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Explicit description of generic representations for quivers of type $\mathbb{A}_n$ or $\mathbb{D}_n$

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Abstract

We describe explicitly a generic representation for Dynkin quivers of type $\mathbb{A}_n$ or $\mathbb{D}_n$ for any dimension vector.

1 Introduction

Let Q be a Dynkin quiver of type $\mathbb{A}_n$ or $\mathbb{D}_n$, i.e. the underlying graph of Q is either

$$1 \overset{\alpha_1}{\text{———}} 2 \overset{\alpha_2}{\text{———}} 3 \qquad \cdots \qquad n-1 \overset{\alpha_{n-1}}{\text{———}} n$$

or

$$\begin{tikzcd} 1 \arrow[dr, dash, "\alpha_1"] & & & & \\ & 3 \arrow[r, dash, "\alpha_3"] & 4 & \cdots & n-1 \arrow[r, dash, "\alpha_{n-1}"] & n \\ 2 \arrow[ur, dash, "\alpha_2"'] & & & & \end{tikzcd}$$

We denote the category of finite dimensional representations of Q by $\operatorname{rep} Q$, and for a dimension vector $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ we consider the space $\operatorname{rep}(Q, \mathbf{d})$ of representations X of Q over an algebraically closed field k with $X(i) = k^{d_i}$ for any vertex i of Q. The affine algebraic group $\operatorname{Gl}(\mathbf{d}) = \prod_{i=1}^n \operatorname{Gl}(d_i)$ acts on the affine space

$$\operatorname{rep}(Q, \mathbf{d}) = \prod_{i=1}^{n-1} \operatorname{Mat}(d_{t\alpha_i} \times d_{s\alpha_i}, k)$$

by $(g \star X)(\alpha_i) = g(t\alpha_i)X(\alpha_i)g(s\alpha_i)^{-1}, i = 1, \cdots, n-1$, for $g = (g_1, \cdots, g_n) \in \operatorname{Gl}(\mathbf{d})$ and $X \in \operatorname{rep}(Q, \mathbf{d})$, where $s\alpha$ and $t\alpha$ denote the starting and terminating vertex of the arrow α. Note that orbits in $\operatorname{rep}(Q, \mathbf{d})$ are just isomorphism classes of representations.

As Q is of finite representation type [2], $\operatorname{rep}(Q, \mathbf{d})$ contains finitely many orbits and thus a unique dense orbit $\operatorname{Gl}(\mathbf{d})T$ with respect to the Zariski topology.

A representation in the dense orbit is called generic; it is unique up to isomorphism. By [5], T is generic if and only if $\mathrm{Ext}^1(T,T) = 0$. Although "most" representations in $\mathrm{rep}(Q,\mathbf{d})$ are generic, it is not obvious to exhibit a generic representation explicitly. Our goal in this paper is to do this for Dynkin quivers of type $\mathbb{A}_n$ and $\mathbb{D}_n$. In fact, the case $\mathbb{A}_n$ has already been treated in Abeasis [1]. A different description from the one presented here in case $\mathbb{D}_n$ as well as several examples (but no proof) can be found in [4].

It is possible but combinatorially more involved to apply the method presented here for Dynkin quivers of type $\mathbb{E}_n, n = 6,7,8$. In fact, any quiver Q of type $\mathbb{E}_n$ can be obtained from a quiver Q' of type either $\mathbb{A}_{n-1}$ or $\mathbb{D}_{n-1}$ by adding one vertex and one arrow. The problem is that the partially ordered set of isomorphism classes of indecomposable representations U of Q', ordered by the existence of a non-zero homomorphism, is more complicated.

The article is accessible for mathematicians with little knowledge in representation theory. In fact, we only use that representations of a quiver without oriented cycles form an abelian category of global dimension at most one and some homological algebra, but no Auslander-Reiten theory. Even Gabriel's theorem [2] mentioned earlier is not necessary.

2 Results

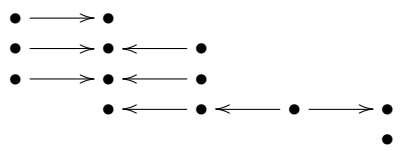
Proposition 2.1. *For a Dynkin quiver Q of type $\mathbb{A}_n$ and a dimension vector $\mathbf{d} = (d_1, \ldots, d_n)$ the following representation $T = T_{Q,\mathbf{d}} \in \mathrm{rep}(Q,\mathbf{d})$ is generic: Choose for $T(\alpha_i)$ the matrix consisting of the identity matrix $\mathbf{1}_{\min\{d_i,d_{i+1}\}}$ of largest possible size in its upper left or lower right corner if α_i points to the right or left, respectively, and zeros otherwise.*

Explicitly, we choose

$$T(\alpha_1) = \begin{pmatrix} \mathbf{1}_{d_1} \\ 0 \end{pmatrix} \text{ if } d_1 \leq d_2 \text{ and } T(\alpha_1) = \begin{pmatrix} \mathbf{1}_{d_2} & 0 \end{pmatrix} \text{ if } d_1 \geq d_2$$

for $\alpha_1 : 1 \to 2$. As an example, we draw $T_{Q,\mathbf{d}}$ for

$$Q = \ 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow 5 \ , \mathbf{d} = (3,4,3,1,2) :$$



Each bullet represents a basis vector. An arrow starting from a bullet means that the basis vector at the start of the arrow is mapped to the one at the terminus. If no arrow starts at a bullet, the corresponding basis vector is mapped to zero.

Proposition 2.2. *Let Q be a Dynkin quiver of type $\mathbb{D}_n, n \geq 4$, with the arrows α_1 and α_2 both starting at the vertex 3, and let $\mathbf{d} = (d_1, \ldots, d_n)$ be a dimension vector with $d_1 \leq d_2 \leq d_3$. The following representation $T = T_{Q,\mathbf{d}} \in \mathrm{rep}(Q, \mathbf{d})$ is generic:*

1. *Choose for $T(\alpha_i)$ the matrix consisting of the identity matrix $\mathbf{1}_{\min\{d_i, d_{i+1}\}}$ of largest possible size in its upper left or lower right corner if α_i points to the right or left, respectively, and zeros otherwise, $i = 3, \ldots, n-1$.*

2. *If $d_1 + d_2 \leq d_3$, set $d_3' = d_3 - d_1 - d_2$ and*

$$T(\alpha_1) = \begin{pmatrix} 0 & \mathbf{1} & 0 & S \end{pmatrix} : k^{d_3'} \oplus k^{d_1} \oplus k^{d_2 - d_1} \oplus k^{d_1} \to k^{d_1},$$

$$T(\alpha_2) = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} : k^{d_3'} \oplus k^{d_1} \oplus k^{d_2 - d_1} \oplus k^{d_1} \to k^{d_2 - d_1} \oplus k^{d_1},$$

3. *If $d_1 + d_2 \geq d_3$, set $d_2' = d_3 - d_2, d_3' = d_1 + d_2 - d_3$ and*

$$T(\alpha_1) = \begin{pmatrix} \mathbf{1} & 0 & S & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} : k^{d_2'} \oplus k^{d_2 - d_1} \oplus k^{d_2'} \oplus k^{d_3'} \to k^{d_2'} \oplus k^{d_3'},$$

$$T(\alpha_2) = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} : k^{d_2'} \oplus k^{d_2 - d_1} \oplus k^{d_2'} \oplus k^{d_3'} \to k^{d_2 - d_1} \oplus k^{d_2'} \oplus k^{d_3'},$$

where $S = (\sigma_{ij}) \in \mathrm{Mat}(m \times m, k)$ is the matrix with $\sigma_{ij} = \delta_{i,m+1-j}$ and $\mathbf{1}$ is the identity matrix of suitable size.

Note that in case $d_1 > d_3$, a generic representation is the direct sum of a generic representation with dimension vector $\mathbf{d} - (d_1 - d_3)\mathbf{dim} S_1$ with $d_1 - d_3$ copies of S_1, where S_1 denotes the one-dimensional representation supported at the vertex 1.

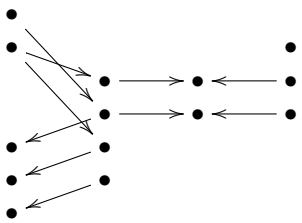
Generic representations for other orientations of Q or other dimension vectors are obtained from the given ones by duality, using the symmetry exchanging the vertices 1 and 2 if possible, or by applying the "reflection functor" [2] corresponding to the vertices 1 or 2. Indeed, suppose α_1 points to the right in Q, let Q' be obtained from Q by reversing α_1, and fix a dimension vector $\mathbf{d}'$ for Q' with $d_1' \leq d_3'$. Then it is easy to check that a representation $X' \in \mathrm{rep}(Q', \mathbf{d}')$ is generic if and only if the representation X of Q obtained from X' by replacing $X'(\alpha_1') : X'(1) \to X'(3)$ by the inclusion $X(\alpha_1) : \ker X'(\alpha_1') \to X'(3)$ (without changing any other vector spaces or arrows) and by choosing $\mathbf{d} = (d_3' - d_1', d_2', \ldots, d_n')$ is generic in $\mathrm{rep}(Q, \mathbf{d})$.

Note that we could fix the direction of α_{n-1} as well by a reflection, but this would limit us to $d_{n-1} \geq d_n$, an inconvenience for our inductive proofs.

Considering the example

$$Q = \begin{tikzcd} 1 \arrow[dr] & & & \\ & 3 \arrow[r] \arrow[dl] & 4 & 5 \arrow[l] \\ 2 & & & \end{tikzcd}, \qquad \mathbf{d} = (2,3,4,2,3),$$

we find



for the generic representation.

3 Proofs

We will prove both results by induction on n. We denote the one-dimensional representation supported at n by S, and we let Q' and $\mathbf{d}'$ be the full subquiver containing all vertices of Q except n and the restriction of $\mathbf{d}$ to Q', respectively. For a representation X of Q', we still denote by X the representation of Q obtained by "extension by zero", i.e. by setting $X(n) = 0, X(\alpha_{n-1}) = 0$.

Without loss of generality we may suppose that the arrow α_{n-1} starts at $n-1$: Consider the opposite quiver if it does not.

The following lemma and its dual are the key to the proofs. For two representations X and Y of a quiver Q, we set

$$[X,Y] = \dim_k \operatorname{Hom}_Q(X,Y), {}^1[X,Y] = \dim_k \operatorname{Ext}^1_Q(X,Y).$$

Let Q be a quiver with a sink s, and denote by Q' the full subquiver containing all vertices except s. Let S be the 1-dimensional representation of Q supported at s, and note that S is projective. For a representation X of Q with ${}^1[X,S] = 1$, denote by $\tilde{X}$ the middle term of a non-split short exact sequence

$$\sigma_X : 0 \to S \to \tilde{X} \to X \to 0$$

and note that $\tilde{X}$ is well-defined up to isomorphism. We set $\tilde{X} = 0$ for $X = 0$.

Lemma 3.1. *Keeping the notations just introduced, let U, V, W be representations of Q' for which $U \oplus V \oplus W$ is generic. Suppose moreover that ${}^1[W,S] = 0$ and $[U,U] = {}^1[U,S] = 1$ and $[V,V] = {}^1[V,S] = 1$, respectively, in case U and V are non-zero. Note that V may be equal to U.*

Then we have

1. $[U \oplus V, \tilde{U} \oplus \tilde{V}] = 0$,

2. *$\tilde{U}$ and $\tilde{V}$ are indecomposable,* ${}^1[\tilde{U}, V \oplus W \oplus \tilde{V} \oplus S] = 0$, ${}^1[U \oplus V \oplus W, \tilde{U}] = 1 - [V, U], [\tilde{U}, \tilde{V}] = [\tilde{U}, V] = [U, V]$, *and* $[S, \tilde{U}] = 1$,

3. *In case $[V, U] = 1$, the representations $\tilde{U} \oplus U \oplus V \oplus W, \tilde{U} \oplus \tilde{V} \oplus V \oplus W$ and $\tilde{U} \oplus \tilde{V} \oplus S \oplus W$ are generic.*

Proof. Note that U and V are indecomposable as their endomorphism algebras are one-dimensional. In order to prove our first claim, we show that, for an indecomposable representation X of Q', the conditions ${}^1[X \oplus U, X \oplus U] = 0$, ${}^1[X, S] = 1$ imply $[X, \tilde{U}] = 0$.

Mapping X to σ_U, we obtain the exact sequence

$$0 \to \mathrm{Hom}(X, S) = 0 \to \mathrm{Hom}(X, \tilde{U}) \to \mathrm{Hom}(X, U) \to$$
$$\to \mathrm{Ext}^1(X, S) = k \to \mathrm{Ext}^1(X, \tilde{U}) \to \mathrm{Ext}^1(X, U) = 0 \to 0.$$

In case $[X, U] = 0$, we obviously have $[X, \tilde{U}] = 0$. Suppose $[X, U] \neq 0$ and fix a non-zero homomorphism $f : X \to U$. It is enough to prove that the connnecting homomorphism $\mathrm{Hom}(X, U) \to \mathrm{Ext}^1(X, S)$ is injective, or equivalently that the pull-back of σ_U by f does not split. From ${}^1[X, S] = \sum \dim X(s\alpha)$, we conclude that

$$\sum \dim X(s\alpha) = \sum \dim U(s\alpha) = 1,$$

where all sums range over all arrows α ending at the sink s. As ${}^1[U, X] = 0$, we know from [3] that f is either a monomorphism or an epimorphism, and thus there is an arrow $\alpha : t \to s$ for which $X(t) = U(t) = k, f(t) \neq 0$, and therefore f cannot factor through the projection $\tilde{U} \to U$. This implies that indeed the pull-back of σ_U by f does not split.

All other claims now follow from considering the long exact sequences obtained by mapping either σ_U or σ_V to suitable representations or by mapping suitable representations to either σ_U or σ_V. □

We are ready to prove proposition 2.1: We have defined T in such a way that there is a decomposition $T = (\oplus_{j=1}^{d_n} T_j) \oplus R$ with T_j indecomposable for all j, $R(n) = 0$ and such that $T_j(n) = ke_j$, where $e_1, \ldots, e_{d_n}$ is the standard basis of $T(n) = k^{d_n}$. We will show by induction on n that ${}^1[T, T] = 0$, that in addition $[T_j, T_l] \leq 1$ for all j, l and that the isomorphism classes in $\{T_j : j = 1, \ldots, d_n\}$ are linearly ordered by the existence of non-zero homomorphisms, i.e. that for T_j and T_l non-isomorphic we have $[T_j, T_l] = 1$ if and only if $j > l$. Note that these conditions are trivially satisfied for $n = 1$ as in this case all indecomposable direct summands of T are isomorphic and there are no non-trivial extensions.

Remember that Q' is the full subquiver of Q containing all vertices except n, and consider the representation $T' = T_{Q', \mathbf{d}'}$ of Q', which is just the restriction of $T = T_{Q, \mathbf{d}}$ to Q'. By induction we know that ${}^1[T', T'] = 0$, that T' has a decomposition $T' = (\oplus_{j=1}^{d_{n-1}} T'_j) \oplus R'$ with T'_j indecomposable, $R'(n-1) = 0$,

that $[T'_j, T'_l] \leq 1$, and that for T'_j and T'_l non-isomorphic we have $[T'_j, T'_l] = 1$ if and only if $j > l$ for $1 \leq j, l \leq d_{n-1}$.

In case $d_n \geq d_{n-1}$, we set $R = R', T_j = \tilde{T}'_j$ for $j = 1, \ldots, d_{n-1}$ and $T_j = S$ for $j = d_{n-1} + 1, \ldots, d_n$. All conditions needed for T then follow from Lemma 3.1. If $d_n \leq d_{n-1}$, we set $R = \oplus_{j=d_n+1}^{d_{n-1}} T'_j \oplus R'$ and $T_j = \widetilde{T'_j}$ for $j \leq d_n$. As ${}^1[R', S] = 0, {}^1[T'_j, S] = 1$ for $j \leq d_{n-1}$ and $[T'_j, T'_l] = 1$ for $l \leq d_n < j \leq d_{n-1}$, Lemma 3.1 yields that T satisfies all conditions stated.

By our inductive procedure we obtain in addition a decomposition of T as a direct sum of indecomposables. Indeed, the summands T'_j of T' with $d_n < j \leq d_{n-1}$ are indecomposable direct summands of T that will not be modified in subsequent steps.

Let us turn to the proof of proposition 2.2. We use the same idea as for $\mathbb{A}_n$, but if Q is of type $\mathbb{D}_n$, the pairwise non-isomorphic indecomposables whose support contains the vertex n are not linearly ordered by the existence of a non-zero homomorphism. Indeed, there is no non-zero homomorphism between the indecomposables with dimension vectors $(1, 0, 1 \ldots, 1)$ and $(0, 1, 1 \ldots, 1)$. It will turn out that these two are the only "incomparable" ones, though. We first prove a supplement to lemma 3.1, treating this case.

Lemma 3.2. *Keeping the notations and hypotheses from lemma 3.1 we assume in addition that* $[U, V] = [V, U] = 0$.

1. *We have* ${}^1[V, \tilde{U}] = {}^1[U, \tilde{V}] = 1$.
2. *The middle term* X *of any non-split short exact sequence*

$$\tau_{U,V} : 0 \to \tilde{U} \to X \to V \to 0$$

   satisfies $[X, X] = 1, {}^1[X, X] = 0$ *and is thus indecomposable and generic.*

3. *The dimensions of the only non-zero homomorphism spaces between any two distinct representations among* $U, \tilde{U}, V, \tilde{V}, X$ *are* $[\tilde{U}, U] = [\tilde{V}, V] = [X, U] = [X, V] = [\tilde{U}, X] = [\tilde{V}, X] = 1$.
4. *The dimensions of the only non-zero extension spaces between any two representations among* $U, \tilde{U}, V, \tilde{V}, X$ *are* ${}^1[V, \tilde{U}] = {}^1[U, \tilde{V}] = 1$. *The middle term of a non-split exact sequence in either one of the spaces* $\mathrm{Ext}^1(V, \tilde{U})$ *or* $\mathrm{Ext}^1(U, \tilde{V})$ *is isomorphic to* X.

Proof. Only the very last statement requires a new idea. Let Y be the middle term of a non-split exact sequence in $\mathrm{Ext}^1(U, \tilde{V})$. Note that

$$\mathbf{dim} Y = \mathbf{dim}(U \oplus \tilde{V}) = \mathbf{dim}(U \oplus V \oplus S) = \mathbf{dim}(V \oplus \tilde{U}) = \mathbf{dim} X.$$

We know that X is generic, and, exchanging U and V, we see that Y is generic as well. But the generic representation is unique up to isomorphism. □

From now on we assume Q to be of type $\mathbb{D}_n, n \geq 4$, with both arrows α_1 and α_2 starting at the vertex 3 and α_{n-1} starting at the vertex $n-1$. We fix $\mathbf{d} \in \mathbb{N}^n$ with $d_1 \leq d_2 \leq d_3$. As a first step, we reformulate Lemma 3.2 for a quiver of type $\mathbb{D}_n$. For $n \geq 3$, we set

$$\tilde{Y}_+ = T_{Q,(1,0,1,\ldots,1,1)}, \tilde{Y}_- = T_{Q,(0,1,1,\ldots,1,1)},$$

and for $n \geq 4$ moreover

$$Y_+ = T_{Q,(1,0,1,\ldots,1,0)}, Y_- = T_{Q,(0,1,1,\ldots,1,0)}.$$

Note that, for $n \geq 4$, $\tilde{Y}_\pm$ is indeed the middle term of a non-split short exact sequence in $\mathrm{Ext}^1(Y_\pm, S)$. By X we denote the middle term of τ_{Y_+,Y_-} or τ_{Y_-,Y_+}.

Corollary 3.3. *1. The dimensions of the only non-zero homomorphism spaces between any two distinct representations among $Y_+, \tilde{Y}_+, Y_-, \tilde{Y}_-, X$ are*

$$[\tilde{Y}_+, Y_+] = [\tilde{Y}_-, Y_-] = [X, Y_+] = [X, Y_-] = [\tilde{Y}_+, X] = [\tilde{Y}_-, X] = 1.$$

2. The dimensions of the only non-zero extension spaces between any two distinct representations among $Y_+, \tilde{Y}_+, Y_-, \tilde{Y}_-, X$ are

$${}^1[Y_-, \tilde{Y}_+] = {}^1[Y_+, \tilde{Y}_-] = 1.$$

The middle term of a non-split exact sequence in either one of the spaces $\mathrm{Ext}^1(Y_-, \tilde{Y}_+)$ or $\mathrm{Ext}^1(Y_+, \tilde{Y}_-)$ is isomorphic to X.

Corollary 3.3 allows us to prove that $T_{Q,\mathbf{d}}$ is generic for special $\mathbf{d}$:

Lemma 3.4. *1. For $n \geq 3$, $r, s \geq 0$ and $\mathbf{e}(r,s) = (r, r+s, 2r+s, \ldots, 2r+s)$, the representation $T_{Q,\mathbf{e}(r,s)}$ has a decomposition $T_{Q,\mathbf{e}(r,s)} = \oplus_{j=1}^{2r+s} T_j$ with T_j indecomposable and*

$$T_j(n) = \begin{cases} ke_j & \text{for } j \leq r+s, \\ k(e_j - e_{j'}) & \text{for } r+s < j \leq 2r+s, \end{cases}$$

where $j' = 2r+s+1-j$. Moreover, T_j equals $\tilde{Y}_+$ for $j \leq r$ and $\tilde{Y}_-$ for $j > r$. In particular, $T_{Q,\mathbf{e}(r,s)}$ is generic.

2. Suppose $n \geq 4$, $r, s \geq 0$, $0 \leq u \leq 2r+s$. For $j > r+s$, we set $j' = 2r+s+1-j$. For $\mathbf{e}(r,s,u) = (r, r+s, 2r+s, \ldots, 2r+s, u)$ the representation $T_{Q,\mathbf{e}(r,s,u)}$ has a decomposition $T_{Q,\mathbf{e}(r,s,u)} = \oplus_{j=1}^{u} T_j \oplus R$ with T_j indecomposable for all j, $R(n) = 0$,

$$T_j(n) = \begin{cases} ke_j & \text{for } j \leq r+s, \\ k(e_j - e_{j'}) & \text{for } r+s < j \leq u \end{cases}$$

Moreover, we have

- *for* $u \leq r, T_j \simeq X, j = 1, \ldots, u,$
- *for* $r \leq u \leq r+s, T_j \simeq \begin{cases} X & \text{for } j = 1, \ldots, r, \\ \tilde{Y}_- & \text{for } j = r+1, \ldots, u, \end{cases}$
- *for* $u \geq r+s, T_j \simeq \begin{cases} X & \text{for } j = 1, \ldots, 2r+s-u, \\ \tilde{Y}_+ & \text{for } j = 2r+s-u+1, \ldots, r, \\ \tilde{Y}_- & \text{for } j = r+1, \ldots, u. \end{cases}$

In particular, $T_{Q,\mathbf{e}(r,s,u)}$ *is generic.*

Proof. In both cases, genericity follows from Corollary 3.3. The decompositions claimed are just a matter of Linear Algebra and some bookkeeping. As $T(\alpha_i) = \mathbf{1}_{2r+s}$ for $i \geq 3$, except for $i = n-1$ in the second case, it is sufficient to prove the decomposition claimed only for $n = 3$ and $n = 4$, respectively.

As Q is of type $\mathbb{A}_3$ for $n = 3$, we know in fact from proposition 2.1 that $(\tilde{Y}_+)^r \oplus (\tilde{Y}_-)^{s+r}$ is generic. The numbering of the vertices is not the same we used in proposition 2.1, however. In particular, indecomposable representations supported at 3 are not linearly ordered by the existence of non-zero homomorphisms. □

The following lemma completes the picture for homomorphisms and extensions involving X:

Lemma 3.5. *Let* U, V, W *be representations of* Q' *with* ${}^1[W, S] = 0, [U, U] = [V, V] = [V, U] = {}^1[U, S] = {}^1[V, S] = 1$ *unless* $U = 0$ *or* $V = 0$. *for which* $U \oplus Y_+ \oplus Y_- \oplus V \oplus W$ *is generic. Suppose that*

- U *and* Y_+ *are not isomorphic and* $[Y_+, U] = 1$ *unless* $U = 0$,
- V *and* Y_+ *are not isomorphic and* $[V, Y_+] = 1$ *unless* $V = 0$.

Then we have that

- $[X, \tilde{U}] = [\tilde{V}, X] = [S, X] = 1$,
- *the following representations are generic:*
  1. $\tilde{U} \oplus X \oplus Y_+ \oplus Y_- \oplus V \oplus W$,
  2. $\tilde{U} \oplus X \oplus \tilde{Y}_- \oplus Y_- \oplus V \oplus W$,
  3. $\tilde{U} \oplus X \oplus \tilde{Y}_- \oplus \tilde{Y}_+ \oplus V \oplus W$.

Proof. We only need to show that

$$[X, \tilde{U}] = [\tilde{V}, X] = 1, {}^1[X, \tilde{U} \oplus W] = {}^1[\tilde{U} \oplus W, X] = 0$$

as all other claims follow from either Lemma 3.1 or corollary 3.3. Note that $[V, Y_-] = [V, Y_+]$ and $[Y_-, U] = [Y_+, U]$. All facts claimed are obtained from considering suitable long exact sequences associated to σ_U, τ_{Y_+,Y_-}, or τ_{Y_-,Y_+}. □

The last claim in the following more technical proposition implies Proposition 2.2 for any dimension vector $\mathbf{d}$ with $d_1 \leq d_2 \leq d_3$. In addition, the proposition describes the decomposition of a generic representation as a direct sum of indecomposables.

Proposition 3.6. *1. For $n \geq 3$, there is a decomposition $d_n = p_n + q_n + 2r_n + s_n + t_n$ with $p_n, q_n, r_n, s_n, t_n \geq 0$ and a decomposition $T = T_{Q,\mathbf{d}} = (\oplus_{j=1}^{d_n} T_j) \oplus R$ with $R(n) = 0$, T_j indecomposable and $T_j(n) = k$ for all j,*

$$T_j = \begin{cases} X & \text{for } 1 \leq j - p_n \leq q_n, \\ \tilde{Y}_+ & \text{for } 1 \leq j - p_n - q_n \leq r_n, \\ \tilde{Y}_- & \text{for } 1 \leq j - p_n - q_n - r_n \leq r_n + s_n, \end{cases}$$

and T_j is not isomorphic to $\tilde{Y}_+, \tilde{Y}_-$ or X for other values of j.

2. For $n \geq 4$ we have $r_n \leq r_{n-1}$. If $r_n > 0, (p_n, q_n, r_n, s_n, t_n)$ satisfies $s_n = s_{n-1}$ and $p_n + q_n + r_n = p_{n-1} + q_{n-1} + r_{n-1}$.

3. $[T_j, T_l] \leq 1$ and, for T_j not isomorphic to T_l, equality holds if and only if $j > l$ and at least one of the following two conditions is satisfied: either $T_j \neq \tilde{Y}_-$ or $T_l \neq \tilde{Y}_+$, for $j, l = 1, \ldots, d_n$.

4. ${}^1[T, T] = 0$.

Proof. For $n = 3$, we choose

$$(p_3, q_3, r_3, s_3, t_3) = \begin{cases} (d_3 - d_1 - d_2, 0, d_1, d_2 - d_1, 0) & \text{if } d_1 + d_2 \leq d_3, \\ (0, 0, d_3 - d_2, d_2 - d_1, d_1 + d_2 - d_3) & \text{if } d_1 + d_2 \geq d_3. \end{cases}$$

We set $R = 0$, and we let T_j be the indecomposable direct summand of $T = T_{Q,\mathbf{d}}$ with

$$T_j(3) = \begin{cases} ke_j & \text{for } 1 \leq j \leq p_3 + q_3 + r_3 + s_3, \\ k(e_j - e_{j'}) & \text{for } 1 \leq j - p_3 - q_3 - r_3 - s_3 \leq r_3, \\ ke_j & \text{for } 1 \leq j - p_3 - q_3 - 2r_3 - s_3, \end{cases}$$

where $j' = 2d_3 - d_1 - d_2 + 1 - j$. Note that $T(\alpha_1)(e_j - e_{j'}) = 0$. We have

$$\mathbf{dim}\, T_j = \begin{cases} (0,0,1) & \text{for } j = 1, \ldots, p_3, \\ (1,0,1) & \text{for } j = p_3 + 1, \ldots, p_3 + r_3, \\ (0,1,1) & \text{for } j = p_3 + r_3 + 1, \ldots, p_3 + 2r_3 + s_3, \\ (1,1,1) & \text{for } j = p_3 + 2r_3 + s_3 + 1, \ldots, p_3 + 2r_3 + s_3 + t_3. \end{cases},$$

and either $p_3 = 0$ or $t_3 = 0$. The representations T_j satisfy all our conditions by lemma 3.1. As $p_3 \cdot t_3 = 0$, the two indecomposable representations with dimension type $(0,0,1)$ and $(1,1,1)$, for which there is a non-split extension, cannot both arise. For genericity we could as well have used proposition 2.1.

We note for further reference that $p_3+q_3+r_3=d_3-d_2, s_3=d_2-d_1$ and that the map $j\mapsto j'$ is the bijection reversing the numbering from

$$\{j : 1\le j-(p_3+q_3+r_3+s_3)\le r_3\}\to\{l : 1\le l-(p_3+q_3)\le r_3\}$$

as we have $2(p_3+q_3+r_3)+s_3=2d_3-d_1-d_2$.

For the inductive step, we start from a decomposition of the generic representation $T_{Q',\mathbf{d}'}=(\oplus_{j=1}^{d_{n-1}}T'_j)\oplus R'$ satisfying the properties listed in our proposition for Q'. So we know that T'_j is indecomposable, $R'(n-1)=0, T'_j(n-1)=k, [T'_j,T'_l]\le 1$ for $j,l=1,\dots,d_{n-1}$ and that

$$T'_j=\begin{cases} Y_+ & \text{for } 1\le j-p_{n-1}-q_{n-1}\le r_{n-1},\\ Y_- & \text{for } 1\le j-p_{n-1}-q_{n-1}-r_{n-1}\le r_{n-1}+s_{n-1}.\end{cases}$$

Moreover, if T'_j is not isomorphic to T'_l and if either j or l do not lie between $p_{n-1}+q_{n-1}+1$ and $p_{n-1}+q_{n-1}+2r_{n-1}+s_{n-1}$, i.e. either T'_j or T'_l is not isomorphic to Y_+ or Y_-, we know that $[T'_j,T'_l]=1$ if and only if $j>l$.

For T_j we have either $\tilde{T}'_j$ or X, and in order to show that all conditions are satisfied for any pair T_j,T_l, we use either lemma 3.1 or a part of lemma 3.5. We distinguish several cases depending on d_n. We have

$$R=\begin{cases}(\oplus_{j=d_n+1}^{d_{n-1}}T'_j)\oplus R' & \text{if } d_n\le d_{n-1},\\ R' & \text{if } d_n\ge d_{n-1}.\end{cases}$$

In case $d_n\le p_{n-1}+q_{n-1}$, we set $(p_n,q_n,r_n,s_n,t_n)=(d_n,0,0,0,0), T_j=\tilde{T}'_j$ for $j=1,\dots,d_n$ and apply Lemma 3.1.

In case $p_{n-1}+q_{n-1}+1\le d_n\le p_{n-1}+q_{n-1}+r_{n-1}$ we choose $(p_n,q_n,r_n,s_n,t_n)=(p_{n-1}+q_{n-1},d_n-p_{n-1}-q_{n-1},0,0,0)$ and

$$T_j=\begin{cases}\tilde{T}'_j & \text{for } j=1,\dots,p_n,\\ X & \text{for } j=p_n+1,\dots,d_n.\end{cases}$$

In order to verify our conditions for two summands T_j,T_l, we use lemma 3.1 $j,l\le p_n$ and lemma 3.5, 1. otherwise.

In case $p_{n-1}+q_{n-1}+r_{n-1}+1\le d_n\le p_{n-1}+q_{n-1}+r_{n-1}+s_{n-1}$, we choose $(p_n,q_n,r_n,s_n,t_n)=(p_{n-1}+q_{n-1},r_{n-1},0,d_n-p_{n-1}-q_{n-1}-r_{n-1},0)$ and

$$T_j=\begin{cases}\tilde{T}'_j & \text{for } j=1,\dots,p_n,\\ X & \text{for } j=p_n+1,\dots,p_n+q_n,\\ \tilde{T}'_j\simeq\tilde{Y}_- & \text{for } j=p_n+q_n+1,\dots,d_n.\end{cases}$$

Now we use lemma 3.5, 2. if one summand under consideration is isomorphic to X and lemma 3.1 if not.

Suppose $p_{n-1}+q_{n-1}+r_{n-1}+s_{n-1}+1 \le d_n \le p_{n-1}+q_{n-1}+2r_{n-1}+s_{n-1}$, and set

$$\begin{aligned}(p_n,q_n,r_n,s_n,t_n) = \\ (p_{n-1}+q_{n-1}, p_{n-1}+q_{n-1}+2r_{n-1}+s_{n-1}-d_n, \\ d_n-p_{n-1}-q_{n-1}-r_{n-1}-s_{n-1}, s_{n-1}, 0),\end{aligned}$$

$T_j = \tilde{T}'_j$ for $j=1,\dots,p_n$, and set $T^{(1)} = \oplus_{j=1}^{p_n} T_j$. Then $T^{(1)}$ does not extend itself, and $T_1,\dots,T_{p_n}$ satisfy all conditions we claim.

The key point is that $r_m \ge r_n > 0$ for $m=3,\dots,n$ and thus

$$s_n = s_m = s_3 = d_2 - d_1,$$

$$p_n+q_n+r_n = p_m+q_m+r_m = p_3+q_3+r_3 = d_3-d_2$$

for $m=3,\dots,n$. Therefore the representation $T_{Q,\mathbf{d}}$ has a direct summand $T^{(2)}$ for which $T^{(2)}(m)$ is the span of the vectors in the standard basis of $T^{(2)}(m)$ with indices

$$j = \begin{cases} d_1-r_{n-1}+1,\dots,d_1 & \text{for } m=1, \\ d_3-d_2+1,\dots,d_3-d_1+r_{n-1} & \text{for } m=2, \\ d_3-d_2-r_{n-1}+1,\dots,d_3-d_1+r_{n-1} & \text{for } m=3,\dots,n-1, \\ d_3-d_2-r_{n-1}+1,\dots,d_3-d_1+r_n & \text{for } m=n. \end{cases}$$

Indeed, the map $j \mapsto j'$ is the bijection reversing the numbering from

$$\{d_3-d_1+1,\dots,d_3-d_1+r_{n-1}\} \to \{d_3-d_2-r_{n-1}+1,\dots,d_3-d_2\}.$$

We remark that $T^{(2)}$ is just the representation $T_{Q,\mathbf{e}(r_{n-1},s_{n-1},u)}$ for $u = d_n - p_n = d_2 - d_1 + r_{n-1} + r_n$, which by lemma 3.4 is isomorphic to $\oplus_{j=p_n+1}^{d_n} T_j$ with

$$T_j = \begin{cases} X & \text{for } 1 \le j-p_n \le q_n, \\ \tilde{Y}_+ & \text{for } 1 \le j-p_n-q_n \le r_n, \\ \tilde{Y}_- & \text{for } 1 \le j-p_n-q_n-r_n \le r_n+s_n. \end{cases}$$

By corollary 3.3, $T^{(2)}$ does not extend itself, and its indecomposable direct summands $T_{p_n+1},\dots,T_{p_n+q_n+2r_n+s_n}$ satisfy the conditions claimed. By lemma 3.1 and lemma 3.5, 3., the summands $T_1,\dots,T_{p_n+q_n+2r_n+s_n}$ of $T = T^{(1)} \oplus T^{(2)}$ satisfy all conditions as well.

In case $1 \le d_n - p_{n-1} - q_{n-1} - 2r_{n-1} - s_{n-1} \le t_{n-1}$ we set

$$(p_n,q_n,r_n,s_n,t_n) = (p_{n-1}+q_{n-1}, 0, r_{n-1}, s_{n-1}, d_n-p_{n-1}-q_{n-1}-2r_{n-1}-s_{n-1}),$$

$T_j = \tilde{T}'_j$ for $j=1,\dots,d_n$. Finally, in case $d_n \ge d_{n-1}$, we take

$$(p_n,q_n,r_n,s_n,t_n) = (p_{n-1}+q_{n-1}, 0, r_{n-1}, s_{n-1}, t_{n-1}+d_n-d_{n-1})$$

and

$$T_j = \begin{cases} \tilde{T}'_j & \text{if } j \leq d_{n-1}, \\ S & \text{if } j > d_{n-1}, \end{cases}$$

and we conclude in both cases by lemma 3.1. □

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