

# FREE DENSE SUBGROUPS OF HOLOMORPHIC AUTOMORPHISMS

RAFAEL B. ANDRIST AND ERLEND FORNÆSS WOLD

ABSTRACT. We show the existence of free dense subgroups, generated by 2 elements, in the holomorphic shear and overshear group of complex-Euklidian space and extend this result to the group of holomorphic automorphisms of Stein manifolds with Density Property, provided there exists a generalized translation. The conjugation operator associated to this generalized translation is hypercyclic on the topological space of holomorphic automorphisms.

## 1. INTRODUCTION

In Functional Analysis and Topological Dynamics, the phenomenon of so-called hypercyclicity of operators and universality of sequences of operators is studied. For a detailed survey, we refer to Grosse-Erdmann [8]. One of the early examples actually comes from Complex Analysis; in 1929 G.D. Birkhoff [4] constructed a holomorphic function  $f : \mathbb{C} \rightarrow \mathbb{C}$  such that for a given sequence  $\{a_m\}_{m=1}^{\infty} \subset \mathbb{R}$  with  $\lim_{m \rightarrow \infty} a_m = \infty$  the set  $\{z \mapsto f(z + a_m), m \in \mathbb{N}\}$  is dense (in compact-open topology) in  $\mathcal{O}(\mathbb{C})$ . It follows in particular that for any translation  $\tau : \mathbb{C} \rightarrow \mathbb{C}$ ,  $\tau \neq \text{id}$ , the set  $\{f \circ \tau^m : m \in \mathbb{N}\}$  is dense in  $\mathcal{O}(\mathbb{C})$ . This motivated the following definition:

**Definition 1.1.** Let  $E$  be a topological space.

A self-map  $T : E \rightarrow E$  is called *hypercyclic* if there exists an  $f \in E$ , called *hypercyclic element* for  $T$ , such that the orbit  $\{T^m(f), m \in \mathbb{N}\}$  is dense in  $E$ .

The result of Birkhoff was later generalized to the case of holomorphic functions on  $\mathbb{C}^n$  and more recently by Zajac [17] also to holomorphic functions on pseudo-convex domains of  $\mathbb{C}^n$ ; he gives (in slightly more general form) the following Theorem which characterizes hypercyclic composition operators:

**Theorem 1.2** (Theorem 3.4. in [17]). *Let  $\Omega \subset \mathbb{C}^n$  be a pseudo-convex domain and let  $\tau \in \mathcal{O}(\Omega, \Omega)$  be a holomorphic selfmap. By  $C_\tau : \mathcal{O}(\Omega) \rightarrow \mathcal{O}(\Omega)$  we denote the composition operator  $f \mapsto f \circ \tau$ . The operator  $C_\tau$  is hypercyclic if and only if  $\tau$  is injective and for every  $\Omega$ -convex compact subset  $K \subset \Omega$  there exists  $m \in \mathbb{N}$  such that  $\tau^m(K) \cap K = \emptyset$  and  $K \cup \tau^m(K)$  is  $\Omega$ -convex.*

We now want to consider groups of holomorphic automorphisms and introduce the following notation:

**Definition 1.3.**

- (1) Let  $X$  be a complex manifold. By  $\text{Aut}(X) := \{f : X \rightarrow X \text{ biholomorphic}\}$  we denote the *group of holomorphic automorphisms of  $X$* , to be understood as topological group with the compact-open topology; and by  $\text{Aut}^*(X)$  its identity component.

---

2010 *Mathematics Subject Classification.* Primary 32M17; Secondary 32E30, 47A16.

*Key words and phrases.* density property, Andersen-Lempert theory, holomorphic automorphisms, free subgroups, hypercyclicity.

- (2) Let  $X$  be a complex manifold of dimension  $n \in \mathbb{N}$ . A holomorphic  $n$ -form  $\omega$  on  $X$  which is everywhere non-degenerate is called a *holomorphic volume form*. By  $\text{Aut}_\omega(X) := \{f : X \rightarrow X \text{ biholomorphic} : f^*(\omega) = \omega\}$  we denote the *group of volume-preserving holomorphic automorphisms of  $X$* , and by  $\text{Aut}_\omega^*(X)$  its identity component.

These topological groups are metrizable with the metric of uniform convergence. However, these metric spaces will not be Cauchy-complete, as one can see for example in case of  $\text{Aut}(\mathbb{C}^n)$  and a sequence of automorphisms  $z \mapsto M_k \cdot z$ , where  $M_k$ ,  $k \in \mathbb{N}$ , is a complex matrix with  $\det M_k \neq 0$  but  $\lim_{k \rightarrow \infty} \det M_k = 0$ . The phenomenon of hypercyclicity is usually studied on separable Fréchet spaces or at least on complete metric spaces.

The results concerning hypercyclicity of compositions operators on  $\mathcal{O}(\mathbb{C}^{n-1})$  can be applied directly to the shear and overshear groups of  $\mathbb{C}^n$ . The importance of these groups lies in the fact that they are dense subgroups of  $\text{Aut}(\mathbb{C}^n)$  resp.  $\text{Aut}_\omega(\mathbb{C}^n)$ ,  $\omega = dz_1 \wedge \cdots \wedge dz_n$ , as shown by Andersén [2] and Andersén–Lempert [3].

A holomorphic overshear  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  has, after possibly a special complex-linear change of coordinates, the form

$$F(z_1, \dots, z_n) = (z_1, z_2, \dots, z_{n-1}, \exp(f(z_1, \dots, z_{n-1})) \cdot z_n + g(z_1, \dots, z_{n-1}))$$

where  $f, g \in \mathcal{O}(\mathbb{C}^{n-1})$ . It is called a shear if  $f \equiv 0$ ; in this case  $\det(dF) = 1$  which is equivalent to  $f^*(\omega) = \omega$ . The (over-)shear group is the group generated by all the (over-)shears.

Using Theorem 1.2 for  $\Omega = \mathbb{C}^{n-1} \subseteq \mathbb{C}^{n-1}$  with  $\tau$  being a translation, and applying this to every holomorphic direction, one obtains immediately that there are  $n+1$  (over-)shears which generate a dense subgroup of the (over-)shear group. However, in Theorem 2.6 we show that it is actually possible to generate dense subgroups by only 2 automorphisms of a special form, one of them being a translation. Moreover we are able to show that this dense subgroup is generated freely by those 2 automorphisms.

The conditions for  $\tau$  which had been used already for Theorem 1.2, motivate the following definition which we use for the formulation of our main theorem.

**Definition 1.4.** Let  $X$  be a complex manifold and  $\tau \in \text{Aut}(X)$ . We call  $\tau$  a *generalized translation* if for any holomorphically convex compact  $K \subsetneq X$  there exists an  $m \in \mathbb{N}$  such that for the iterate  $\tau^m$  it holds that

- (1)  $\tau^m(K) \cap K = \emptyset$  and
- (2)  $\tau^m(K) \cup K$  is  $\mathcal{O}(X)$ -convex.

An obvious example for  $X = \mathbb{C}^n$  is the translation  $z \mapsto z + a$ ,  $a \neq 0$ .

**Theorem (3.6).** *Let  $X$  be a Stein manifold with density property and assume there exists a generalized translation  $\tau \in \text{Aut}^*(X)$ . Then there exists an  $F \in \text{Aut}^*(X)$  such that  $\langle \tau, F \rangle$  is a free and dense (in compact-open topology) subgroup of  $\text{Aut}^*(X)$ .*

From the point of view of hypercyclicity we can also state

**Theorem (3.9).** *Let  $X$  be a Stein manifold with density property and assume there exists a generalized translation  $\tau \in \text{Aut}^*(X)$ . Then the associated conjugation operator  $\tilde{C}_\tau$ , defined by  $\tilde{C}_\tau(f) = \tau \circ f \circ \tau^{-1}$  is hypercyclic on  $\text{Aut}^*(X)$ .*

The notion of Density Property and so-called Andersén–Lempert Theory is introduced in Section 3. Manifolds with density property in particular include  $\mathbb{C}^n$ ,  $n \geq 2$ , certain homogeneous spaces and all linear algebraic groups except those with connected components  $\mathbb{C}$  or  $(\mathbb{C}^*)^n$ . The density property of Stein manifolds implies

that the automorphism group is large, in particular infinite dimensional and acts  $m$ -transitively for any  $m \in \mathbb{N}$  (Varolin [16]).

## 2. COMPLEX EUKLIDEAN SPACE

Shears as a special kind of holomorphic automorphisms of  $\mathbb{C}^n$  have been introduced by Rosay and Rudin [14]. For a more recent treatment we refer to the textbook of Forsternič [7].

**Definition 2.1.** A map

$$F(z_1, \dots, z_n) = (z_1, z_2, \dots, z_{n-1}, \exp(f(z_1, \dots, z_{n-1})) \cdot z_n + g(z_1, \dots, z_{n-1}))$$

where  $f, g \in \mathcal{O}(\mathbb{C}^{n-1})$ , is called an *overshear in direction of the  $n$ -th coordinate axis* (or a *shear in direction of the  $n$ -th coordinate axis* if  $f \equiv 0$ ). For  $A \in \text{SL}(\mathbb{C}^n)$  the conjugates  $A^{-1} \circ F \circ A$  are called *overshears* (or *shears* if  $f \equiv 0$ ). The group generated by the shears resp. overshears is called the *shear group* resp. *overshear group*.

**Remark 2.2.** The group generated by (over-)shears in all coordinate directions coincides with the (over)-shear group.

For our purposes it will be convenient to introduce the following notion:

**Definition 2.3.** Set  $I : \mathbb{C}^n \rightarrow \mathbb{C}^n$ ,  $n \geq 2$ , to be

$$I(z_1, \dots, z_n) := (z_2, \dots, z_n, -(-1)^n z_1)$$

A map  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  which can be written as  $F = G \circ I$ , where  $F$  is an (over)-shear is called a *twisted (over)-shear*.

**Lemma 2.4.** *The groups generated by (over)-shears and twisted (over)-shears coincide.*

*Proof.* By definition,  $I$  is a twisted shear, hence every (over)-shear  $G$  can be written as  $G = F \circ I^{-1}$  where  $F = G \circ I$  is a twisted (over)-shear. It remains to show that  $I$  is a composition of shears: it is sufficient to show that any transposition of coordinates with a sign change can be written as a composition of shears, for simplicity  $t(z_1, \dots, z_{n-1}, z_n) = (z_1, \dots, z_n, -z_{n-1})$ . Let

$$A(z_1, \dots, z_{n-1}, z_n) := (z_1, \dots, z_{n-1}, z_n + z_{n-1})$$

$$B(z_1, \dots, z_{n-1}, z_n) := (z_1, \dots, z_{n-1} - z_n, z_n)$$

Then  $t = A^{-1} \circ B \circ A$ . □

For the following application of hypercyclicity we will need a slightly different statement than in Theorem 1.2. Despite that the method of proof is essentially the same and follows the original idea of G.D. Birkhoff [4], we include the proof for a lack of reference.

**Proposition 2.5.** *Let  $X$  be a Stein manifold and let  $\tau : X \rightarrow X$  be a generalized translation. Then there exists a pair of holomorphic functions  $f, g \in \mathcal{O}(X)$  such that for any pair of holomorphic functions  $\hat{f}, \hat{g} \in \mathcal{O}(X)$  there is a subsequence  $\{m_\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{N}$  such that*

$$\begin{aligned} f \circ \tau^{m_{2\ell-1}} &\rightarrow \hat{f}, \\ g \circ \tau^{m_{2\ell-1}} &\rightarrow 0, \\ f \circ \tau^{m_{2\ell}} &\rightarrow 0 \text{ and} \\ g \circ \tau^{m_{2\ell}} &\rightarrow \hat{g} \end{aligned}$$

*in compact-open topology.*

*Proof.* Let  $\{h_j\}_{j \in \mathbb{N}} \subset \mathcal{O}(X)$  be a sequence such that  $\{h_{2j}\}_{j \in \mathbb{N}}$  and  $\{h_{2j-1}\}_{j \in \mathbb{N}}$  are both dense (in compact-open topology) in  $\mathcal{O}(X)$ . Because  $X$  is Stein, there exists an exhaustion of  $X$  with  $\mathcal{O}(X)$ -convex compacts  $\{K_j\}_{j \in \mathbb{N}}$ . By definition of the generalized translation  $\tau$ , there exists an  $m_j \in \mathbb{N}$  such that  $K_j \cap \tau^{m_j}(K_j) = \emptyset$  and  $K_j \cup \tau^{m_j}(K_j)$  is  $\mathcal{O}(X)$ -convex. By omitting certain compacts of this exhaustion we may further assume that  $K_j \cup \tau^{m_j}(K_j) \subset K_{j+1}$  after renumbering. Therefore there exist by Runge approximation two sequences  $f_j, g_j \in \mathcal{O}(X), j \in \mathbb{N}$ , such that

$$\begin{aligned} \sup_{\tau^{m_{2j-1}}(K_{2j-1})} d(f_{2j-1}, h_{2j-1} \circ \tau^{-m_{2j-1}}) &< \frac{1}{2^j}, & \sup_{K_{2j-1}} d(f_{2j-1}, 0) &< \frac{1}{2^j} \quad \forall j \in \mathbb{N} \\ \sup_{\tau^{m_{2j-1}}(K_{2j-1})} d(f_{2j-1}, 0) &< \frac{1}{2^{2\ell_j}}, \quad \forall j < \ell_j \\ \sup_{\tau^{m_{2j-1}}(K_{2j-1})} d(g_{2j-1}, 0) &< \frac{1}{2^j}, & \sup_{K_{2j-1}} d(g_{2j-1}, 0) &< \frac{1}{2^j} \quad \forall j \in \mathbb{N} \\ \sup_{\tau^{m_{2j-1}}(K_{2j-1})} d(g_{2j-1}, 0) &< \frac{1}{2^{2\ell_j}}, \quad \forall j < \ell_j \\ \sup_{\tau^{m_{2j}}(K_{2j})} d(f_{2j}, 0) &< \frac{1}{2^j}, & \sup_{K_{2j}} d(f_{2j}, 0) &< \frac{1}{2^j} \quad \forall j \in \mathbb{N} \\ \sup_{\tau^{m_{2j}}(K_{2j})} d(f_{2j}, 0) &< \frac{1}{2^{2\ell_j}}, \quad \forall j < \ell_j \\ \sup_{\tau^{m_{2j}}(K_{2j})} d(g_{2j}, h_{2j} \circ \tau^{-m_{2j}}) &< \frac{1}{2^j}, & \sup_{K_{2j}} d(g_{2j}, 0) &< \frac{1}{2^j} \quad \forall j \in \mathbb{N} \\ \sup_{\tau^{m_{2j}}(K_{2j})} d(g_{2j}, 0) &< \frac{1}{2^{2\ell_j}}, \quad \forall j < \ell_j \end{aligned}$$

Set  $f := \sum_{j \in \mathbb{N}} f_j$  and  $g := \sum_{j \in \mathbb{N}} g_j$ . From the above estimates it is clear that the series  $f$  and  $g$  converge uniformly on compacts of  $X$ . Picking a subsequence of  $\{h_j\}_{j \in \mathbb{N}}$  such that  $h_{2m_\ell-1} \rightarrow \hat{f}$  and  $h_{2m_\ell} \rightarrow \hat{g}$  for  $\ell \rightarrow \infty$  in compact-open topology, the following estimate holds uniformly on  $K_{j\ell}$ :

$$\begin{aligned} d(f \circ \tau^{m_{2j_\ell-1}}, \hat{f}) &\leq \sum_{j=1}^{2j_\ell-2} d(f_j \circ \tau^{m_{2j_\ell-1}}, 0) \\ &\quad + d(f_{2j_\ell-1} \circ \tau^{m_{2j_\ell-1}}, h_{2j_\ell-1}) + d(h_{2j_\ell-1}, \hat{f}) \\ &\quad + \sum_{j=2j_\ell}^{\infty} d(f \circ \tau^{m_{2j_\ell-1}}, 0) \\ &\leq 2^{-j_\ell} + 2^{-j_\ell} + d(h_{2j_\ell-1}, \hat{f}) + 2^{-j_\ell} \rightarrow 0 \end{aligned}$$

Analogous estimates hold for the other three cases.  $\square$

### Theorem 2.6.

- (1) For any translation  $\tau \neq 0$  of  $\mathbb{C}^n, n \geq 2$ , there exists a holomorphic automorphism  $F$  of  $\mathbb{C}^n$  which is conjugate to a twisted shear map such that  $\langle \tau, F \rangle$  is a free and dense (in compact-open topology) subgroup of the shear group of  $\mathbb{C}^n$ .
- (2) For any translation  $\tau \neq 0$  of  $\mathbb{C}^n, n \geq 2$ , there exists a holomorphic automorphism  $f$  of  $\mathbb{C}^n$  which is conjugate to a twisted overshear map such that  $\langle \tau, F \rangle$  is a free and dense (in compact-open topology) subgroup of the overshear group of  $\mathbb{C}^n$ .

*Proof.* After conjugating with a special complex-linear map, we may assume that

$$\tau(z) = \tau(z_1, \dots, z_n) = (z_1 + b, \dots, z_n + b), \quad b \in \mathbb{R}_{>0}$$

We then define the following elements of the overshear group of  $\mathbb{C}^n$ :

$$\begin{aligned} I(z) &:= (z_2, \dots, z_n, -(-1)^n z_1) \\ F_{f,g}(z) &:= (z_2, \dots, z_n, \\ &\quad -(-1)^n \exp(f(z_2, \dots, z_n)) \cdot z_1 + g(z_2, \dots, z_n) + (1 - (-1)^n) z_n) \end{aligned}$$

where  $f, g \in \mathcal{O}(\mathbb{C}^{n-1})$ . Note that for  $f \equiv 0$ , these two elements actually belong to the shear group. The different signs depending on the dimension ensure that the Jacobian is equal to 1.

- (1) We first treat the shear group, where  $f \equiv 0$ . By Theorem 1.2 we may choose  $g$  to be a hypercyclic element for the composition operator associated to the translation  $(z_2, \dots, z_n) \mapsto (z_2 + b, \dots, z_n + b)$  of  $\mathbb{C}^{n-1}$ . For the conjugates of  $F_{0,g}$  by the  $m$ -th iteration of  $\tau$  we obtain:

$$\begin{aligned} (\tau^{-m} \circ F_{0,g} \circ \tau^m)(z_1, \dots, z_n) = \\ (z_2, \dots, z_n, -(-1)^n z_1 + g(z_2 + mb, \dots, z_n + mb) + 2z_n) \end{aligned}$$

Thus, by hypercyclicity of the composition operator we obtain that the set  $\{\tau^{-m} \circ F_{0,g} \circ \tau^m, m \in \mathbb{N}\}$  is dense in  $\{F_{0,h}, h \in \mathcal{O}(\mathbb{C}^{n-1})\}$ . In particular it is possible to approximate the map  $I$  by compositions of  $F_{0,g}$  and  $\tau$ , hence also  $I^n = -(-1)^n \text{id}$  and  $I^{-1}(z) = I^{2n-1}(z) = (-(-1)^n z_n, z_1, \dots, z_{n-1})$ . Conjugating with  $I^1, \dots, I^{n-1}$  will give twisted shears in all directions, and  $F_{0,h} \circ I^{-1}$  is an ordinary shear. Therefore all shears and their compositions can be approximated this way.

- (2) For the overshear group we proceed in exactly the same way. By Proposition 2.5, using an ordinary translation  $\tau : \mathbb{C}^n \rightarrow \mathbb{C}^n$  it is possible to approximate all twisted overshears of the form  $F_{0,\hat{g}}$  and  $F_{\hat{f},0}$  and hence also  $F_{\hat{f},\hat{g}} = F_{0,\hat{g}} \circ I^{-1} \circ F_{\hat{f},0}$ .

To ensure that the generated group is actually a free group, we need to show that no reduced word formed by  $F_{f,g}$  and  $\tau$  can equal the identity. This follows from an application of Nevanlinna theory similar to a degree argument as in Ahern and Rudin [1].  $\square$

### Corollary 2.7.

- (1) For any translation  $\tau \neq 0$  of  $\mathbb{C}^n, n \geq 2$ , there exists an  $F \in \text{Aut}_\omega(\mathbb{C}^n)$  such that  $\langle \tau, F \rangle$  is a free and dense (in compact-open topology) subgroup of  $\text{Aut}_\omega(\mathbb{C}^n)$ , where  $\omega = dz_1 \wedge \dots \wedge dz_n$ .
- (2) For any translation  $\tau \neq 0$  of  $\mathbb{C}^n, n \geq 2$ , there exists an  $F \in \text{Aut}(\mathbb{C}^n)$  such that  $\langle \tau, F \rangle$  is a free and dense (in compact-open topology) subgroup of  $\text{Aut}(\mathbb{C}^n)$ .

*Proof.* By a result of Andersén [2], the group of holomorphic shears is dense in  $\text{Aut}_\omega(\mathbb{C}^n)$ ,  $\omega = dz_1 \wedge \dots \wedge dz_n$ , and by the corresponding result of Andersén–Lempert [3], the group of holomorphic overshears is dense in  $\text{Aut}(\mathbb{C}^n)$ .  $\square$

## 3. DENSITY PROPERTY

The density property was introduced in Complex Geometry by Varolin [15], [16]. For a survey about the current state of research related to density property and Andersén–Lempert theory, we refer to Kaliman and Kutzschebauch [9].

**Definition 3.1.** A complex manifold  $X$  has the *density property* if in the compact-open topology the Lie algebra  $Lie_{hol}(X)$  generated by completely integrable holomorphic vector fields on  $X$  is dense in the Lie algebra  $VF_{hol}(X)$  of all holomorphic vector fields on  $X$ .

**Definition 3.2.** Let a complex manifold  $X$  be equipped with a holomorphic volume form  $\omega$ . We say that  $X$  has the *volume density property* with respect to  $\omega$  if in the compact-open topology the Lie algebra  $Lie_{hol}^\omega(X)$  generated by completely integrable holomorphic vector fields  $\nu$  such that  $\nu(\omega) = 0$ , is dense in the Lie algebra  $VF_{hol}^\omega(X)$  of all holomorphic vector fields that annihilate  $\omega$ .

The following theorem is the central result of Andersén–Lempert theory (originating from works of Andersén and Lempert [2], [3]), has been stated by Varolin [15, 16] after introducing the density property and is given in the following form in [9] by Kaliman and Kutzschebauch, but essentially (for  $\mathbb{C}^n$ ) proven already in [6] by Forstnerič and Rosay.

**Theorem 3.3** (Theorem 2 in [9]). *Let  $X$  be a Stein manifold with the density (resp. volume density) property and let  $\Omega$  be an open subset of  $X$ . In case of volume density property further assume that in de-Rham cohomology  $H^{n-1}(\Omega, \mathbb{C}) = 0$ . Suppose that  $\Phi : [0, 1] \times \Omega \rightarrow X$  is a  $\mathcal{C}^1$ -smooth map such that*

- (1)  $\Phi_t : \Omega \rightarrow X$  is holomorphic and injective (and resp. volume preserving) for every  $t \in [0, 1]$
- (2)  $\Phi_0 : \Omega \rightarrow X$  is the natural embedding of  $\Omega$  into  $X$
- (3)  $\Phi_t(\Omega)$  is a Runge subset of  $X$  for every  $t \in [0, 1]$

*Then for each  $\varepsilon > 0$  and every compact subset  $K \subset \Omega$  there is a continuous family  $\alpha : [0, 1] \rightarrow \text{Aut}(X)$  of holomorphic (and resp. volume preserving) automorphisms of  $X$  such that*

$$\alpha_0 = \text{id} \text{ and } \sup_K d(\alpha_t, \Phi_t) < \varepsilon$$

*for every  $t \in [0, 1]$ .*

**Examples 3.4.**

- (1)  $\mathbb{C}^n$ ,  $n \geq 2$ , have the density property.
- (2)  $\mathbb{C}^* \times \mathbb{C}^*$  has the volume density property for the holomorphic volume form  $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$ . Whether it has the density property is not clear.
- (3) Homogeneous Stein manifolds  $X = G/K$ , where  $G$  is a semi-simple Lie group have the density property. (see [5]).
- (4) Linear algebraic groups except those with  $\mathbb{C}$  and  $(\mathbb{C}^*)^n$  as a connected component have the density property, and all linear algebraic groups have the volume density property with respect to the Haar form (see [12], [10]). Note that examples 1 and 2 are special cases of linear algebraic groups.
- (5) A hypersurface  $H \subset \mathbb{C}^{n+2}$  of the form  $H = \{(x, u, v) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} : f(x) = u \cdot v\}$  where  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  is a polynomial with smooth zero fiber, has both the density property and the volume density property with respect to a unique algebraic volume form (see [12], [11]).

We will also need the following lemma which makes the distinction between connected and path-connected component superfluous for groups of holomorphic automorphisms.

**Lemma 3.5.** *The connected component of the group of holomorphic automorphisms resp. volume-preserving holomorphic automorphisms of a complex manifold resp. complex manifold with a holomorphic volume form is  $\mathcal{C}^1$ -path-connected.*

*Proof.* See Lind [13, Remark 6.6]. □

**Theorem 3.6.** *Let  $X$  be a Stein manifold with density property and assume there exists a generalized translation  $\tau \in \text{Aut}^*(X)$ . Then there exists an  $F \in \text{Aut}^*(X)$  such that  $\langle \tau, F \rangle$  is a free and dense (in compact-open topology) subgroup of  $\text{Aut}^*(X)$ .*

*Proof.* Because  $X$  is Stein, there exists an exhaustion  $\{L_j\}_{j \in \mathbb{N}}$  of  $X$  with  $\mathcal{O}(X)$ -convex compacts.

We choose a countable dense subset (in compact-open topology)  $\{g_j\}_{j \in \mathbb{N}} \subset \text{Aut}^*(X)$ . Such a set exists, since we can think of  $\text{Aut}^*(X) \subset \mathcal{O}(X, X) \subset \mathcal{O}(\mathbb{C}^N, \mathbb{C}^N)$  as metric spaces, where we take  $X$  to be properly embedded into  $\mathbb{C}^N$ , and  $\mathcal{O}(\mathbb{C}^N, \mathbb{C}^N)$  is known to be a separable Fréchet space.

We now construct  $F$  inductively as a sequence of maps  $F_j$ ,  $j \in \mathbb{N}$ , together with sequences  $\{\varepsilon_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_{>0}$  with  $\sum_{j=1}^{\infty} \varepsilon_j < \infty$ ,  $\{k(j)\}_{j \in \mathbb{N}}, k(j) \in \mathbb{N}, k(j) < k(j+1)$ , and  $m_j \in \mathbb{N}$ , such that the following hold at step  $j$ :

$$(a_j) \sup_{L_{k(i)}} d(\tau^{-m_i} \circ F_j \circ \tau^{m_i}, g_i) < \varepsilon_i, \forall i \leq j,$$

$$(b_j) \sup_{L_{k(j)}} (d(F_j, F_{j-1}) + d(F_j^{-1}, F_{j-1}^{-1})) < \varepsilon_j.$$

We set  $F_1 = \text{Id}$ ,  $\varepsilon_1 = k(1) = m(1) = 1$ . Assuming that  $g_1 = \text{id}$  we have  $(a_1)$  and  $(a_2)$  is void.

Assume now that we have constructed  $F_{j-1}$  along with the integers for some  $j \geq 2$ .

Choose  $k(j)$  so large such that

$$\tau^{m_i}(L_{k(i)}) \subset L_{k(j)}^\circ \text{ for all } i < j.$$

By Lemma 3.5 there exist  $\mathcal{C}^1$ -paths  $[0, 1] \ni t \rightarrow \varphi_t^j, \psi_t^j$  in  $\text{Aut}^*(X)$  connecting  $F_{j-1}$  and  $g_j$  respectively to the identity. Let  $K_j$  be a large enough holomorphically convex compact set containing both the complete  $\varphi_t^j$ -orbit of  $L_{k(j)} \cup F_{j-1}^{-1}(L_{k_j})$  and the complete  $\psi_t^j$ -orbit of  $L_{k(j)}$  in its interior. Choose  $m_j$  large enough such that  $K_j$  and  $\widehat{\tau^{m_j}(K_j)}$  are disjoint and such that their union is  $\mathcal{O}(X)$ -convex. Let  $C_j := [L_{k(j)} \cup F_{j-1}^{-1}(L_{k(j)})]$ . We define an isotopy  $\Phi_t^j$  of maps near  $C_j \cup \tau^{m_j}(L_{k(j)})$  by setting it equal to  $\varphi_t^j$  near  $C_j$  and  $\tau^{m_j} \circ \psi_t^j \circ \tau^{-m_j}$  near  $\tau^{m_j}(L_{k(j)})$ .

For any  $\tilde{\varepsilon}_j$  we may now apply the Andersén–Lempert Theorem and obtain an  $F_j \in \text{Aut}^*(X)$  such that

$$\sup_{L_{k(j)}} d(F_j, F_{j-1}) + d(F_j^{-1}, F_{j-1}^{-1}) < \tilde{\varepsilon}_j$$

and

$$\sup_{\tau^{m_j}(L_{k(j)})} d(F_j, \tau^{m_j} \circ g_j \circ \tau^{-m_j}) < \varepsilon_j$$

We see that we may choose  $\tilde{\varepsilon}_j < 2^{-j}$  small enough such that  $(a_j)$  and  $(b_j)$  are satisfied.

To ensure that the group generated by  $\tau$  and  $F$  is free, we need to avoid that there is a non-trivial reduced word (formed by  $\tau$  and  $F$ ) equal to the identity. For induction step  $j$  there are finitely many words of length  $\leq j$ . By changing the choice of  $f_j$  within the required bounds we can make sure that on the compact  $L_{k(j)}$  no word of length  $\leq j$  equals the identity. Set  $\delta_j := \min\{\sup_{L_{k(j)}} d(w, \text{id}) : w \neq \text{id} \text{ reduced word of length } \leq j, \text{ consisting of } \tau \text{ and } F_j\}$  and choose all subsequent  $\varepsilon_k, k \geq j+1$ , small enough such that  $\sum_{k=j+1}^{\infty} C_k \varepsilon_k < \delta_j$  where the constants  $C_k$  take into account the number of possible words.

We have now obtained a sequence  $F_j$  of holomorphic automorphisms of  $X$  and it follows from the conditions  $(b_j)$  the  $F_j \rightarrow F \in \text{Aut}^*(X)$ . It is immediate from

the condition  $(a_j)$  that  $\{\tau^{-m_j} \circ F_j \circ \tau^{m_j}\}_{j \in \mathbb{N}}$  is dense in  $\text{Aut}^*(X)$ . It follows that  $\tau^{-m_j} \circ F \circ \tau^{m_j} = (\tau^{-m_j} \circ F_j \circ \tau^{m_j}) \circ (\tau^{-m_j} \circ (F_j^{-1} \circ F) \circ \tau^{m_j})$ ,  $j \in \mathbb{N}$ , is a dense sequence in  $\text{Aut}^*(X)$ , since  $\tau^{-m_j} \circ (F_j^{-1} \circ F) \circ \tau^{m_j} \rightarrow \text{id}$  provided  $\varepsilon_j$  is decreasing fast enough.  $\square$

**Remark 3.7.** In case  $(X, \omega)$  would be a Stein manifold of dimension  $n \in \mathbb{N}$  with volume density property and an exhaustion of compacts  $\{L_j\}_{j \in \mathbb{N}}$  with de Rham cohomology  $H^{n-1}(L_j, \mathbb{C}) = 0$ , one has a similar statement as in Theorem 3.6. Actually, a careful investigation of the proof of the Andersén–Lempert Theorem shows that the condition  $H^{n-1}(L_j, \mathbb{C}) = 0$  can be dropped for components of  $L_j$  where one wants to approximate the identity and for components where one wants to approximate an already globally defined automorphism, provided that  $H^{n-1}(X, \mathbb{C}) = 0$  holds.

**Examples 3.8.** In the following examples of Stein manifolds with volume density property, the method proof of the preceding theorem fails for various reasons:

- The manifold  $\mathbb{C}^*$  with volume form  $\omega = \frac{dz}{z}$  has the volume density property;  $\text{Aut}(\mathbb{C}^*) = \text{Aut}_\omega(\mathbb{C}^*) = \{z \mapsto a \cdot z^{\pm 1} : a \in \mathbb{C}^*\}$ . Therefore it is obvious that no generalized translation exists on  $\mathbb{C}^*$ . In addition there is due to dimensional reasons ( $n - 1 = 0$ ) the obstruction  $\dim H^0(\mathbb{C}^*, \mathbb{C}) = 1$ .
- The manifold  $\mathbb{C}$  with volume form  $\omega = dz$  has the volume density property as well, and actually all volume-preserving automorphisms consist of translations. However again due to dimensional reasons there is the obstruction  $\dim H^0(\mathbb{C}, \mathbb{C}) = 1$ .
- The manifolds  $X := \mathbb{C}^* \times \mathbb{C}^*$  and  $Y := \mathbb{C} \times \mathbb{C}^*$  with volume forms  $\omega_X = \frac{dz}{z} \wedge \frac{dw}{w}$  resp.  $\omega_Y = dz \wedge \frac{dw}{w}$  are known to have the volume density property. Generalized translations, e.g.  $\tau_X(z, w) = (2z, \frac{1}{2}w)$ ,  $\tau_Y(z, w) = (z + 1, w)$ , exist, but in these cases ( $n - 1 = 1$ ) there is a topological obstruction  $\dim H^1(X, \mathbb{C}) = 2$  resp.  $\dim H^1(Y, \mathbb{C}) = 1$ .

The proof of Theorem 3.6 reveals as well that:

**Theorem 3.9.** *Let  $X$  be a Stein manifold with density property and assume there exists a generalized translation  $\tau \in \text{Aut}^*(X)$ . Then the associated conjugation operator  $\tilde{C}_\tau$ , defined by  $\tilde{C}_\tau(f) = \tau \circ f \circ \tau^{-1}$ , is hypercyclic on  $\text{Aut}^*(X)$ .*

#### 4. EXISTENCE OF GENERALIZED TRANSLATIONS

In the last section we give a couple of examples of manifolds with Density Property with a generalized translation.

**Example 4.1.** Let  $Y$  be a Stein manifold, then there exists a generalized translation on  $X = \mathbb{C} \times Y$ , given by  $(z, y) \mapsto (z + a, y)$  for  $a \in \mathbb{C}^*$ . In case  $Y$  is a complex Lie group,  $Y$  has the Density Property (Varolin [15]).

**Example 4.2.** Let  $Y$  be a Stein manifold, then there exists a generalized translation on  $X = \mathbb{C}^* \times \mathbb{C}^* \times Y$ , given by  $(z, w, y) \mapsto (az, w/a, y)$ . In case  $Y$  is a complex Lie group with Volume Density Property,  $Y$  has the Volume Density Property (Varolin [15]).

**Example 4.3.** Let  $p(z) \in \mathbb{C}[z]$  be a polynomial with simple roots. Then the so-called *Danielewski surface*

$$D_p := \{(x, y, z) \in \mathbb{C}^3 : x \cdot y = p(z)\}$$



has the Density Property (Kaliman and Kutzschebauch [11]). The following map is a generalized translation  $\tau_a$ ,  $a \in \mathbb{C}^*$ , on  $D_p$ :

$$\begin{aligned} (x, y, z) &\mapsto \left( x, \frac{p(z + a \cdot x)}{x}, z + a \cdot x \right) \\ &= \left( x, y + \frac{p(z + a \cdot x) - p(z)}{x}, z + a \cdot x \right) \\ &= \left( x, y + \frac{p(z + a \cdot x) - p(z)}{x}, z + a \cdot x \right) \\ &= \left( x, y + p'(z) \cdot a + \sum_{k=2}^{\infty} p^{(k)}(z) \cdot a^k \cdot x^{k-1}, z + a \cdot x \right) \end{aligned}$$

Note that any compact of  $D_p$  can be moved as far away as possible by iterations of  $\tau_a^m = \tau_{am}$ . The only crucial case one has to check is for  $x = 0$ : this implies  $p(z) = 0$ ; but since  $p$  has only simple and finitely many zeros,  $p'(z)$  is bounded away from 0, and so is the shift in  $y$ -direction.

#### REFERENCES

1. Ahern, P. and Rudin, W.: *Periodic Automorphisms of  $\mathbb{C}^n$* . Indiana Univ. Math. J. **44** (1985), No. 2., 173–208
2. Andersén, E.: *Volume-preserving automorphisms of  $\mathbb{C}^n$* . Complex Variables Theory Appl. **14** (1990), no. 1–4, 223–235.
3. Andersén, W. and Lempert, L.: *On the group of holomorphic automorphisms of  $\mathbb{C}^n$* . Invent. Math. **110** (1992), no. 2, 371–388.
4. Birkhoff, G. D.: *Démonstration d’un théorème élémentaire sur les fonctions entières*, C. R. Acad. Sci. Paris **189** (1929) 473–475.
5. Donzelli, F.; Dvorsky, A.; Kaliman, S.: *Algebraic density property of homogeneous spaces*. Transform. Groups **15** (2010), no. 3, 551–576
6. Forstnerič, F. and Jean-Pierre Rosay, J.-P.: *Approximation of biholomorphic mappings by automorphisms of  $\mathbb{C}^n$* , Invent. Math.
7. Forstnerič: *Stein Manifolds and Holomorphic Mappings*. Ergebnisse der Mathematik und ihrer Grenzgebiete. **56**. Springer 2011
8. Grosse-Erdmann, K.-G.: *Universal Families and Hypercyclic Operators*. Bull. Amer. Soc. **36** No. 3 (1999), 324–381
9. Kaliman, Sh. and Kutzschebauch, F.: *On the present state of the Andersén-Lempert theory*. Proc. of the Conf. on Affine Algebraic Geometry held in Professor Russel’s honour. McGill Univ., Montreal, 2009.
10. Kaliman, Sh. and Kutzschebauch, F.: *Criteria for the Density Property of Complex Manifolds*. Invent. Math. **172** (2008), 71–87
11. Kaliman, Sh. and Kutzschebauch, F.: *The Density property for hypersurfaces  $UV = P(\bar{X})$* . Math. Z. **258** (2008), 115–131
12. Kaliman, Sh. and Kutzschebauch, F.: *Algebraic volume density property of affine algebraic manifolds*. Invent. Math. **181** (2010), no. 3, 605–647
13. Lind, A.: *Holomorphic Automorphisms of Danielewski Surfaces*. Mid Sweden University Doctoral Thesis 76. (2009) **112** (1993), no. 2, 323–349.
14. Rosay, J.-P. and Rudin, W.: *Holomorphic Maps from  $C^n$  to  $C^n$* . Tran. AMS **310**, No. 1 (Nov., 1988), 47–86
15. Varolin, D.: *The Density Property for Complex Manifolds and Geometric Structures*. J. Geom. Anal. **11** (2001), 135–160
16. Varolin, D.: *The Density Property for Complex Manifolds and Geometric Structures II*. Internat. J. Math. **11**, No. 6 (2000), 837–847
17. Zajac, S.: *Hypercyclicity of composition operators in pseudoconvex domains*. arXiv:1202.6638