) convex set, $\xi_0 \in G_0$ and $c_v < c_b \leq 0$. The slicing cone $V : \mathbb{R}^{2n+1} \to \mathbb{R}$ with vertex (ξ_0, c_v) and base $G_0 \cap H_{\xi_0}$ with the value c_b on $\partial G_0 \cap H_{\xi_0}$ has the following properties:*

(i) $B_{\mathbb{R}^{2n}}(0, r_0) \subset \partial_H V(\xi_0)$, *where* $r_0 = \frac{c_b - c_v}{\mathrm{diam}_H(G_0 \cap H_{\xi_0})}$;

(ii) *for every* $p \in \mathrm{int}(\partial_H V(\xi_0))$, *we have*

$$V(\xi) > V(\xi_0) + p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)), \qquad \forall \xi \in \overline{G_0} \cap H_{\xi_0} \setminus \{\xi_0\}. \tag{3.14}$$

Proof. Let us prove first (i). By definition, $p \in \partial_H V(\xi_0)$ is equivalent to the inequality

$$V(\xi) \geq V(\xi_0) + p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)), \qquad \forall \xi \in G_0 \cap H_{\xi_0}. \tag{3.15}$$

We shall use that V on $G_0 \cap H_{\xi_0}$ is defined by (3.13), with $\xi^\perp = \xi$. Applying a group multiplication to the relation (3.11) by ξ_0^{-1} from the left and applying the projection map Pr_1 to both sides we obtain

$$\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0) = \lambda^\xi(\mathrm{Pr}_1(\xi^\partial) - \mathrm{Pr}_1(\xi_0)).$$

Therefore, (3.15) is equivalent to the inequality

$$c_b - c_v \geq p \cdot (\mathrm{Pr}_1(\xi^\partial) - \mathrm{Pr}_1(\xi_0)), \qquad \forall \xi \in G_0 \cap H_{\xi_0}. \tag{3.16}$$

Since

$$|\mathrm{Pr}_1(\xi^\partial) - \mathrm{Pr}_1(\xi_0)| = N(\xi_0^{-1} \circ \xi^\partial) \leq \mathrm{diam}_H(G_0 \cap H_{\xi_0}),$$

by the definition of the number $r_0 > 0$ it is easy to see that for all $p \in B_{\mathbb{R}^{2n}}(0, r_0)$, relation (3.16) holds.

Now, we are going to prove (ii). Since $\partial_H V(\xi_0)$ is convex and $0 \in B_{\mathbb{R}^{2n}}(0, r_0) \subset \partial_H V(\xi_0)$ (cf. (i)), $\partial_H V(\xi_0)$ is a star-shaped set with respect to the origin of $\mathbb{R}^{2n}$. Moreover,

$$\mathrm{int}(\partial_H V(\xi_0)) = \bigcup\{\alpha p : \alpha \in [0,1), p \in \partial_H V(\xi_0)\}.$$

Let $\alpha \in (0,1)$ and $p \in \partial_H V(\xi_0)$ be fixed. The latter relation implies that for every $\beta \in (0,1)$ we have that $\beta p \in \partial_H V(\xi_0)$, and for every $\xi \in \overline{G}_0 \cap H_{\xi_0}$,

$$V(\xi) \geq V(\xi_0) + \beta p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)). \tag{3.17}$$

If $p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)) > 0$, we set $\beta = (\alpha+1)/2$ and (3.17) implies

$$V(\xi) > V(\xi_0) + \alpha p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)). \tag{3.18}$$

If $p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)) < 0$, we set $\beta = \alpha/2$ and (3.17) implies

$$V(\xi) > V(\xi_0) + \alpha p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)). \tag{3.19}$$

The third possibility is the case when $p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)) = 0$ for some $\xi \in \overline{G_0} \cap H_{\xi_0} \setminus \{\xi_0\}$. Since $V(\xi) = c_v \left(1 - \left(1 - \frac{c_b}{c_v}\right)\lambda^\xi\right) > c_v = V(\xi_0)$ we obtain again the inequality

$$V(\xi) > V(\xi_0) = V(\xi_0) + \alpha p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)).$$

Combining the latter relation with (3.18) and (3.19), we have that for all $\xi \in \overline{G_0} \cap H_{\xi_0} \setminus \{\xi_0\}$,

$$V(\xi) > V(\xi_0) + \alpha p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)),$$

which concludes the proof. □

Proof of Theorem 1.1. Let $\xi_0 \in \Omega_0$ be fixed. Without loss of generality, we may assume that $u(\xi_0) < v(\xi_0) < 0$; otherwise, we subtract a sufficiently large number from both functions. Let us fix $q \in \partial_H v(\xi_0)$ and consider the function $U : \overline{\Omega} \to \mathbb{R}$ defined by

$$U(\xi) = u(\xi) - q \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)).$$

Clearly, U is $H-$convex, $U(\xi_0) = u(\xi_0)$, and

$$\begin{aligned} U(\xi) &= u(\xi) - q \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)) \\ &= v(\xi) - q \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)) \\ &\geq v(\xi_0) = u(\xi_0) + m_0, \qquad \forall \xi \in \partial\Omega_0 \cap H_{\xi_0} \end{aligned} \tag{3.20}$$

where $m_0 = v(\xi_0) - u(\xi_0) > 0$. We notice that for every $\xi \in \Omega_0$,

$$\partial_H U(\xi) = \partial_H u(\xi) - q. \tag{3.21}$$

Now let us denote here and in the sequel by Ω_0^{conv} the Euclidean convex hull of Ω_0 and we consider the slicing cone $V : \mathbb{R}^{2n+1} \to \mathbb{R}$ with vertex $(\xi_0, u(\xi_0))$ and base $\Omega_0^{\text{conv}} \cap H_{\xi_0}$ with the value $v(\xi_0) = u(\xi_0) + m_0$ on $\partial\Omega_0^{\text{conv}} \cap H_{\xi_0}$; see (3.12). We know that V is Euclidean convex and hence $H-$convex.

Since $\Omega_0 \subset \Omega_0^{\text{conv}}$, from (3.20) we have

$$U(\xi_0) = u(\xi_0) = V(\xi_0) \qquad \text{and} \qquad U(\xi) \geq u(\xi_0) + m_0 \geq V(\xi), \quad \forall \xi \in \partial\Omega_0 \cap H_{\xi_0}. \tag{3.22}$$

In addition, by applying Proposition 3.1 with $G_0 = \Omega_0^{\text{conv}}$, $c_b = u(\xi_0) + m_0$ and $c_v = u(\xi_0)$, and taking into account that $\partial\Omega_0 \subset \overline{\Omega_0^{\text{conv}}}$, we have

(i) $B_{\mathbb{R}^{2n}}(0, r_{\xi_0}) \subset \partial_H V(\xi_0)$, where $r_{\xi_0} = \frac{m_0}{\text{diam}_{HS}(\Omega_0^{\text{conv}})}$;

(ii) for every $p \in \text{int}(\partial_H V(\xi_0))$, we have

$$V(\xi) > V(\xi_0) + p \cdot (\text{Pr}_1(\xi) - \text{Pr}_1(\xi_0)), \qquad \forall \xi \in \partial\Omega_0 \cap H_{\xi_0}. \tag{3.23}$$

Taking into consideration (3.22) and (ii) we can apply Theorem 3.1 for the functions U and V on the open bounded set $\Omega_0 \subset \Omega$ to conclude that for any $p \in \text{int}(\partial_H V(\xi_0))$, we have $p \in \partial_H U(\Omega_0 \cap H_{\xi_0})$. Consequently, one has

$$\text{int}(\partial_H V(\xi_0)) \subset \partial_H U(\Omega_0 \cap H_{\xi_0}). \tag{3.24}$$

By using (i) and (3.21) we deduce the following chain of inclusions:

$$0 \in B_{\mathbb{R}^{2n}}(0, r_{\xi_0}/2) \subset \text{int}(\partial_H V(\xi_0)) \subset \partial_H U(\Omega_0 \cap H_{\xi_0}) = \partial_H u(\Omega_0 \cap H_{\xi_0}) - q. \tag{3.25}$$

In particular, $q \in \partial_H u(\Omega_0 \cap H_{\xi_0})$, which concludes the proof. □

The following result is a direct consequence of Theorem 3.1.

Theorem 3.2 *Let $\Omega \subset \mathbb{H}^n$ be an open, horizontally bounded and $H-$convex set, $u : \Omega \to \mathbb{R}$ be an $H-$convex function, and $v : \Omega \to \mathbb{R}$ be a strictly $H-$convex function. Let $\Omega_0 \subset \mathbb{H}^n$ be open such that $\overline{\Omega_0} \subset \Omega$ and assume that $u \leq v$ in Ω_0 and $u = v$ on $\partial\Omega_0$. Then*

$$\partial_H v(\Omega_0) \subset \partial_H u(\Omega_0).$$

Remark 3.1 The two consequences of Theorem 3.1, i.e. the statements of Theorem 1.1 and Theorem 3.2, can be merged once we replace $u < v$ by $u \leq v$ in Ω_0 in the former, and the strict $H-$convexity by the $H-$convexity in the latter result. We think that such a general statement is still valid in our context but the method of Theorem 3.1 does not seem to work. However, Theorem 3.1 is sufficient to prove the Aleksandrov-type estimate.

Another consequence of Theorem 3.1 is the Heisenberg comparison principle which corresponds to the Euclidean one, see Gutiérrez [17, Theorem 1.4.6].

Proof of Theorem 1.2. Without loss of generality, we may assume that u and v are strictly negative in $\overline{\Omega}$ and that $\min_{\xi\in\partial\Omega}(v(\xi)-u(\xi))=0$. Otherwise, we may replace v by $\tilde{v}=v+A-\min_{\xi\in\partial\Omega}(v(\xi)-u(\xi))$ and u by $\tilde{u}=u+A$, where A is a sufficiently small negative number.

Suppose that there exists $\xi_0\in\Omega$ such that $v(\xi_0)<u(\xi_0)<0$. Let us fix $\alpha\in(0,1)$ such that $v(\xi_0)<\alpha v(\xi_0)<u(\xi_0)$ and consider the set

$$\Omega_0=\{\xi\in\Omega:\ \alpha v(\xi)<u(\xi)\}.$$

Since u and v are continuous functions on Ω, and $\xi_0\in\Omega_0$, it follows that Ω_0 is a non-empty open set.

We first notice that $\overline{\Omega_0}\subset\Omega$. Indeed, if we assume by contradiction that there exists $\zeta\in\partial\Omega\cap\overline{\Omega_0}$, then $\alpha v(\zeta)\le u(\zeta)$. Since $\min_{\xi\in\partial\Omega}(v(\xi)-u(\xi))=0$, we have that $v(\zeta)\ge u(\zeta)$, a contradiction with the facts that $\alpha\in(0,1)$ and u,v are strictly negative.

We can apply Theorem 1.1 to functions $\alpha v<u$ in Ω_0 obtaining that $\partial_H u(\Omega_0)\subset\partial_H(\alpha v)(\Omega_0)=\alpha\partial_H v(\Omega_0)$. We notice that from the proof of Theorem 1.1, by replacing u by αv and v by u, respectively, it also follows that $\mathcal{L}^{2n}(\partial_H v(\Omega_0))>0$, see relation (3.25). Moreover, by Proposition 2.1 one also has that $\mathcal{L}^{2n}(\partial_H v(\Omega_0))<+\infty$. Therefore, we obtain

$$\mathcal{L}^{2n}(\partial_H u(\Omega_0))\le\alpha^{2n}\mathcal{L}^{2n}(\partial_H v(\Omega_0))<\mathcal{L}^{2n}(\partial_H v(\Omega_0)),$$

which contradicts the assumption. □

Proof of Corollary 1.1. It follows directly from Theorem 1.2. □

4 Aleksandrov-type maximum principles

In this section we prove the main result of the paper, i.e., the Heisenberg version of Aleksandrov's maximum principle in Theorem 1.3. The proof of Theorem 1.3 is based on a strategy following three arguments:

- Using the basic comparison principle we shall prove first an Aleksandrov-type estimate with respect to *horizontal planes*, i.e.,

  $$|u(\xi_0)|^{2n}\le C_n'\mathrm{dist}_H(\xi_0,\partial\Omega\cap H_{\xi_0})\mathrm{diam}_H(\Omega\cap H_{\xi_0})^{2n-1}\mathcal{L}^{2n}(\partial_H u(\Omega\cap H_{\xi_0})),\ \forall\xi_0\in\Omega,\quad(4.1)$$

  where $C_n'>0$ depends only on n, see Theorem 4.1. Observe that for bounded cylindrical-type domains (which have 'flat faces' close, but parallel to horizontal planes at a given point) one may occur that $\mathrm{dist}_H(\xi_0,\partial\Omega\cap H_{\xi_0})\not\to 0$ in spite of the fact that $\xi_0\to\partial\Omega$. In such cases the estimate (4.1) is much weaker than the desired (1.6). The solution to this problem is to compare the values $u(\xi_0)$ and $u(\zeta)$ where $\zeta\in\Omega$ are close enough to ξ_0 and a better estimate for $\mathrm{dist}_H(\zeta,\partial\Omega\cap H_\zeta)$ is available.

- We establish a Harnack-type inequality by proving that there exists a constant $C_1 > 1$ such that if $B_H(\xi_0, 3R) \subset \Omega$ for some $\xi_0 \in \Omega$ and $R > 0$, then

$$\frac{1}{C_1} u(\xi) \geq u(\zeta) \geq C_1 u(\xi), \quad \forall \xi, \zeta \in B_H(\xi_0, R),$$

see Theorem 6.3 in the Appendix. Now, from (4.1) and Harnack estimate we have that

$$|u(\xi_0)|^{2n} \leq C''_n \mathcal{D}(\xi_0) \mathrm{diam}_{HS}(\Omega)^{2n-1} \mathcal{L}^{2n}_{HS}(\partial_H u(\Omega)), \ \forall \xi_0 \in \Omega,$$

where $C''_n = (C_1)^{2n} C'_n$ and

$$\mathcal{D}(\xi_0) = \min\{\mathrm{dist}_H(\zeta, \partial\Omega \cap H_\zeta) : \zeta \in \overline{B_H(\xi_0, \mathrm{dist}_H(\xi_0, \partial\Omega)/3)}\}.$$

- Finally, by exploiting a typically Heisenberg phenomenon, i.e., the twirling effect of the horizontal planes from one point to another, we prove that there is a constant $C_2 > 0$ such that

$$\mathcal{D}(\xi_0) \leq C_2 \mathrm{dist}_H(\xi_0, \partial\Omega), \quad \forall \xi_0 \in \Omega.$$

4.1 Maximum principle on horizontal planes

The first step in our strategy consists of the following statement:

Theorem 4.1 (Aleksandrov-type maximum principle on horizontal planes) *Let $\Omega \subset \mathbb{H}^n$ be an open, horizontally bounded and convex set. If $u : \overline{\Omega} \to \mathbb{R}$ is a continuous $H-$convex function which verifies $u = 0$ on $\partial\Omega$, then*

$$|u(\xi_0)|^{2n} \leq C'_n \mathrm{dist}_H(\xi_0, \partial\Omega \cap H_{\xi_0}) \mathrm{diam}_H(\Omega \cap H_{\xi_0})^{2n-1} \mathcal{L}^{2n}(\partial_H u(\Omega \cap H_{\xi_0})), \qquad \forall \xi_0 \in \Omega, \quad (4.2)$$

where $C'_n > 0$ depends only on the dimension n.

Proof. By Proposition 2.2, we know that either $u \equiv 0$ on $\overline{\Omega}$, or $u < 0$ in Ω. In the first case, relation (4.2) is trivial; thus, we assume that $u < 0$ in Ω. Let $\xi_0 \in \Omega$ be fixed; thus, $u(\xi_0) < 0$. The main ingredient of the proof is the application of Theorem 3.1 for an appropriately constructed comparison function to our function u. The proof is divided into three steps.

Step 1. Let $\varepsilon > 0$ be small enough and let Ω_ε be an open and convex set (in the Euclidean sense) such that $\overline{\Omega_\varepsilon} \subset \Omega$ and $\lim_{\varepsilon \to 0^+} \Omega_\varepsilon = \Omega$. The strategy is to prove (in step 2) the Aleksandrov-type estimate for the function u restricted to Ω_ε by means of a comparison function; in step 3, we let $\varepsilon \to 0$. To do this, let us define first the quantity

$$\tau_{\xi_0}(\varepsilon) = \min\{u(\xi) : \xi \in \partial\Omega_\varepsilon \cap H_{\xi_0}\}. \quad (4.3)$$

Since $u = 0$ on $\partial\Omega$ and u is continuous on $\overline{\Omega}$, we may consider ε so small such that $\xi_0 \in \Omega_\varepsilon$, and $|\tau_{\xi_0}(\varepsilon)| < |u(\xi_0)|/2$. Let

$$t_{\xi_0}(\varepsilon) = 1 - \frac{\tau_{\xi_0}(\varepsilon)}{u(\xi_0)}.$$

Note that $1/2 < t_{\xi_0}(\varepsilon) \leq 1$ and $t_{\xi_0}(\varepsilon) \to 1$ as $\varepsilon \to 0$. We shall choose v_ε to be the slicing cone $v_\varepsilon : \mathbb{R}^{2n+1} \to \mathbb{R}$ with vertex $(\xi_0, u(\xi_0))$ and base $\Omega_\varepsilon \cap H_{\xi_0}$ with the value $\tau_{\xi_0}(\varepsilon)$ on $\partial\Omega_\varepsilon \cap H_{\xi_0}$; see (3.12). We know v_ε is Euclidean convex, then it is $H-$convex.

For further use, let us choose ξ_ε^- on $\partial\Omega_\varepsilon \cap H_{\xi_0}$ with the property that

$$N(\xi_0^{-1} \circ \xi_\varepsilon^-) = \min_{\xi' \in \partial\Omega_\varepsilon \cap H_{\xi_0}} N(\xi_0^{-1} \circ \xi').$$

Note that the point ξ_ε^- that realizes the previous minimum, in general, is not unique. Similarly to (3.11), for every $\xi \in \overline{\Omega_\varepsilon} \cap H_{\xi_0}$ with $\xi \neq \xi_0$, we define $\xi_\varepsilon^\partial = \xi_\varepsilon^\partial(\xi)$ the unique point in $\partial\Omega_\varepsilon \cap H_{\xi_0}$ such that ξ belongs to the horizontal segment from ξ_0 to ξ_ε^∂; let $\lambda_\varepsilon := \lambda_\varepsilon^\xi$ be the unique number in $(0, 1]$ such that

$$\xi = \xi_0 \circ \delta_{\lambda_\varepsilon}(\xi_0^{-1} \circ \xi_\varepsilon^\partial). \tag{4.4}$$

For $\xi = \xi_0$ we set $\lambda_\varepsilon^{\xi_0} = 0$, furthermore we set ξ_ϵ^∂ to be an arbitrary point in $\partial\Omega_\epsilon \cap H_{\xi_0}$. Similarly to (3.13), the restriction of v_ε to $\overline{\Omega_\varepsilon} \cap H_{\xi_0}$ is explicitly given by the formula

$$v_\varepsilon(\xi) = u(\xi_0)\left(1 - t_{\xi_0}(\varepsilon)\lambda_\varepsilon^\xi\right), \ \xi \in \overline{\Omega_\varepsilon} \cap H_{\xi_0}. \tag{4.5}$$

Step 2. On account of (4.5) and (4.3) we observe that

$$u(\xi_0) = v_\varepsilon(\xi_0) \qquad \text{and} \qquad u(\xi) \geq \tau_{\xi_0}(\varepsilon) = v_\varepsilon(\xi), \quad \forall \xi \in \partial\Omega_\varepsilon \cap H_{\xi_0}. \tag{4.6}$$

We claim the following properties hold:

(i) $B_{\mathbb{R}^{2n}}(0, r_\varepsilon) \subset \partial_H v_\varepsilon(\xi_0)$ for $r_\varepsilon = -t_{\xi_0}(\varepsilon)\dfrac{u(\xi_0)}{\operatorname{diam}_H(\Omega_\varepsilon \cap H_{\xi_0})}$;

(ii) for every $p \in \operatorname{int}(\partial_H v_\varepsilon(\xi_0))$, we have

$$v_\varepsilon(\xi) > v_\varepsilon(\xi_0) + p \cdot (\operatorname{Pr}_1(\xi) - \operatorname{Pr}_1(\xi_0)), \qquad \forall \xi \in \partial\Omega_\varepsilon \cap H_{\xi_0}. \tag{4.7}$$

(iii) $p_\varepsilon^- = -u(\xi_0)t_{\xi_0}(\varepsilon)\dfrac{\operatorname{Pr}_1(\xi_\varepsilon^-) - \operatorname{Pr}_1(\xi_0)}{|\operatorname{Pr}_1(\xi_\varepsilon^-) - \operatorname{Pr}_1(\xi_0)|^2} \in \partial_H v_\varepsilon(\xi_0)$.

Properties (i) and (ii) follow directly from Proposition 3.1. It remains to prove (iii). To do that, $p \in \partial_H v_\varepsilon(\xi_0)$ is equivalent to the inequality

$$v_\varepsilon(\xi) \geq v_\varepsilon(\xi_0) + p \cdot (\operatorname{Pr}_1(\xi) - \operatorname{Pr}_1(\xi_0)), \qquad \forall \xi \in \Omega_\varepsilon \cap H_{\xi_0}. \tag{4.8}$$

By (4.4) and (4.5), the latter inequality reduces to

$$-u(\xi_0)t_{\xi_0}(\varepsilon) \geq p \cdot (\operatorname{Pr}_1(\xi_\varepsilon^\partial) - \operatorname{Pr}_1(\xi_0)), \qquad \forall \xi \in \Omega_\varepsilon \cap H_{\xi_0}. \tag{4.9}$$

By inserting $p = p_\varepsilon^-$ in (4.9), we obtain that

$$(\operatorname{Pr}_1(\xi_\varepsilon^-) - \operatorname{Pr}_1(\xi_0)) \cdot (\operatorname{Pr}_1(\xi_\varepsilon^\partial) - \operatorname{Pr}_1(\xi_0)) \leq |\operatorname{Pr}_1(\xi_\varepsilon^-) - \operatorname{Pr}_1(\xi_0)|^2, \qquad \forall \xi \in \Omega_\varepsilon \cap H_{\xi_0}.$$

From general properties of convex domains, see Rockafellar [27], it follows that the above inequality holds; (iii) is proven.

By relation (4.6) and (ii), due to Theorem 3.1, we have that

$$\operatorname{int}(\partial_H v_\varepsilon(\xi_0)) \subseteq \partial_H u(\Omega_\varepsilon \cap H_{\xi_0}). \tag{4.10}$$

Step 3. By (i) and (iii) and since $\partial_H v_\varepsilon(\xi_0)$ is convex, we have that

$$\{\{p_\varepsilon^-\} \cup B_{\mathbb{R}^{2n}}(0, r_\varepsilon)\}^{\text{conv}} \subseteq \partial_H v_\varepsilon(\xi_0). \tag{4.11}$$

Consequently, combining (4.11) and relation (4.10), it yields that

$$\text{int } \{\{p_\varepsilon^-\} \cup B_{\mathbb{R}^{2n}}(0, r_\varepsilon)\}^{\text{conv}} \subseteq \partial_H u(\Omega_\varepsilon \cap H_{\xi_0}) \subseteq \partial_H u(\Omega \cap H_{\xi_0}).$$

Therefore, we have

$$\mathcal{L}^{2n}(\partial_H u(\Omega \cap H_{\xi_0})) \geq \mathcal{L}^{2n}\left(\{\{p_\varepsilon^-\} \cup B_{\mathbb{R}^{2n}}(0, r_\varepsilon)\}^{\text{conv}}\right) \geq c_n \cdot |p_\varepsilon^-| r_\varepsilon^{2n-1}$$

for some constant $c_n > 0$ depending only on n, i.e., from the definition of r_ε and p_ε^-, one has

$$|u(\xi_0)|^{2n} \leq C_n' \frac{1}{t_{\xi_0}(\varepsilon)^{2n}} |\text{Pr}_1(\xi_\varepsilon^-) - \text{Pr}_1(\xi_0)| \text{diam}_H(\Omega_\varepsilon \cap H_{\xi_0})^{2n-1} \mathcal{L}^{2n}(\partial_H u(\Omega \cap H_{\xi_0})),$$

with $C_n' = 1/c_n > 0$. Since $\text{diam}_H(\Omega_\varepsilon \cap H_{\xi_0}) \leq \text{diam}_H(\Omega \cap H_{\xi_0})$ and $\xi_\varepsilon^- \in \partial\Omega_\varepsilon \cap H_{\xi_0} \subset \Omega$, we have that $|\text{Pr}_1(\xi_\varepsilon^-) - \text{Pr}_1(\xi_0)| \leq \text{dist}_H(\xi_0, \partial\Omega \cap H_{\xi_0})$ which gives

$$|u(\xi_0)|^{2n} \leq C_n' \frac{1}{t_{\xi_0}(\varepsilon)^{2n}} \text{dist}_H(\xi_0, \partial\Omega \cap H_{\xi_0}) \text{diam}_H(\Omega \cap H_{\xi_0})^{2n-1} \mathcal{L}^{2n}(\partial_H u(\Omega \cap H_{\xi_0})).$$

Since $t_{\xi_0}(\varepsilon) \to 1$ as $\varepsilon \to 0$, we obtain the desired estimate. The proof is complete. $\square$

Corollary 4.1 *Under the same assumptions as in Theorem 4.1, we have*

$$|u(\xi_0)|^{2n} \leq C_n' \text{dist}_H(\xi_0, \partial\Omega \cap H_{\xi_0}) \text{diam}_{HS}(\Omega)^{2n-1} \mathcal{L}_{HS}^{2n}(\partial_H u(\Omega)), \qquad \forall \xi_0 \in \Omega. \tag{4.12}$$

4.2 Maximum principle in convex domains

As we already pointed out at the beginning of the section, it can happen, that $\text{dist}_H(\xi, \partial\Omega \cap H_\xi) \not\to 0$ in spite of the fact that $\xi \to \partial\Omega$, thus the estimate in (4.2) is not enough accurate. However, by combining Theorem 4.1 (see also Corollary 4.1) and a Harnack type estimate (see Theorem 6.3 in the Appendix), we obtain

Theorem 4.2 *Let $\Omega \subset \mathbb{H}^n$ be an open, horizontally bounded and convex set. If $u : \overline{\Omega} \to \mathbb{R}$ is a continuous $H-$convex function such that $u = 0$ on $\partial\Omega$, then*

$$|u(\xi)|^{2n} \leq C_n'' \mathcal{D}(\xi) \text{diam}_{HS}(\Omega)^{2n-1} \mathcal{L}_{HS}^{2n}(\partial_H u(\Omega)), \qquad \forall \xi \in \Omega, \tag{4.13}$$

where $C_n'' > 0$ depends only on the dimension n, and

$$\mathcal{D}(\xi) = \min\{\text{dist}_H(\zeta, \partial\Omega \cap H_\zeta) : \zeta \in \overline{B_H(\xi, \text{dist}_H(\xi, \partial\Omega)/3)}\}.$$

To deduce Theorem 1.3 from Theorem 4.2 we need the following geometric result, which exploits the twirling character of the horizontal planes in the Heisenberg framework.

Proposition 4.1 *Let $\Omega \subset \mathbb{H}^n$ be an open, horizontally bounded and convex set. Then,*

$$\mathcal{D}(\xi) \leq \left(\frac{\sqrt[4]{97}}{2} + \frac{1}{3} \right) \mathrm{dist}_H(\xi, \partial\Omega), \ \forall \xi \in \Omega. \tag{4.14}$$

Proof. After a left-translation argument, it is enough to prove inequality (4.14) for $\xi = 0$. Let $d = \mathrm{dist}_H(0, \partial\Omega) > 0$ and fix an element $\xi_0 = (x_0, y_0, t_0) \in \partial\Omega$ such that $d = d_H(0, \xi_0)$. Since Ω is convex, we can fix a supporting hyperplane π_{ξ_0} at $\xi_0 \in \partial\Omega$ which is represented by

$$\pi_{\xi_0} = \{(x, y, t) \in \mathbb{H}^n : A \cdot (x - x_0) + B \cdot (y - y_0) + c(t - t_0) = 0\},$$

for some $A, B \in \mathbb{R}^n$ and $c \in \mathbb{R}$. For the sake of notations, we set $a_k = (a, ..., a) \in \mathbb{R}^k$ for every $a \in \mathbb{R}$ and $k \in \{1, ..., n\}$.

Case 1. $A = B = 0$. In this particular case, the horizontal plane $H_{(0_n, 0_n, 0)}$ and π_{ξ_0} are parallel. Let $\zeta_0 = \left(\left(\frac{d_0}{\sqrt{n}} \right)_n, 0_n, 0 \right) \in \partial B_H(0, d_0)$ where $d_0 = d/3$. Let us denote by L_0 the $(2n-1)-$dimensional plane, which is the intersection of the horizontal plane $H_{\zeta_0} = \{(x, y, t) \in \mathbb{H}^n : t = -2\frac{d_0}{\sqrt{n}}(y_1 + ... + y_n)\}$ and π_{ξ_0}. Note that $\mathrm{Pr}_1(L_0)$ is a hyperplane in $\mathbb{R}^{2n}$ whose equation is given by

$$y_1 + ... + y_n + \frac{t_0\sqrt{n}}{2d_0} = 0. \tag{4.15}$$

Since $L_0 \subset H_{\zeta_0}$, on the account of equation (4.15), we have that

$$\begin{aligned} \mathrm{dist}_H(\zeta_0, L_0) &= \inf_{\zeta \in L_0} d_H(\zeta_0, \zeta) = \inf_{\zeta \in L_0} |\mathrm{Pr}_1(\zeta) - \mathrm{Pr}_1(\zeta_0)| \\ &= \inf_{\tilde{\zeta} \in \mathrm{Pr}_1(L_0)} |\tilde{\zeta} - \mathrm{Pr}_1(\zeta_0)| = \frac{\frac{|t_0|\sqrt{n}}{2d_0}}{\sqrt{n}} \\ &= \frac{|t_0|}{2d_0}. \end{aligned}$$

First, since π_{ξ_0} is a supporting hyperplane at $\xi_0 \in \partial\Omega$ to the convex set Ω, we have that

$$\mathrm{dist}_H(\zeta_0, \partial\Omega \cap H_{\zeta_0}) \leq \mathrm{dist}_H(\zeta_0, L_0) = \frac{|t_0|}{2d_0}.$$

On the other hand, since $d = d_H(0, \xi_0) = N(\xi_0) = N(x_0, y_0, t_0)$, then $|t_0| \leq d^2 = 9d_0^2$. Thus,

$$\mathcal{D}(0) = \min\left\{ \mathrm{dist}_H(\zeta, \partial\Omega \cap H_\zeta) : \zeta \in \overline{B_H(0, d_0)} \right\} \leq \mathrm{dist}_H(\zeta_0, \partial\Omega \cap H_{\zeta_0}) \leq \frac{|t_0|}{2d_0} \leq \frac{3}{2}d.$$

Case 2. $|A|^2 + |B|^2 \neq 0$. Clearly, after a normalization, we may assume that $|A|^2 + |B|^2 = 1$. Let $\zeta_0 = (d_0 A, d_0 B, 0) \in \partial B_H(0, d_0)$ where $d_0 = d/3$ as above. A simple computation shows that the plane π_{ξ_0} is not parallel to the horizontal plane in ζ_0,

$$H_{\zeta_0} = \{(x, y, t) : t = 2d_0(B \cdot x - A \cdot y)\}.$$

Let $L_{AB} = \pi_{\xi_0} \cap H_{\zeta_0}$, which is a $(2n-1)-$dimensional plane. One has that $\mathrm{Pr}_1(L_{AB})$ is a hyperplane in $\mathbb{R}^{2n}$ whose equation is obtained after the elimination of t from π_{ξ_0} and H_{ζ_0}, i.e.,

$$(A + 2cd_0 B) \cdot x + (B - 2cd_0 A) \cdot y - A \cdot x_0 - B \cdot y_0 - ct_0 = 0. \tag{4.16}$$

Note that

$$|A + 2cd_0B|^2 + |B - 2cd_0A|^2 = |A|^2 + |B|^2 + 4c^2d_0^2 = 1 + 4c^2d_0^2 > 0.$$

Taking into account that $L_{AB} \subset H_{\zeta_0}$, we have that

$$\begin{aligned}
\text{dist}_H(\zeta_0, L_{AB}) &= \inf_{\zeta \in L_{AB}} d_H(\zeta_0, \zeta) = \inf_{\zeta \in L_{AB}} |\text{Pr}_1(\zeta) - \text{Pr}_1(\zeta_0)| \\
&= \inf_{\tilde{\zeta} \in \text{Pr}_1(L_{AB})} |\tilde{\zeta} - \text{Pr}_1(\zeta_0)| \\
&= \frac{|d_0(A + 2cd_0B) \cdot A + d_0(B - 2cd_0A) \cdot B - A \cdot x_0 - B \cdot y_0 - ct_0|}{\sqrt{1 + 4c^2d_0^2}} \\
&= \frac{|d_0 - A \cdot x_0 - B \cdot y_0 - ct_0|}{\sqrt{1 + 4c^2d_0^2}} \\
&\leq d_0 + \frac{|A \cdot x_0 + B \cdot y_0 + ct_0|}{\sqrt{1 + 4c^2d_0^2}}.
\end{aligned}$$

By Schwartz inequality and from the fact that $|t_0| \leq d^2 = 9d_0^2$, it is clear that

$$\begin{aligned}
\frac{|A \cdot x_0 + B \cdot y_0 + ct_0|}{\sqrt{1 + 4c^2d_0^2}} &\leq \sqrt{|x_0|^2 + |y_0|^2 + \frac{t_0^2}{4d_0^2}} \leq \sqrt[4]{1 + \frac{t_0^2}{16d_0^4}} \sqrt[4]{(|x_0|^2 + |y_0|^2)^2 + t_0^2} \\
&\leq \frac{\sqrt[4]{97}}{2} N(x_0, y_0, t_0) = \frac{\sqrt[4]{97}}{2} d.
\end{aligned}$$

The rest of the proof is similar to the Case 1. The proof is concluded. □

Proof of Theorem 1.3. It follows from Theorem 4.2 and Proposition 4.1. □

We conclude this section showing that the estimate (1.6) in Theorem 1.3 has the natural scaling invariance property with respect to Heisenberg dilations δ_λ :

Remark 4.1 Let $\Omega \subset \mathbb{H}^n$ and $u : \overline{\Omega} \to \mathbb{R}$ be as in Theorem 1.3. Let $\lambda > 0$ and $\delta_\lambda \Omega$ be the Heisenberg dilation of the set Ω. We define the function $u^\lambda : \overline{\delta_\lambda \Omega} \to \mathbb{R}$ by $u^\lambda(\xi) = u(\delta_{\frac{1}{\lambda}}(\xi))$. Then Theorem 1.3 gives that

$$|u^\lambda(\xi_1)|^{2n} \leq C_n \text{dist}_H(\xi_1, \partial(\delta_\lambda \Omega)) \ \text{diam}_{HS}(\delta_\lambda \Omega)^{2n-1} \mathcal{L}^{2n}_{HS}(\partial_H u(\delta_\lambda \Omega)), \qquad \forall \xi_1 \in \delta_\lambda \Omega. \quad (4.17)$$

If we consider $\xi_0 = \delta_\lambda(\xi_1)$ and taking into account that

- $\text{diam}_{HS}(\delta_\lambda \Omega) = \lambda \text{diam}_{HS}(\Omega)$,
- $p \in \partial_H u^\lambda(\xi)$ if and only if $\lambda p \in \partial_H u(\delta_{\frac{1}{\lambda}}(\xi))$,
- $\text{dist}_H(\xi_1, \partial(\delta_\lambda \Omega)) = \lambda \text{dist}_H(\xi_0, \partial \Omega)$,

we obtain that (4.17) coincides with (1.6).

5 Examples: sharpness of the results

In this final section we provide explicit examples showing the sharpness of our results.

5.1 Failure of comparison principles in the absence of convexity

In this subsection we provide an example which shows the failure of the comparison principle for the horizontal normal mapping in the absence of the convexity of functions. Let

$$\Omega = \left\{(x, y, t) \in \mathbb{H}^1 : x^2 + y^2 < 1, |t| < 1\right\},$$

and $u, v \in \Gamma^\infty(\mathbb{H}^1)$ be defined by

$$u(x, y, t) = t - (1 - t^2)g(x, y), \qquad v(x, y, t) = t,$$

where

$$\begin{cases} g \in C^\infty(\mathbb{R}^2) \text{ is radial, } 0 \le g \le \frac{1}{4}, \\ g > 0 \text{ on } A\left(\frac{1}{4}, \frac{3}{4}\right) =: S, \text{ and } g = 0 \text{ on } \mathbb{R}^2 \setminus S. \end{cases}$$

Here, $A(r, R) \subset \mathbb{R}^2$ is the standard open annulus with center 0 between the radii r and R.

It is clear that u is neither convex nor $H-$convex, while $u = v$ on $\partial\Omega$ and $u \le v$ in Ω. We shall prove that

$$B_{\mathbb{R}^2}(0, 1/4) \subset \partial_H v(\Omega) \setminus \partial_H u(\Omega). \tag{5.1}$$

First of all, since v is regular and $H-$convex, for every $\xi = (x, y, t) \in \Omega$ one has

$$\partial_H v(\xi) = \{(X_1 v(\xi), Y_1 v(\xi))\} = \{(2y, -2x)\}.$$

Therefore, $\partial_H v(\Omega) = B_{\mathbb{R}^2}(0, 2)$.

Now, we show that $\partial_H u(\xi) = \emptyset$ for every $\xi = (x, y, t) \in \Omega$ with $(x, y) \in B_{\mathbb{R}^2}(0, 1/4)$. By contradiction, if $p_0 \in \partial_H u(\xi_0)$ for some $\xi_0 = (x_0, y_0, t_0) \in \Omega$ with $(x_0, y_0) \in B_{\mathbb{R}^2}(0, 1/4)$, one has in particular that

$$u(\xi) \ge u(\xi_0) + p_0 \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)), \qquad \xi \in \Omega \cap H_{\xi_0} \cap H_{(0,0,t_0)} := L_0. \tag{5.2}$$

Note that $u(\xi) = u(\xi_0) = t_0$ for every $\xi = (x, y, t) \in L_0$ with $(x, y) \notin S$; thus, by (5.2) it follows that $p_0 \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)) = 0$ for every $\xi \in L_0$. Now, if we consider $\xi = (x, y, t) \in L_0$ such that $(x, y) \in S$, then (5.2) yields the contradiction $t_0 > t_0 - (1 - t_0^2)g(x, y) = u(\xi) \ge u(\xi_0) = t_0$. This proves that $\partial_H u(\xi_0) = \emptyset$.

Finally, we study $\partial_H u(\xi)$ for $\xi = (x, y, t) \in \Omega$ such that $(x, y) \notin B_{\mathbb{R}^2}(0, 1/4)$. Since u is smooth in Ω, if $\partial_H u(\xi) \ne \emptyset$, then $\partial_H u(\xi) = \{\nabla_H u(\xi)\}$: hence one has

$$X_1 u = -(1 - t^2)g_x(x, y) + 2y(1 + 2tg(x, y)), \quad Y_1 u = -(1 - t^2)g_y(x, y) - 2x(1 + 2tg(x, y)).$$

Since g is radial, we have $g(x, y) = g(r)$ with $r = \sqrt{x^2 + y^2}$, thus for $\xi = (x, y, t) \in \Omega$, we have

$$(X_1 u(\xi))^2 + (Y_1 u(\xi))^2 = (1 - t^2)^2 g'(r)^2 + 4r^2(1 + 2tg(r))^2. \tag{5.3}$$

Now, for every $\xi = (x, y, t) \in \Omega$ such that $\partial_H u(\xi) \ne \emptyset$ and $(x, y) \notin B_{\mathbb{R}^2}(0, 1/4)$, since $0 \le g \le 1/4$, we have

$$(X_1 u(\xi))^2 + (Y_1 u(\xi))^2 \ge \frac{1}{16}.$$

Consequently, $\partial_H u(\Omega) \cap B_{\mathbb{R}^2}(0, 1/4) = \emptyset$, which proves the claim.

Remark 5.1 We cannot expect even to have $\mathcal{L}^2_{HS}(\partial_H v(\Omega)) \leq \mathcal{L}^2_{HS}(\partial_H u(\Omega))$ for functions u and v with $u = v$ on $\partial\Omega$ and $u \leq v$ in Ω in the absence of convexity. Indeed, with respect to the previous example we assume in addition that $|g'| \leq c$ and $0 \leq g \leq c$ for some $c > 0$. While $\mathcal{L}^2_{HS}(\partial_H v(\Omega)) = \mathcal{L}^2(\partial_H v(\Omega)) = 4\pi$, by relations (5.1) and (5.3) we have

$$\mathcal{L}^2_{HS}(\partial_H u(\Omega)) \leq \mathcal{L}^2(\partial_H u(\Omega)) \leq \left(c^2 + 4(1+2c)^2 - \frac{1}{16}\right)\pi$$

which is smaller than 4π for $c > 0$ sufficiently small.

5.2 Sharpness of the Aleksandrov-type maximum principle

In this subsection we shall study the sharpness of the Aleksandrov-type maximum principle for the first Heisenberg group $\mathbb{H}^1$. More precisely, under the assumptions of Theorem 1.3, let us assume that for some $s \geq 1$ we have

$$|u(\xi)|^2 \leq C_1 \mathrm{dist}_H(\xi, \partial\Omega)^s \ \mathrm{diam}_{HS}(\Omega)\mathcal{L}^2_{HS}(\partial_H u(\Omega)), \qquad \forall \xi \in \Omega. \tag{A_s}$$

Theorem 5.1 *(A_1) is sharp, i.e., the exponent s in (A_s) cannot be greater than* 1.

Proof. By Theorem 1.3, (A_1) holds for every horizontally bounded, open and convex set $\Omega \subset \mathbb{H}^1$ and every continuous $H-$convex function $u : \overline{\Omega} \to \mathbb{R}$ which verifies $u = 0$ on $\partial\Omega$.

Let $\varepsilon \in (0,1)$ be arbitrarily fixed. Our claim is proved once we construct a bounded, convex domain Ω and a function $u : \overline{\Omega} \to \mathbb{R}$ with the above properties such that $\mathcal{L}^2_{HS}(\partial_H u(\Omega)) < \infty$, and

$$\sup_{\xi \in \Omega} \frac{|u(\xi)|^2}{\mathrm{dist}_H(\xi, \partial\Omega)^{1+\varepsilon}} = +\infty. \tag{5.4}$$

To do this, let us choose $\alpha < 1$ and $\beta > 1$ such that

$$\alpha = \frac{2\beta}{4\beta - 1} + \frac{\epsilon}{4} < \frac{1}{2} + \frac{\epsilon}{2}. \tag{5.5}$$

With these choices of α and β, we consider the domain

$$\Omega_+ := \left\{(x,y,t) \in \mathbb{H}^1 : \ x \in (0,1], \ (y^2+t^2)^\beta - x^\alpha < 0\right\},$$

and its reflection over the plane $x = 1$ defined as

$$\Omega_- := \left\{(x,y,t) \in \mathbb{H}^1 : \ (2-x,y,t) \in \Omega_+\right\}. \tag{5.6}$$

We shall define the functions $u_\pm : \overline{\Omega}_\pm \to \mathbb{R}$ as

$$u_+(x,y,t) := (y^2+t^2)^\beta - x^\alpha \quad \text{and} \quad u_-(x,y,t) := u_+(2-x,y,t). \tag{5.7}$$

Finally, let Ω be the open and convex set $\Omega = \Omega_+ \cup \Omega_-$; we define $u : \overline{\Omega} \to \mathbb{R}$ by

$$u(x,y,t) = u_\pm(x,y,t) \qquad \text{for} \ \ (x,y,t) \in \overline{\Omega}_\pm. \tag{5.8}$$

By definition, it is immediate that $u \in C(\overline{\Omega})$ is a convex function such that $u = 0$ on $\partial\Omega$ and $u < 0$ in Ω. Moreover, $u \in \Gamma^\infty(\mathrm{int}(\Omega_+))$ and according to Theorem 2.1, for every $\xi = (x, y, t) \in \mathrm{int}(\Omega_+)$ we have that

$$\begin{aligned} \partial_H u_+(\xi) &= \{\nabla_H u_+(\xi)\} = \{(X_1 u_+(\xi), Y_1 u_+(\xi))\} \\ &= \Big\{\Big(-\alpha x^{\alpha-1} + 4\beta yt(y^2+t^2)^{\beta-1}, 2\beta y(y^2+t^2)^{\beta-1} - 4\beta xt(y^2+t^2)^{\beta-1}\Big)\Big\}. \end{aligned} \tag{5.9}$$

Similarly, for every $\xi = (x, y, t) \in \mathrm{int}(\Omega_-)$ we have that

$$\partial_H u_-(\xi) = \Big\{\Big(\alpha(2-x)^{\alpha-1} + 4\beta yt(y^2+t^2)^{\beta-1}, 2\beta y(y^2+t^2)^{\beta-1} - 4\beta xt(y^2+t^2)^{\beta-1}\Big)\Big\}. \tag{5.10}$$

For every $\xi = (x, y, t) \in \mathrm{int}(\Omega_+)$ with $0 < x \le \frac{\alpha}{2\beta}$ we have

$$\begin{aligned} |X_1 u_+(\xi)| &\le \alpha x^{\alpha-1} + 2\beta x^\alpha \le 2\alpha x^{\alpha-1}, \\ |Y_1 u_+(\xi)| &\le 2\beta x^{\alpha\cdot\frac{\beta-1}{\beta}}|y - 2xt| \le 6\beta x^{\frac{\alpha}{\beta}\cdot(\beta-\frac{1}{2})}. \end{aligned}$$

We deduce that

$$\partial_H u_+(\mathrm{int}(\Omega_+)) \subseteq A_1 \cup A_2,$$

where

$$A_1 = X_1 u_+\Big([\alpha/(2\beta), 1], [-1,1], [-1,1]\Big) \times Y_1 u_+\Big([\alpha/(2\beta), 1], [-1,1], [-1,1]\Big)$$

and

$$A_2 = \Big\{(-v, w) : v \in [\gamma, \infty), |w| \le C v^{(\beta-\frac{1}{2})\cdot\frac{1}{\alpha-1}\cdot\frac{\alpha}{\beta}}\Big\},$$

where γ and C are positive constants. Clearly, the measure of A_1 is finite while for A_2, we have

$$\mathcal{L}^2(A_2) = C\int_\gamma^\infty v^{(\beta-\frac{1}{2})\cdot\frac{1}{\alpha-1}\cdot\frac{\alpha}{\beta}}\,dv$$

that converges if and only if $\alpha > \frac{2\beta}{4\beta-1}$. According to our choice from (5.5) the above condition holds, proving that $\mathcal{L}^2(\partial_H u_+(\mathrm{int}(\Omega_+))) < \infty$. The fact that $\mathcal{L}^2(\partial_H u_-(\mathrm{int}(\Omega_-))) < \infty$ works similarly. Moreover, if $\xi = (x, y, t) \in \Omega_+ \cap \Omega_-$, then $x = 1$ and $\partial_H u(\xi)$ is not a singleton: more precisely, taking into account (5.9) and (5.10), we have that

$$\partial_H u(\xi) = \Big[-\alpha + 4\beta yt(y^2+t^2)^{\beta-1}, \alpha + 4\beta yt(y^2+t^2)^{\beta-1}\Big] \times Y_1 u_+(\xi)$$

that implies $\partial_H u(\xi) \subset [-\alpha - 2\beta, \alpha + 2\beta] \times [-6\beta, 6\beta]$. Therefore,

$$\mathcal{L}^2_{HS}(\partial_H u(\Omega)) \le \mathcal{L}^2(\partial_H u(\Omega)) = \mathcal{L}^2(\partial_H u_+(\mathrm{int}(\Omega_+))) + \mathcal{L}^2(\partial_H u_-(\mathrm{int}(\Omega_-))) + \mathcal{L}^2(\partial_H u(\Omega_+ \cap \Omega_-)) < \infty.$$

Let us choose $(0,0,0) \in \partial\Omega$ and $\xi = (x, 0, 0) \in \Omega$ such that $x \to 0^+$. Since $\mathrm{dist}_H(\xi, \partial\Omega)$ is comparable to $x > 0$ and $2\alpha < 1 + \varepsilon$ (cf. (5.5)), it follows that

$$\frac{|u(\xi)|^2}{\mathrm{dist}_H(\xi,\partial\Omega)^{1+\varepsilon}} = \frac{x^{2\alpha}}{\mathrm{dist}_H(\xi,\partial\Omega)^{1+\varepsilon}} \sim x^{2\alpha-1-\varepsilon} \to +\infty \quad \text{as } x \to 0^+,$$

concluding the proof of (5.4). $\square$

Remark 5.2 Instead of (5.5), let us choose the parameters α and β as

$$\alpha = \frac{\beta}{3\beta - 1} + \frac{\epsilon}{4} < \frac{1}{3} + \frac{\epsilon}{3},$$

for some $\varepsilon \in (0, 1)$. Then, the domain and function introduced in Theorem 5.1 can be used to prove the sharpness of the Aleksandrov-type maximum principle in the Euclidean case $\mathbb{R}^3$ as well (see relation (1.1) for $n = 3$), i.e.,

$$|u(\xi)|^3 \leq C_1 \text{dist}(\xi, \partial\Omega) \ \text{diam}(\Omega)^2 \mathcal{L}^2(\partial u(\Omega)), \qquad \forall \xi \in \Omega.$$

The details are left as an exercise to the interested reader.

5.3 Horizontal Monge-Ampère operator versus horizontal normal mapping

Let $\Omega \subset \mathbb{H}^1$ be an open, bounded and convex set. We consider the horizontal Monge-Ampère operator

$$\mathcal{S}_{ma}(u)(\xi) = \det[\text{Hess}_H(u)(\xi)]^* + 12(Tu(\xi))^2, \tag{5.11}$$

where $u \in C^2(\Omega)$ and $[\text{Hess}_H(u)(\xi)]^*$ is the symmetrized horizontal Hessian:

$$[\text{Hess}_H(u)(\xi)]^* = \begin{bmatrix} X_1^2 u & (X_1Y_1u + Y_1X_1u)/2 \\ (X_1Y_1u + Y_1X_1u)/2 & Y_1^2 u \end{bmatrix} (\xi), \ \ \xi \in \Omega.$$

Having in our mind relation (1.2) from the Euclidean case, we are interested to study the connection between the quantities $\int_\Omega \mathcal{S}_{ma}(u)(\xi)d\xi$ and $\mathcal{L}^2(\partial_H u(\Omega))$ (or $\mathcal{L}^2_{HS}(\partial_H u(\Omega))$) whenever $u \in C^2(\Omega)$ is an $H-$convex function. Some initial information are available as follows:

- In [9] the authors prove that

$$\int_{\partial_H u(\Omega)} \mathcal{S}^2_{\mathbb{H}}(\{\xi \in \Omega : \nabla_H u(\xi) = v\}) \, dv = \int_\Omega \left(\det[\text{Hess}_H(u)(\xi)]^* + 4(Tu(\xi))^2\right) d\xi,$$

  where $\mathcal{S}^2_{\mathbb{H}}$ denotes the 2–dimensional spherical Hausdorff measure. Note that if $u \in \Gamma^2(\Omega)$ is $H-$convex, the matrix $[\text{Hess}_H(u)(\xi)]^*$ is positive semi-definite for every $\xi \in \Omega$ (see Danielli, Garofalo and Nhieu [14]), thus the latter integral and $\int_\Omega \mathcal{S}_{ma}(u)(\xi)d\xi$ are comparable.

- By the oscillation estimate of Gutiérrez and Montanari [18, Theorem 1.4], we know that for any compact domain $A \subset \Omega$ there exists a constant $C = C(A, \Omega) > 0$ such that

$$\int_A \mathcal{S}_{ma}(u)(\xi)d\xi \leq C(\sup_\Omega u - \inf_\Omega u)^2$$

  for every $H-$convex function $u \in C^2(\Omega)$. By combining this result with our Aleksandrov-type maximum principle in (1.6), one has that for every compact set $A \subset \Omega$ and for every $H-$convex function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ with $u = 0$ on $\partial\Omega$,

$$\int_A \mathcal{S}_{ma}(u)(\xi)d\xi \leq C_1 C \text{diam}_{HS}(\Omega) \mathcal{L}^2_{HS}(\partial_H u(\Omega)).$$

Clearly, if $\int_\Omega \mathcal{S}_{ma}(u)(\xi)d\xi$ were comparable to $\mathcal{L}^2_{HS}(\partial_H u(\Omega))$, then our Aleksandrov-type maximum principle would provide an estimate of the form

$$|u(\xi_0)|^2 \leq C\text{dist}_H(\xi_0, \partial\Omega)\text{diam}_{HS}(\Omega)\int_\Omega \mathcal{S}_{ma}(u)(\xi)d\xi, \qquad \xi_0 \in \Omega.$$

Unfortunately, this turns out to be only a wishful thinking as shown by the following:

Proposition 5.1 *There exists an open, bounded and convex set* $\Omega \subset \mathbb{H}^1$, *and an* $H-$*convex function* $u : \overline{\Omega} \to \mathbb{R}$ *with* $u \in C(\overline{\Omega}) \cap C^2(\Omega)$, $u = 0$ *on* $\partial\Omega$, *such that*

(i) $\mathcal{L}^2_{HS}(\partial_H u(\Omega)) = \infty$;

(ii) $\int_\Omega \mathcal{S}_{ma}(u)(\xi)d\xi < \infty$.

Proof. The construction is similar to (5.7) and (5.8). More precisely, let us consider $\beta > 1$ with

$$\frac{1}{2} < \alpha \leq \frac{2\beta}{4\beta - 1},$$

the new domain

$$\Omega_+ := \left\{\xi = (x, y, t) \in \mathbb{H}^1 : \ x \in (0, 1], \ (y^2 + t^2)^\beta - x^\alpha + \frac{\alpha}{2}x^2 < 0\right\},$$

and its reflection Ω_- over the plane $x = 1$ defined as in (5.6). The functions $u_\pm : \overline{\Omega}_\pm \to \mathbb{R}$ are defined as

$$u_+(x, y, t) := (y^2 + t^2)^\beta - x^\alpha + \frac{\alpha}{2}x^2 \quad \text{and} \quad u_-(x, y, t) := u_+(2 - x, y, t).$$

Let $\Omega = \Omega_+ \cup \Omega_-$, which is an open and convex set; we define $u : \overline{\Omega} \to \mathbb{R}$ in the same way as in (5.8). It is a straighforward computation to see that $u \in C(\overline{\Omega})$, and $u \in C^2(\Omega)$ since

$$\frac{\partial u_+}{\partial x}(1, y, t) = \frac{\partial u_-}{\partial x}(1, y, t) = 0 \qquad \text{and} \qquad \frac{\partial^2 u_+}{\partial x^2}(1, y, t) = \frac{\partial^2 u_-}{\partial x^2}(1, y, t) = -\alpha^2 + 2\alpha.$$

Moreover, u is a convex function on Ω such that $u = 0$ on $\partial\Omega$ and $u < 0$ in Ω.

(i) First of all, note that

$$\mathcal{L}^2_{HS}(\partial_H u(\Omega)) \geq \limsup_{k\to\infty} \mathcal{L}^2\left(\partial_H u\left(A_+ \cap H_{(\frac{1}{2k},0,0)}\right)\right),$$

where $A_+ = \{\xi = (x, y, t) \in \text{int}(\Omega_+) : \ y \geq 0)\}$. Since u_+ is regular and $H-$convex in $\text{int}(\Omega_+)$, we have

$$\begin{aligned}\partial_H u(\xi) &= \{\nabla_H u_+(\xi)\} = \{(X_1 u_+(\xi), Y_1 u_+(\xi))\} \\ &= \left\{\left(-\alpha x^{\alpha-1} + \alpha x + 4\beta yt(y^2 + t^2)^{\beta-1}, 2\beta(y - 2xt)(y^2 + t^2)^{\beta-1}\right)\right\}, \ \xi \in \text{int}(\Omega_+).\end{aligned}$$

Therefore, for every $k \geq 1$, one has

$$\begin{aligned}
S_k &:= \partial_H u\left(A_+ \cap H_{(\frac{1}{2k},0,0)}\right) \\
&= \Bigg\{\left(\alpha x(1-x^{\alpha-2}) - \frac{4\beta}{k}y^{2\beta}\left(1+\frac{1}{k^2}\right)^{\beta-1}, 2\beta y^{2\beta-1}\left(1+\frac{2x}{k}\right)\left(1+\frac{1}{k^2}\right)^{\beta-1}\right): \\
&\qquad 0 < x < 1,\ 0 \leq y < \left(x^\alpha - \frac{\alpha}{2}x^2\right)^{\frac{1}{2\beta}}\left(1+\frac{1}{k^2}\right)^{-1/2}\Bigg\} \\
&\supset \Bigg\{\left(\alpha x(1-x^{\alpha-2}) - \frac{4\beta}{k}y^{2\beta}\left(1+\frac{1}{k^2}\right)^{\beta-1}, 2\beta y^{2\beta-1}\left(1+\frac{2x}{k}\right)\left(1+\frac{1}{k^2}\right)^{\beta-1}\right): \\
&\qquad 0 < x < 1,\ 0 \leq y < \left(\frac{x^\alpha}{2^{\beta+1}}\right)^{\frac{1}{2\beta}}\Bigg\} \\
&\supset \Bigg\{\left(\alpha x(1-x^{\alpha-2}) - \frac{4\beta}{k}y^{2\beta}\left(1+\frac{1}{k^2}\right)^{\beta-1}, 2\beta y^{2\beta-1}\left(1+\frac{2x}{k}\right)\left(1+\frac{1}{k^2}\right)^{\beta-1}\right): \\
&\qquad 0 \leq y \leq 2^{-\frac{\beta+1}{2\beta}},\ (2^{\beta+1}y^{2\beta})^{\frac{1}{\alpha}} < x < 1\Bigg\}.
\end{aligned}$$

By the Fatou lemma, we have that

$$\liminf_{k\to\infty} \mathcal{L}^2(S_k) \geq \mathcal{L}^2(S),$$

where

$$S = \left\{\left(\alpha x(1-x^{\alpha-2}), 2\beta y^{2\beta-1}\right) : 0 \leq y \leq 2^{-\frac{\beta+1}{2\beta}},\ (2^{\beta+1}y^{2\beta})^{\frac{1}{\alpha}} < x < 1\right\}.$$

On the other hand, we have that

$$\mathcal{L}^2(S) \geq \int_0^\gamma \alpha 2^{\frac{\beta+1}{\alpha}}\left(\frac{s}{2\beta}\right)^{\frac{2\beta}{(2\beta-1)\alpha}}\left(-1 + 2^{\frac{(\alpha-2)(\beta+1)}{\alpha}}\left(\frac{s}{2\beta}\right)^{\frac{2\beta(\alpha-2)}{(2\beta-1)\alpha}}\right) ds,$$

where γ is a positive constant depending only on β. The latter integral is $+\infty$ since $\alpha \leq \frac{2\beta}{4\beta-1}$.

(ii) By symmetry, it is enough to prove the claim for u_+. Since $\int_{\Omega_+}(Tu_+)^2 d\xi < \infty$, by (5.11) we only need to consider the integral $\int_{\Omega_+} \det[\mathrm{Hess}_H(u_+)(\xi)]^* d\xi$ which is clearly finite if $\int_{\Omega_+}(X_1^2 u_+ Y_1^2 u_+)(\xi) d\xi < \infty$. The singular term in the integral is coming from

$$X_1^2 u_+(x,y,t) = -\alpha(\alpha-1)x^{\alpha-2} + \alpha + 8\beta y^2 \frac{\partial}{\partial t}\left(t(y^2+t^2)^{\beta-1}\right).$$

Calculating the term $Y_1^2 u_+$, since $0 < x < 1$, we obtain

$$|Y_1^2 u_+(x,y,t)| \leq C(y^2+t^2)^{\beta-1},$$

for some constant $C = C(\beta) > 0$. Using integration in polar coordinates in the $(y,t)-$plane, we have

$$\int_{\Omega_+} |X_1^2 u_+ Y_1^2 u_+|(\xi) d\xi \leq C' \int_0^1 x^{\alpha-2} \int_0^{x^{\frac{\alpha}{2\beta}}} r^{2\beta-1} dr dx + C' = \frac{C'}{2\beta} \int_0^1 x^{2\alpha-2} dx + C',$$

for some constant $C' = C'(\alpha,\beta) > 0$. Since $\alpha > \frac{1}{2}$, the above integral converges. □

Remark 5.3 Unlike in the Euclidean case (see relation (1.2) versus Proposition 5.1), the horizontal normal mapping does not play the same role as the Euclidean normal mapping in the study of the Monge-Ampère equation via the operator $\mathcal{S}_{ma}$ given by (5.11). Furthermore, if $\Omega \subset \mathbb{H}^n$ is an open, bounded and convex set, and $u : \overline{\Omega} \to \mathbb{R}$ is a continuous $H-$convex function, we may consider for every $E \subset \Omega$ the function $\nu_u(E) = \mathcal{L}^{2n}_{HS}(\partial_H u(E))$, which is a natural candidate for the Monge-Ampère measure in the Heisenberg setting. This defines an outer measure, however ν_u is not a Borel measure in general. Indeed, let $\Omega \subset \mathbb{H}^1$ be the cylinder introduced in §5.1 and let $D_i = \{(x,y,t) \in \Omega : t = t_i\}$, $i \in \{1,2\}$, be two discs with $-1 < t_1 < t_2 < 1$. If $u(x,y,t) = t$, then $\nu_u(D_1 \cup D_2) = \nu_u(D_1) = \nu_u(D_2) = 4\pi$, i.e., the additivity on Borel sets of ν_u fails.

6 Appendix

6.1 Degree theory for set-valued maps

We recall some facts from the degree theory for upper semicontinuous set-valued maps, see Hu and Papageorgiou [20]. Note that the degree theory developed in [20] is also valid for infinite-dimensional spaces, which is a generalization of the Brouwer, Browder and Leray-Schauder degree theories. In our context, it is enough to consider the finite-dimensional version.

Let us start with the definition of Brouwer degree $\deg_B$ for a continuous function:

Theorem 6.1 (see [22]) *Let*

$$M = \left\{(f,U,y) :\ U \subset \mathbb{R}^n \text{ open and bounded},\ f \in C(\overline{U}, \mathbb{R}^n),\ y \in \mathbb{R}^n \backslash f(\partial U)\right\}.$$

There exists a function, called the Brouwer degree, $\deg_B : M \to \mathbb{Z}$, *that satisfies the following properties:*

- *if* $\deg_B(f,U,y) \neq 0$, *then there exists* $x \in U$ *such that* $f(x) = y$;
- $\deg_B(Id,U,y) = 1$ *if* $y \in U$;
- *if* $\mathcal{F} : [0,1] \times \overline{U} \to \mathbb{R}^n$ *is a homotopy such that* $y \in \mathbb{R}^n \backslash \mathcal{F}([0,1] \times \partial U)$, *then* $t \mapsto \deg_B(\mathcal{F}(t,\cdot),U,y)$ *is constant:*
- $\deg_B(f,U,y) = \deg_B(f-y,U,0)$.

In order to work with the degree of set-valued maps, we need the following notion.

Definition 6.1 (see [20, Definition 3]) *Let X be a finite-dimensional normed space and $U \subset X$ be an open bounded set. A set-valued map $F : \overline{U} \to 2^X \setminus \{\emptyset\}$ is said to belong to the* class (P) *if:*

(i) *it maps bounded sets into relatively compact sets;*

(ii) *for every $x \in \overline{U}$, $F(x)$ is closed and convex in X;*

(iii) *F is upper semicontinuous on $\overline{U}$.*

A parameter-depending version of Definition 6.1 reads as follows, which will be used to exploit homotopy properties of certain set-valued maps.

Definition 6.2 (see [20, Definition 9]) *Let X be a finite-dimensional normed space and $U \subset X$ be an open bounded set. A one-parameter family of set-valued maps $\mathcal{F}_\lambda : \overline{U} \to 2^X \setminus \{\emptyset\}$, $\lambda \in [0, 1]$ is said to be a* homotopy of class (P) *if:*

(i) *$\overline{\{\cup \mathcal{F}_\lambda(x) : (\lambda, x) \in [0, 1] \times \overline{U}\}}$ is compact in X;*

(ii) *for every $(\lambda, x) \in [0, 1] \times \overline{U}$, $\mathcal{F}_\lambda(x)$ is closed and convex in X;*

(iii) *$(\lambda, x) \mapsto \mathcal{F}_\lambda(x)$ is upper semicontinuous from $[0, 1] \times \overline{U}$ into $2^X \setminus \{\emptyset\}$.*

For the set-valued degree of upper semicontinuous set-valued map certain selectors are needed:

Proposition 6.1 (see [12]) *If X, V are Banach spaces, $U \subset X$ is an open bounded set and $F : \overline{U} \to 2^V \setminus \{\emptyset\}$ is an upper semicontinuous set-valued map with closed and convex values then for every $\varepsilon > 0$ there exists a* continuous approximate selector *$f_\varepsilon : \overline{U} \to V$ such that*

$$f_\varepsilon(y) \in F((y + B_X(0, \varepsilon)) \cap \overline{U}) + B_V(0, \varepsilon), \quad \forall y \in \overline{U}.$$

The next result is a set-valued version of Theorem 6.1 and it plays a fundamental role in our degree theoretical argument from Section 3.

Theorem 6.2 (see [20, Definition 11 and Theorem 12]) *Let X be a finite-dimensional normed space. Let*

$$M_{SV} = \left\{ (F, U, y) : \begin{array}{l} U \subset X \text{ open and bounded}, \\ F : \overline{U} \to 2^X \setminus \{\emptyset\} \text{ belongs to the class (P)}, \ y \in X \setminus F(\partial U) \end{array} \right\}.$$

There exists a function, called as set-valued degree function, *$\deg_{SV} : M_{SV} \to \mathbb{Z}$, that is defined as the common value*

$$\deg_{SV}(F, U, y) = \deg_B(f_\varepsilon, U, y)$$

for every small $\varepsilon > 0$, where f_ε comes from Proposition 6.1. The function $\deg_{SV}$ verifies the properties of

- normalization*: $\deg_{SV}(Id, U, y) = \deg_B(Id, U, y) = 1$ for all $y \in U$;*
- additivity on domain*: If $U_1, U_2 \subset U$ are disjoint open sets and $y \notin F(\overline{U} \setminus (U_1 \cup U_2))$, then*

$$\deg_{SV}(F, U, y) = \deg_{SV}(F, U_1, y) + \deg_{SV}(F, U_2, y);$$

- homotopy invariance*: if $\mathcal{F}_\lambda : \overline{U} \to 2^X$ is a homotopy of class (P) and $\gamma : [0, 1] \to X$ is such that $\gamma(\lambda) \notin \mathcal{F}_\lambda(\partial U)$ for all $\lambda \in [0, 1]$, then $\deg_{SV}(\mathcal{F}_\lambda, U, \gamma(\lambda))$ is independent of $\lambda \in [0, 1]$.*

6.2 Quantitative Harnack-type inequality for $H-$convex functions

Lemma 6.1 *Let Ω be an open convex domain such that $B_H(0, cR) \subset \Omega$ for some constants $c, R > 0$. Let $u : \overline{\Omega} \to \mathbb{R}$ be an $H-$convex function with $u \leq 0$ in Ω. Let $\xi_1, \xi_2 \in B_H(0, cR)$ with $\xi_2 \in H_{\xi_1}$ and some constants $c_1, c_2 \geq 0$ and $c_3 > 0$ such that*

$$N(\xi_1) \leq c_1 R;\; N(\xi_2) \leq c_2 R,\; d_H(\xi_1, \xi_2) \leq c_3 R$$

and

$$c_1 + c_3 < c;\; c_2 + c_3 < c.$$

Then

$$\frac{c - c_1 - c_3}{c - c_1} u(\xi_1) \geq u(\xi_2) \geq \frac{c - c_2}{c - c_2 - c_3} u(\xi_1).$$

Proof. The idea of the proof is close to Lemma 5.2 from Gutiérrez and Montanari [18]. Let $\xi'_\lambda = \xi_1 \circ \delta_\lambda(\xi_1^{-1} \circ \xi_2) \in H_{\xi_1}$ for $\lambda > 0$. If $\xi'_\lambda \in \partial B_H(0, cR)$, then we have that

$$\begin{aligned} cR &= N(\xi'_\lambda) = N(\xi_1 \circ \delta_\lambda(\xi_1^{-1} \circ \xi_2)) \\ &\leq N(\xi_1) + N(\delta_\lambda(\xi_1^{-1} \circ \xi_2)) \\ &= N(\xi_1) + \lambda N(\xi_1^{-1} \circ \xi_2) \\ &\leq c_1 R + \lambda c_3 R. \end{aligned}$$

Therefore,

$$\lambda \geq \frac{c - c_1}{c_3} > 1.$$

Now, the relation $\xi'_\lambda = \xi_1 \circ \delta_\lambda(\xi_1^{-1} \circ \xi_2)$ can be written into the form $\xi_2 = \xi_1 \circ \delta_{1/\lambda}(\xi_1^{-1} \circ \xi'_\lambda)$. The $H-$convexity of u and the fact that $u \leq 0$ yields that

$$u(\xi_2) \leq \left(1 - \frac{1}{\lambda}\right) u(\xi_1) + \frac{1}{\lambda} u(\xi'_\lambda) \leq \left(1 - \frac{1}{\lambda}\right) u(\xi_1).$$

Consequently,

$$u(\xi_2) \leq \left(1 - \frac{1}{\lambda}\right) u(\xi_1) \leq \left(1 - \frac{c_3}{c - c_1}\right) u(\xi_1) = \frac{c - c_1 - c_3}{c - c_1} u(\xi_1).$$

Now, changing the roles of ξ_1 and ξ_2, by taking into account that $\xi_2 \in H_{\xi_1}$ (thus, $\xi_1 \in H_{\xi_2}$), we obtain in a similar manner that

$$u(\xi_1) \leq \frac{c - c_2 - c_3}{c - c_2} u(\xi_2),$$

which ends the proof. □

Theorem 6.3 (Harnack-type inequality) *Let $\Omega \subset \mathbb{H}^n$ be an open, horizontally bounded and convex set. If $u : \overline{\Omega} \to \mathbb{R}$ is an $H-$convex function with $u = 0$ on $\partial\Omega$, and $B_H(\xi_0, 3R) \subset \Omega$ for some $\xi_0 \in \Omega$ and $R > 0$, then*

$$\frac{1}{31} u(\xi) \geq u(\zeta) \geq 31 u(\xi), \qquad \forall \xi, \zeta \in B_H(\xi_0, R). \tag{6.1}$$

Proof. The proof is similar to Gutiérrez and Montanari [18, Proposition 5.3]. After a left-translation by ξ_0^{-1}, it is enough to prove (6.1) for every $\xi, \zeta \in B_H(0,R)$.

By the first part of Proposition 2.2 one has that $u \leq 0$ on $\overline{\Omega}$. Let us fix $\xi = (x_0, y_0, t_0) \in B_H(0,R)$ arbitrarily, i.e., $N(\xi) \leq R$, with $x_0 = (x_1^0, ..., x_n^0) \in \mathbb{R}^n$ and $y_0 = (y_1^0, ..., y_n^0) \in \mathbb{R}^n$. In particular, we have that $\sqrt{|t_0|} \leq R$. For simplicity, we assume that $t_0 \geq 0$ (the case $t_0 < 0$ works similarly).

Step 1. Let $\xi_1 = \exp\left(-\sum_{j=1}^n (x_j^0 X_j + y_j^0 Y_j)\right) \circ \xi = (0_n, 0_n, t_0) \in H_\xi$. It is clear that

$$N(\xi) \leq R;\ N(\xi_1) = \sqrt{t_0} \leq R;\ d_H(\xi, \xi_1) = \sqrt{|x_0|^2 + |y_0|^2} \leq R.$$

Thus, we may apply Lemma 6.1 with $c_1 = c_2 = c_3 = 1$ and $c = 3$, obtaining

$$\frac{1}{2}u(\xi) \geq u(\xi_1) \geq 2u(\xi).$$

Step 2. Let $\xi_2 = \exp\left(\sigma \sum_{j=1}^n X_j\right) \circ \xi_1 = (\sigma_n, 0_n, t_0) \in H_{\xi_1}$, where

$$\sigma = \frac{\sqrt{t_0}}{2\sqrt{n}}.$$

Note that

$$N(\xi_1) \leq R;\ N(\xi_2) = (n^2\sigma^4 + t_0^2)^{\frac{1}{4}} = 17^{\frac{1}{4}}\sigma\sqrt{n} \leq \frac{17^{\frac{1}{4}}}{2}R;\ d_H(\xi_1, \xi_2) = \sigma\sqrt{n} = \frac{\sqrt{t_0}}{2} \leq \frac{R}{2}.$$

Therefore, we apply Lemma 6.1 with $c_1 = 1$, $c_2 = \frac{17^{\frac{1}{4}}}{2}$, $c_3 = \frac{1}{2}$ and $c = 3$, obtaining

$$\frac{\frac{3}{2}}{2}u(\xi_1) \geq u(\xi_2) \geq \frac{3 - \frac{17^{\frac{1}{4}}}{2}}{\frac{5}{2} - \frac{17^{\frac{1}{4}}}{2}}u(\xi_1).$$

Step 3. Let $\xi_3 = \exp\left(\sigma \sum_{j=1}^n Y_j\right) \circ \xi_2 = (\sigma_n, \sigma_n, t_0 - 2\sigma^2 n) \in H_{\xi_2}$. Note that

$$N(\xi_2) \leq \frac{17^{\frac{1}{4}}}{2}R;\ N(\xi_3) = \sqrt[4]{4\sigma^4 n^2 + (t_0 - 2\sigma^2 n)^2} = 8^{\frac{1}{4}}\sigma\sqrt{n} \leq \frac{8^{\frac{1}{4}}}{2}R;\ d_H(\xi_2, \xi_3) = \sigma\sqrt{n} \leq \frac{R}{2}.$$

Now, we apply Lemma 6.1 with $c_1 = \frac{17^{\frac{1}{4}}}{2}$, $c_2 = \frac{8^{\frac{1}{4}}}{2}$, $c_3 = \frac{1}{2}$ and $c = 3$, obtaining

$$\frac{\frac{5}{2} - \frac{17^{\frac{1}{4}}}{2}}{3 - \frac{17^{\frac{1}{4}}}{2}}u(\xi_2) \geq u(\xi_3) \geq \frac{3 - \frac{8^{\frac{1}{4}}}{2}}{\frac{5}{2} - \frac{8^{\frac{1}{4}}}{2}}u(\xi_2).$$

Step 4. Let $\xi_4 = \exp\left(-\sigma \sum_{j=1}^n X_j\right) \circ \xi_3 = (0_n, \sigma_n, t_0 - 4\sigma^2 n) = (0_n, \sigma_n, 0) \in H_{\xi_3}$. Note that

$$N(\xi_3) \leq \frac{8^{\frac{1}{4}}}{2}R;\ N(\xi_4) = \sigma\sqrt{n} \leq \frac{R}{2};\ d_H(\xi_3, \xi_4) = \sigma\sqrt{n} \leq \frac{R}{2}.$$

We apply Lemma 6.1 with $c_1 = \frac{8^{\frac{1}{4}}}{2}$, $c_2 = c_3 = \frac{1}{2}$ and $c = 3$, obtaining

$$\frac{\frac{5}{2} - \frac{8^{\frac{1}{4}}}{2}}{3 - \frac{8^{\frac{1}{4}}}{2}} u(\xi_3) \geq u(\xi_4) \geq \frac{\frac{5}{2}}{2} u(\xi_3).$$

Step 5. Let $\xi_5 = \exp\left(-\sigma \sum_{j=1}^n Y_j\right) \circ \xi_4 = (0_n, 0_n, 0) \in H_{\xi_4}$. Note that

$$N(\xi_4) \leq \frac{1}{2}R;\ N(\xi_5) = 0;\ d_H(\xi_4, \xi_5) = \sigma\sqrt{n} \leq \frac{R}{2}.$$

We may apply Lemma 6.1 with $c_1 = \frac{1}{2}$, $c_2 = 0$, $c_3 = \frac{1}{2}$ and $c = 3$, obtaining

$$\frac{2}{\frac{5}{2}} u(\xi_4) \geq u(\xi_5) = u(0) \geq \frac{3}{\frac{5}{2}} u(\xi_4).$$

By the Steps 1-5 we conclude that $u(\xi) = 0$ if and only if $u(0) = 0$. Therefore, if $u(0) = 0$, the arbitrariness of $\xi \in B_H(0, R)$ shows that $u \equiv 0$ in $B_H(0, R)$.

If $u(0) \neq 0$ then $u < 0$ in $B_H(0, R)$, and by multiplying the estimates from the above five steps, we have that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{\frac{5}{2} - \frac{17^{\frac{1}{4}}}{2}}{3 - \frac{17^{\frac{1}{4}}}{2}} \cdot \frac{\frac{5}{2} - \frac{8^{\frac{1}{4}}}{2}}{3 - \frac{8^{\frac{1}{4}}}{2}} \cdot \frac{4}{5} u(\xi) \geq u(0) \geq 2 \cdot \frac{3 - \frac{17^{\frac{1}{4}}}{2}}{\frac{5}{2} - \frac{17^{\frac{1}{4}}}{2}} \cdot \frac{3 - \frac{8^{\frac{1}{4}}}{2}}{\frac{5}{2} - \frac{8^{\frac{1}{4}}}{2}} \cdot \frac{5}{4} \cdot \frac{6}{5} u(\xi).$$

Repeating the above argument for another point $\zeta \in B_H(0, R)$ and combining the two estimates, it yields that

$$\tilde{c}^{-1} u(\xi) \geq u(\zeta) \geq \tilde{c} u(\xi),$$

where

$$\tilde{c} = 10 \cdot \left(\frac{3 - \frac{17^{\frac{1}{4}}}{2}}{\frac{5}{2} - \frac{17^{\frac{1}{4}}}{2}} \cdot \frac{3 - \frac{8^{\frac{1}{4}}}{2}}{\frac{5}{2} - \frac{8^{\frac{1}{4}}}{2}} \right)^2 \approx 30.26,$$

which concludes the proof. □

Proof of Proposition 2.2 (second part). Let $\xi_0 \in \Omega$ be such that $u(\xi_0) < 0$ and fix $\xi \in \Omega$ arbitrarily. Let $L = \{(1-\lambda)\xi_0 + \lambda\xi : \lambda \in [0, 1]\}$ be the Euclidean segment connecting these two points. From the convexity of Ω we conclude that the Euclidean tubular neighborhood around L with radius $0 < r < \min\{\text{dist}(\xi_0, \partial\Omega), \text{dist}(\xi, \partial\Omega)\}$, i.e., $N_L(r) = \{\xi \in \mathbb{H}^n : \text{dist}(\xi, L) < r\}$, is contained in Ω. [Here, 'dist' is the Euclidean distance.] Now, we consider the covering $\bigcup_{\zeta \in L} B_H(\zeta, R_\zeta)$ of the set L where $R_\zeta > 0$ is such that $B_H(\zeta, 3R_\zeta) \subset N_L(r)$ for every $\zeta \in L$. By the compactness of L, there exists $k \in \mathbb{N}$ such that $L \subset \bigcup_{i=1}^k B_H(\zeta_{\lambda_i}, R_{\zeta_{\lambda_i}})$ where $\zeta_{\lambda_i} = (1 - \lambda_i)\xi_0 + \lambda_i \xi$ with $0 \leq \lambda_1 < ... < \lambda_k \leq 1$. If $k = 1$, we are done by (6.1), obtaining that $0 > \frac{1}{31} u(\xi_0) \geq u(\xi)$. If $k \geq 2$, since $B_H(\zeta_{\lambda_i}, R_{\zeta_{\lambda_i}}) \cap B_H(\zeta_{\lambda_{i+1}}, R_{\zeta_{\lambda_{i+1}}}) \neq \emptyset$ for every $i = 1, ..., k-1$, we may repeatedly apply (6.1) on the balls $B_H(\zeta_{\lambda_i}, R_{\zeta_{\lambda_i}})$, by obtaining that $0 > \frac{1}{31^k} u(\xi_0) \geq u(\xi)$. □

References

[1] J.-P. Aubin, A. Cellina, Differential inclusions. Set-valued maps and viability theory. Springer-Verlag, Berlin, 1984.

[2] J.-P. Aubin, H. Frankowska, Set-Valued Analysis. Systems & Control: Foundations & Applications, 2. Birkhäuser Boston, Inc., Boston, MA, 1990.

[3] Z. M. Balogh, A. Calogero, A. Kristály, E. Vecchi, Aleksandrov-type estimates on Carnot groups, in preparation.

[4] Z. M. Balogh, M. Rickly, Regularity of convex functions on Heisenberg groups. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 2 (2003), no. 4, 847–868.

[5] M. Bardi, F. Dragoni, Convexity and semiconvexity along vector fields. *Calc. Var. Partial Differential Equations*, 42 (2011), 405–427.

[6] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Stratified Lie groups and potential theory for their sub-Laplacians. *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 2007.

[7] L. A. Caffarelli, Interior $W^{2,p}-$estimates for solutions of the Monge-Ampère equation. *Ann. Math.* 131 (1990), 135–150.

[8] L. A. Caffarelli, Some regularity properties of solutions of Monge-Ampère equation. *Comm. Pure Appl. Math.* 44 (1991), 965–969.

[9] A. Calogero, R. Pini, Horizontal normal map on the Heisenberg group. *J. Nonlinear Convex Anal.* 12 (2011), no. 2, 287–307.

[10] A. Calogero, R. Pini, c horizontal convexity on Carnot groups. *J. Convex Analysis* 19 (2012), no. 3, 541–567.

[11] L. Capogna, D. Maldonado, A note on the engulfing property and the $\Gamma^{1+\alpha}-$regularity of convex functions in Carnot groups. *Proc. Amer. Math. Soc.* 134 (2006), no. 11, 3191–3199.

[12] A. Cellina, Approximations of set-valued functions and fixed point theorems. *Ann. Mat. Pura Appl.* 82 (1969), 17–24.

[13] J. Cygan, Subadditivity of homogeneous norms on certain nilpotent Lie groups. *Proc. Amer. Math. Soc.* 83, (1981), 69–70.

[14] D. Danielli, N. Garofalo, D. M. Nhieu, Notions of convexity in Carnot groups. *Comm. Anal. Geom.* 11 (2003), no. 2., 263–341.

[15] D. Danielli, N. Garofalo, D. M. Nhieu, F. Tournier, The theorem of Busemann-Feller-Alexandrov in Carnot groups *Comm. Anal. Geom.* 12 (2004), no. 4., 853–886.

[16] N. Garofalo, F. Tournier, New properties of convex functions in the Heisenberg group. *Trans. Amer. Math. Soc.* 358 (2006), no. 5, 2011–2055.

[17] C. E. Gutiérrez, The Monge-Ampère Equation. Birkhäuser, Boston, MA, 2001.

[18] C. E. Gutiérrez, A. Montanari, Maximum and comparison principles for convex functions on the Heisenberg group. *Comm. Partial Differential Equations* 29 (2004), no. 9-10, 1305–1334.

[19] C. E. Gutiérrez, A. Montanari, On the second order derivatives of convex functions on the Heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 3 (2004), 349–366.

[20] S. Hu, N. S. Papageorgiou, Generalizations of Browder's degree theory. *Trans. Amer. Math. Soc.* 347 (1995), no. 1, 233–259.

[21] P. Juutinen, G. Lu, J. J. Manfredi, B. Stroffolini, Convex functions on Carnot groups. *Rev. Mat. Iberoam.* 23 (2007), 191–200.

[22] N. G. Lloyd, Degree Theory. Cambridge University Press, 1978.

[23] G. Lu, J. J. Manfredi, B. Stroffolini, Convex functions on the Heisenberg group. *Calc. Var. Partial Differential Equations*, 19 (2004), no. 1, 1–22.

[24] V. Magnani, Lipschitz continuity, Aleksandrov theorem and characterizations for $H-$convex functions. *Math. Ann.* 334 (2006), 199–233.

[25] V. Magnani, M. Scienza, Regularity estimates for convex functions in Carnot-Carathéodory spaces. *Preprint* (2012).

[26] M. Rickly, First order regularity of convex functions on Carnot groups. *J. Geom. Anal.* 16 (2006), no. 4, 679–702.

[27] R.T. Rockafellar, Convex Analysis. Princeton University Press, 1969.

[28] N. S. Trudinger, X.-J. Wang, Hessian measures I. *Topol. Methods Nonlinear Analysis,* 10 (1997), no. 2, 225–239.

[29] N. S. Trudinger, X.-J. Wang, Hessian measures II. *Ann. of Math.* (2) 150 (1999), no. 2, 579–604.

[30] N. S. Trudinger, X.-J. Wang, Hessian measures III. *J. Funct. Anal.* 193 (2002), no. 1, 1–23.

[31] N. S. Trudinger, W. Zhang, Hessian measures on the Heisenberg group. *J. Funct. Anal.* 264 (2013), no. 10, 2335–2355.

Mathematisches Institute, Universität Bern, Sidlerstrasse 5, 3012 Bern, Switzerland. Email: zoltan.balogh@math.unibe.ch

Dipartimento di Matematica e Applicazioni, Universitá di Milano Bicocca, Via Cozzi 53, 20125 Milano, Italy. Email: andrea.calogero@unimib.it

Department of Economics, Babeş-Bolyai University, 400591 Cluj-Napoca, Romania. Email: alexandru.kristaly@econ.ubbcluj.ro