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A POSTERIORI ERROR ANALYSIS OF hp-FEM FOR SINGULARLY PERTURBED PROBLEMS

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ABSTRACT. We consider the approximation of singularly perturbed linear second-order boundary value problems by hp-finite element methods. In particular, we include the case where the associated differential operator may not be coercive. Within this setting we derive an *a posteriori* error estimate for a natural residual norm. The error bound is robust with respect to the perturbation parameter and fully explicit with respect to both the local mesh size h and the polynomial degree p.

1. INTRODUCTION

A posteriori error estimation and adaptivity for low-order methods has seen a significant development in the last decades as witnessed by several monographs [1, 3, 28] on *a posteriori* error estimation, and on convergence and optimality of adaptive algorithms; see, e.g., [7, 13, 26]. The situation is less developed for high-order finite element methods (hp-FEM), where both the local mesh size can be reduced and the local approximation order can be increased to improve the accuracy.

In an hp-context, several adaptive strategies and algorithms have been proposed (see [23] for an overview and comparison). The first work on hp-adaptive strategies for finite element approximations of elliptic problems was presented in [25]. In addition, methods based on smoothness estimation techniques were proposed in [11, 15, 16, 19], or in the recent approach [12, 29, 30] involving Sobolev embeddings, which will also be exploited in the present article. Moreover, a prediction technique was developed in [22]. Further hp-adaptive approaches in the literature include, for example, the use of *a priori* knowledge, mesh optimization strategies, the Texas-3-step algorithm, or the application of reference solution strategies; see, e.g., [2, 8, 9, 14, 24]. Research focusing on the convergence of hp-adaptive FEM has been developed only recently in [5, 6].

In spite of the practical success of these hp-adaptive algorithms, *a posteriori* error estimation in hp-FEM is still a topic of active research, and several, structurally different *a posteriori* error estimators for hp-FEM for standard elliptic problems are available in the literature. We mention in particular the one of residual type, featuring a reliability-efficiency gap in the approximation order [10, 22], and the p-robust estimators of [4], which is particularly suited for H^1-elliptic formulations.

Here, we present an *a posteriori* error estimator for hp-FEM that is suitable for singularly perturbed problems; it is of residual type and results from merging the techniques of [27] for singular perturbations with p-explicit estimators from [22]. More precisely, on an interval $\Omega = (a, b) \subset \mathbb{R}$, $a < b$, we consider the singularly perturbed boundary value problem

$$-\varepsilon u''(x) + d(x)u(x) = f(x), \qquad x \in \Omega, \tag{1}$$

$$u(a) = u(b) = 0. \tag{2}$$

Here, $\varepsilon > 0$ is a possibly small constant, $d \in L^\infty(\Omega)$ is a given function, and $f \in L^2(\Omega)$ is the right-hand side. We use standard notation: For an open set $D \subseteq \Omega$, we let $L^2(D)$ be the standard Lebesgue space of all square-integrable functions on D with norm $\|\cdot\|_{L^2(D)}$, and $L^\infty(D)$ is the space of all essentially bounded functions on D with norm $L^\infty(D)$.

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We propose the following variational formulation of (1)–(2): Find $u \in H_0^1(\Omega)$, the standard L^2-based Sobolev space of first order with vanishing trace, such that

$$a(u,v) := \varepsilon \int_\Omega u'(x)v'(x)\,\mathsf{d}x + \int_\Omega d(x)u(x)v(x)\,\mathsf{d}x = \int_\Omega f(x)v(x)\,\mathsf{d}x \qquad \forall v \in H_0^1(\Omega). \tag{3}$$

Throughout this paper, we make the general assumption that the solution of (3) exists and is unique. Evidently, this the case if $d \geq 0$.

The article is organized as follows: In the following Section 2 we provide the *hp*-framework and *hp*-FEM for the discretization of (1)–(2). Furthermore, Section 3 contains some *hp*-interpolation results, and the *hp*-*a posteriori* error analysis. In addition, we present some numerical tests in Section 4. Finally, we summarize our work in Section 5.

2. *hp*-FEM Discretization

In order to discretize the boundary value problem (1)–(2) by means of an *hp*-finite element method, let us introduce a partition $\mathcal{T} = \{K_j\}_{j=1}^N$ of $N \geq 1$ (open) elements $K_j = (x_{j-1}, x_j)$, $j = 1, 2, \ldots, N$ on $\Omega = (a,b)$, with

$$a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b.$$

The length of an element K_j is denoted by $h_j = x_j - x_{j-1}$, $j = 1, 2, \ldots, N$. For each element $K_j \in \mathcal{T}$, it will be convenient to introduce the patch $\widetilde{K}_j = \bigcup\{K_i \in \mathcal{T} \,|\, \overline{K_i} \cap \overline{K_j} \neq \emptyset\}$ as the union of K_j and of the elements adjacent to it. In addition, to each element K_j we associate a polynomial degree $p_j \geq 1$, $j = 1, 2, \ldots, N$. These numbers are stored in a polynomial degree vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_N)$. Then, we define an *hp*-finite element space by

$$V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p}) = \left\{ v \in H_0^1(\Omega) : v|_{K_j} \in \mathbb{P}_{p_j}(K_j),\, j = 1, 2, \ldots, N \right\},$$

where, for $p \geq 1$, we denote by $\mathbb{P}_p$ the space of all polynomials of degree at most p. We say that the pair $(\mathcal{T}, \boldsymbol{p})$ of a partition $\mathcal{T}$ and of a degree vector $\boldsymbol{p}$ is μ-shape regular, for some constant $\mu > 0$ independent of j, if

$$\mu^{-1} h_{j+1} \leq h_j \leq \mu h_{j+1}, \qquad \mu^{-1} p_{j+1} \leq p_j \leq \mu p_{j+1}, \qquad j = 1, \ldots, N-1, \tag{4}$$

i.e., if both the element sizes and polynomial degrees of neighboring elements are comparable.

We can now discretize the variational formulation (3) by finding a numerical approximation $u_{\mathrm{hp}} \in V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p})$ such that

$$a(u_{\mathrm{hp}}, v) = \int_\Omega f v \,\mathsf{d}x \qquad \forall v \in V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p}). \tag{5}$$

As in the continuous case, we generally suppose that, for a given *hp*-space $V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p})$, a unique numerical solution $u_{\mathrm{hp}} \in V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p})$ of (5) exists.

Furthermore, let us introduce the following norm on $H_0^1(\Omega)$:

$$|\!|\!| v |\!|\!|^2 := \sum_{j=1}^N |\!|\!| v |\!|\!|_{K_j}^2 := \sum_{j=1}^N \left(\varepsilon \left\| v' \right\|_{L^2(K_j)}^2 + \left\| \sqrt{|d|}\, v \right\|_{L^2(K_j)}^2 \right). \tag{6}$$

We note that, if $d \geq 0$ on Ω, then the norm $|\!|\!|\cdot|\!|\!|$ equals the natural energy norm corresponding to the bilinear form $a(\cdot,\cdot)$ from (3). More precisely, in that case we have that $a(v,v) = |\!|\!| v |\!|\!|^2$ for any $v \in H_0^1(\Omega)$.

3. Robust A Posteriori Error Analysis

The goal of this section is to derive an *a posteriori* error analysis for the *hp*-FEM (5) with respect to the residual

$$\mathsf{R}_{\mathrm{hp}}[e_{\mathrm{hp}}] := \sup_{\substack{v \in H_0^1(\Omega) \\ v \not\equiv 0}} \frac{|a(u - u_{\mathrm{hp}}, v)|}{|\!|\!| v |\!|\!|},$$

where $u \in H_0^1(\Omega)$ and $u_{\mathrm{hp}} \in V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p})$ are the exact and numerical solutions of (3) and (5), respectively, and $e_{\mathrm{hp}} = u - u_{\mathrm{hp}}$ signifies the error. Again, let us notice that, if $d \geq 0$, then the residual $\mathsf{R}_{\mathrm{hp}}[e_{\mathrm{hp}}]$ equals the norm $|\!|\!| e_{\mathrm{hp}} |\!|\!|$ of the error.

In order to state our main result, let us denote by Π_{K_j}, for $j = 1, 2, \ldots, N$, the elementwise L^2-projection onto $\mathbb{P}_{p_j}(K_j)$. Moreover, let

$$[\![u_{\mathrm{hp}}']\!](x_j) = u_{\mathrm{hp}}'(x_j^+) - u_{\mathrm{hp}}'(x_j^-) = \lim_{x \searrow x_j} u'(x) - \lim_{x \nearrow x_j} u'(x), \qquad 1 \leq j \leq N-1,$$

signify the jump of u_{hp}' at the mesh point x_j, and define $[\![u_{\mathrm{hp}}']\!](x_0) = [\![u_{\mathrm{hp}}']\!](x_N) = 0$.

3.1. **Main Result.** We shall prove the following *a posteriori* error bound:

Theorem 3.1. *For the error $e_{\mathrm{hp}} = u - u_{\mathrm{hp}}$ between the exact solution $u \in H_0^1(\Omega)$ of* (3) *and its numerical approximation $u_{\mathrm{hp}} \in V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p})$ from* (5), *there holds the following a posteriori error estimate:*

$$\mathsf{R}_{\mathrm{hp}}[e_{\mathrm{hp}}]^2 \leq C \sum_{j=1}^{N} \eta_{K_j}^2. \tag{7}$$

Here, for $j = 1, 2, \ldots, N$,

$$\begin{aligned} \eta_{K_j}^2 := \alpha_j & \left(\left\| \Pi_{K_j} f + \varepsilon u_{\mathrm{hp}}'' - d u_{\mathrm{hp}} \right\|_{L^2(K_j)}^2 + \left\| f - \Pi_{K_j} f \right\|_{L^2(K_j)}^2 \right) \\ & + \frac{1}{2} \varepsilon^2 \gamma_{j-1} \left| [\![u_{\mathrm{hp}}']\!](x_{j-1}) \right|^2 + \frac{1}{2} \varepsilon^2 \gamma_j \left| [\![u_{\mathrm{hp}}']\!](x_j) \right|^2 \end{aligned} \tag{8}$$

are local error indicators, where we let

$$\alpha_j = \begin{cases} \min \left\{ \varepsilon^{-1} h_j^2 p_j^{-2}, \|{}^1\!/\!{}_d\|_{L^\infty(\widetilde{K}_j)} \right\}, & \text{if } {}^1\!/\!{}_d \in L^\infty(\widetilde{K}_j), \\ \varepsilon^{-1} h_j^2 p_j^{-2}, & \text{otherwise}, \end{cases} \tag{9}$$

(with obvious modifications if $j = 0$ or $j = N$), and

$$\beta_j = \alpha_j h_j^{-1} + 2\sqrt{\varepsilon^{-1} \alpha_j}. \tag{10}$$

Moreover,

$$\gamma_j = \frac{\beta_j \beta_{j+1}}{\beta_j + \beta_{j+1}}, \tag{11}$$

for $1 \leq j \leq N-1$, and $\gamma_0 = \gamma_N = 0$. The constant $C > 0$ is independent of u, u_{hp}, f, ε, $\mathcal{T}$, and of $\boldsymbol{p}$.

Remark 3.2. We emphasize that the constants α_j (provided that $\|{}^1\!/\!{}_d\|_{L^\infty(\widetilde{K}_j)} < \infty$) and $\varepsilon^2 \alpha_j \gamma_j$ appearing in the error indicators η_{K_j} from (8) remain bounded as $h_j, \varepsilon \to 0$ (and $p_j \to \infty$). We also note that $\frac{1}{2} \min\{\beta_j, \beta_{j+1}\} \leq \gamma_j \leq \min\{\beta_j, \beta_{j+1}\}$ and that $2\sqrt{\varepsilon^{-1}\alpha_j} \leq \beta_j \leq 3\sqrt{\varepsilon^{-1}\alpha_j}$.

3.2. ***hp*-Interpolation.** For the proof of the above Theorem 3.1 the construction of a suitable *hp*-interpolation operator is crucial. In particular, in order to derive an (upper) *a posteriori* error estimate on the error e_{hp} that is robust with respect to the singular perturbation parameter ε as well as optimally scaled with respect to the local element sizes h_j and polynomial degrees p_j, an interpolant that is *simultaneously* L^2- and H^1-stable is required. This will be accomplished in the current section (Proposition 3.3 and Corollary 3.4).

Proposition 3.3. *Let the pair $(\mathcal{T}, \boldsymbol{p})$ be μ-shape regular (see* (4)*) and $v \in H_0^1(\Omega)$. Then, there exists an interpolant $\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v \in V_{\mathrm{hp}}(\mathcal{T}, \boldsymbol{p})$ of v such that, for any $j = 1, 2, \ldots, N$, there holds*

$$\begin{aligned} &\left\| v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v \right\|_{L^2(K_j)} \leq C_I \left\| v \right\|_{L^2(\widetilde{K}_j)}, \qquad \left\| v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v \right\|_{L^2(K_j)} \leq C_I \frac{h_j}{p_j} \left\| v' \right\|_{L^2(\widetilde{K}_j)}, \\ &\left\| (v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})} v)' \right\|_{L^2(K_j)} \leq C_I \left\| v' \right\|_{L^2(\widetilde{K}_j)}. \end{aligned} \tag{12}$$

Furthermore, we have the nodal estimates

$$|(v-\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v)(x_i)|^2 \le C_I\Big[\frac{1}{h_i+h_{i+1}}\|v-\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v\|^2_{L^2(K_i\cup K_{i+1})} + \|v-\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v\|^2_{L^2(K_i\cup K_{i+1})}\|(v-\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v)'\|^2_{L^2(K_i\cup K_{i+1})}\Big].$$

Here, $C_I>0$ is a constant that depends solely on μ; in particular, it is independent of v, $\mathcal{T}$, and of $\boldsymbol{p}$.

Proof. Let us, without loss of generality, assume that $\Omega=(0,1)$. The result can be shown with the techniques developed for the higher-dimensional case in [17, 18]. In the present, one-dimensional case, a simpler argument can be brought to bear. Let $x_{-1}=-h_1$ and $x_{N+1}=1+h_N$ and φ_i, $i=0,\dots,N+1$ be the standard piecewise linear hat functions associated with the nodes x_i, $i=-1,\dots,N+1$. The extra nodes x_{-1} and x_{N+1} define in a natural way the elements K_0 and K_{N+1}. The (open) patches ω_i, $i=0,\dots,N$, are given by the supports of the functions φ_i, i.e., $\omega_i=(\operatorname{supp}\varphi_i)^\circ=K_i\cup K_{i+1}\cup\{x_i\}$.

Polynomial approximation (see, e.g., [20, Proposition A.2]) gives the existence of a interpolation operator $J_p: L^2(-1,1)\to\mathbb{P}_p(-1,1)$ that is uniformly (in $p\ge 0$) stable, i.e., $\|J_pv\|_{L^2(-1,1)}\le C\|v\|_{L^2(-1,1)}$ for all $v\in L^2(-1,1)$ and has the following properties for $v\in H^1(-1,1)$:

$$(p+1)\|v-J_pv\|_{L^2(-1,1)}+\|(v-J_pv)'\|_{L^2(-1,1)}\le C\|v'\|_{L^2(-1,1)}.$$

Furthermore, if v is antisymmetric with respect to the midpoint $x=0$, then J_pv can be assumed to be antisymmetric as well, i.e., $(J_pv)(0)=0$ (this follows from studying the antisymmetric part of the original function J_pv).

The approximation $\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v$ is now constructed with the aid of a "partition of unity argument" as described in [21, Theorem 2.1]. For ω_0 and ω_N, extend v anti-symmetrically, i.e., $v(x):=-v(-x)$ for $x\in K_0$ and $v(x):=-v(1-x)$ for $x\in K_{N+1}$. Then v is defined on each patch ω_i, $i=0,\dots,N$. For each patch ω_i, let $p_i':=\min\{p_i,p_{i+1}\}$ (with the understanding $p_0=p_1$ and $p_{N+1}=p_N$). The above operator J_p then induces for each patch ω_i by scaling an operator $J^i: L^2(\omega_i)\to\mathcal{P}_{p_i'-1}(\omega_i)$ with the following properties:

$$\frac{p_i'+1}{h_i}\|v-J^iv\|_{L^2(\omega_i)}+\|(v-J^iv)'\|_{L^2(\omega_i)}\le C\|v'\|_{L^2(\omega_i)};$$

here, we have exploited the μ-shape regularity of the mesh. We note that $(J^0v)(0)=0$ and $(J^Nv)(1)=0$. Also, the operators J^i are uniformly (in the polynomial degree) stable in $L^2(\omega_i)$. The approximation $\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v$ is now taken to be $\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v:=\sum_{i=0}^N\varphi_iJ^iv$. The desired approximation properties follow now from [21, Theorem 2.1].

Finally, the nodal estimate results from the observation that at the mesh nodes, there holds the identity $\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v(x_i)=(J^iv)(x_i)$, and from a multiplicative trace inequality (see Appendix A, Lemma A.1). ☐

The above proposition implies the following bounds.

Corollary 3.4. *For $v\in H_0^1(\Omega)$, the interpolant from Proposition 3.3 satisfies*

$$\left\|v-\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v\right\|^2_{L^2(K_j)}\le C_I^2\alpha_j\,|||v|||^2_{\widetilde{K}_j},\qquad j=1,2,\dots,N,$$

and

$$\left|(v-\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2\le C_I^2\gamma_j\left(|||v|||^2_{\widetilde{K}_j}+|||v|||^2_{\widetilde{K}_{j+1}}\right),\qquad j=1,2,\dots,N-1,$$

where α_j and γ_j are defined in (9) *and* (11)*, respectively, and C_I is the constant from* (12).

Proof. We proceed along the lines of [27]. Using the bounds from Proposition 3.3, we have for each element $K_j\in\mathcal{T}$ that

$$\left\|v-\pi_{V_{\rm hp}(\mathcal{T},\boldsymbol{p})}v\right\|^2_{L^2(K_j)}\le C_I^2\frac{h_j^2}{\varepsilon p_j^2}\varepsilon\,\|v'\|^2_{L^2(\widetilde{K}_j)}.$$

Furthermore, if ${}^1\!/d \in L^\infty(\widetilde{K}_j)$, then

$$\left\|v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|^2_{L^2(K_j)} \leq C_I^2 \left\|v\right\|^2_{L^2(\widetilde{K}_j)} \leq C_I^2 \left\|{}^1\!/d\right\|_{L^\infty(\widetilde{K}_j)} \left\|\sqrt{|d|}v\right\|^2_{L^2(\widetilde{K}_j)}.$$

Combining these two estimates, yields the first bound.

In order to prove the second estimate, we apply, for $1 \leq j \leq N-1$, a multiplicative trace inequality (see Appendix A, Lemma A.1):

$$\begin{aligned}&\left|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2 \\ &\quad\leq h_j^{-1}\left\|v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|^2_{L^2(K_j)} + 2\left\|v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|_{L^2(K_j)}\left\|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)'\right\|_{L^2(K_j)}.\end{aligned}$$

Then, invoking the above bounds as well as the estimates from Proposition 3.3, we get

$$\begin{aligned}\left|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2 &\leq C_I^2\left(\alpha_j h_j^{-1}\left\|\!\left|v\right|\!\right\|^2_{\widetilde{K}_j} + 2\sqrt{\alpha_j}\left\|\!\left|v\right|\!\right\|_{\widetilde{K}_j}\left\|v'\right\|_{L^2(\widetilde{K}_j)}\right) \\ &\leq C_I^2\left(\alpha_j h_j^{-1}\left\|\!\left|v\right|\!\right\|^2_{\widetilde{K}_j} + 2\sqrt{\varepsilon^{-1}\alpha_j}\left\|\!\left|v\right|\!\right\|^2_{\widetilde{K}_j}\right) \\ &\leq C_I^2\beta_j\left\|\!\left|v\right|\!\right\|^2_{\widetilde{K}_j},\end{aligned}$$

with β_j from (10). Since x_j is also a boundary point of K_{j+1}, we similarly obtain that

$$\left|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2 \leq C_I^2\beta_{j+1}\left\|\!\left|v\right|\!\right\|^2_{\widetilde{K}_{j+1}}.$$

Therefore,

$$\begin{aligned}\left|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2 &= \frac{\beta_{j+1}}{\beta_j+\beta_{j+1}}\left|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2 + \frac{\beta_j}{\beta_j+\beta_{j+1}}\left|(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|^2 \\ &\leq C_I^2\gamma_j\left(\left\|\!\left|v\right|\!\right\|^2_{\widetilde{K}_j} + \left\|\!\left|v\right|\!\right\|^2_{\widetilde{K}_{j+1}}\right),\end{aligned}$$

with γ_j from (11). Thus, we have shown the second estimate. □

3.3. **Proof of Theorem 3.1.** We are now in a position to prove the *hp-a posteriori* error bound (7).

From the definitions of the exact solution u from (3) and the numerical solution u_{hp} defined in (5), it follows that, for any $v \in H_0^1(\Omega)$ and any $v_{\mathrm{hp}} \in V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})$,

$$\begin{aligned}a(u,v) - a(u_{\mathrm{hp}},v) &= a(u, v - v_{\mathrm{hp}}) - a(u_{\mathrm{hp}}, v - v_{\mathrm{hp}}) \\ &= \int_\Omega f(v - v_{\mathrm{hp}})\,\mathsf{d}x - \varepsilon\int_\Omega u_{\mathrm{hp}}'(v - v_{\mathrm{hp}})'\,\mathsf{d}x - \int_\Omega d u_{\mathrm{hp}}(v - v_{\mathrm{hp}})\,\mathsf{d}x.\end{aligned}$$

Integrating by parts elementwise in the second integral leads to

$$\begin{aligned}\int_\Omega u_{\mathrm{hp}}'(v - v_{\mathrm{hp}})'\,\mathsf{d}x &= \sum_{j=1}^N\int_{K_j} u_{\mathrm{hp}}'(v - v_{\mathrm{hp}})'\,\mathsf{d}x \\ &= -\sum_{j=1}^N\int_{K_j} u_{\mathrm{hp}}''(v - v_{\mathrm{hp}})\,\mathsf{d}x + \sum_{j=1}^N\left(u_{\mathrm{hp}}'(x_j^-)(v - v_{\mathrm{hp}})(x_j) - u_{\mathrm{hp}}'(x_{j-1}^+)(v - v_{\mathrm{hp}})(x_{j-1})\right) \\ &= -\sum_{j=1}^N\int_{K_j} u_{\mathrm{hp}}''(v - v_{\mathrm{hp}})\,\mathsf{d}x - \sum_{j=1}^{N-1}[\![u_{\mathrm{hp}}']\!](x_j)(v - v_{\mathrm{hp}})(x_j),\end{aligned}$$

and thus, choosing $v_{\mathrm{hp}} = \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v$ to be the hp-interpolant from Section 3.2, we arrive at

$$\begin{aligned}a(u,v) - a(u_{\mathrm{hp}},v) &= \sum_{j=1}^N\left(\Pi_{K_j}f + \varepsilon u_{\mathrm{hp}}'' - d u_{\mathrm{hp}}\right)(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)\,\mathsf{d}x \\ &\quad + \sum_{j=1}^N\left(f - \Pi_{K_j}f\right)(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)\,\mathsf{d}x + \varepsilon\sum_{j=1}^{N-1}[\![u_{\mathrm{hp}}']\!](x_j)(v - \pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j).\end{aligned}$$

Hence, applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|a(u,v)-a(u_{\mathrm{hp}},v)| \leq &\sum_{j=1}^{N}\left\|\Pi_{K_j}f+\varepsilon u_{\mathrm{hp}}''-du_{\mathrm{hp}}\right\|_{L^2(K_j)}\left\|v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|_{L^2(K_j)}\\ &+\sum_{j=1}^{N}\left\|f-\Pi_{K_j}f\right\|_{L^2(K_j)}\left\|v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v\right\|_{L^2(K_j)}\\ &+\sum_{j=1}^{N-1}\varepsilon\left|[\![u_{\mathrm{hp}}']\!](x_j)\right|\left|(v-\pi_{V_{\mathrm{hp}}(\mathcal{T},\boldsymbol{p})}v)(x_j)\right|.\end{aligned}$$

The bounds from Corollary 3.4 lead to

$$\begin{aligned}|a(u,v)-a(u_{\mathrm{hp}},v)| \leq &\, C_I\sum_{j=1}^{N}\sqrt{\alpha_j}\left\|\Pi_{K_j}f+\varepsilon u_{\mathrm{hp}}''-du_{\mathrm{hp}}\right\|_{L^2(K_j)}|\!|\!|v|\!|\!|_{\widetilde{K}_j}\\ &+C_I\sum_{j=1}^{N}\sqrt{\alpha_j}\left\|f-\Pi_{K_j}f\right\|_{L^2(K_j)}|\!|\!|v|\!|\!|_{\widetilde{K}_j}\\ &+C_I\sum_{j=1}^{N-1}\left(|\!|\!|v|\!|\!|_{\widetilde{K}_j}^2+|\!|\!|v|\!|\!|_{\widetilde{K}_{j+1}}^2\right)^{1/2}\varepsilon\sqrt{\gamma_j}\left|[\![u_{\mathrm{hp}}']\!](x_j)\right|.\end{aligned}$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned}&|a(u,v)-a(u_{\mathrm{hp}},v)|\\ &\leq C_I\left(\sum_{j=1}^{N}\alpha_j\left\|\Pi_{K_j}f+\varepsilon u_{\mathrm{hp}}''-du_{\mathrm{hp}}\right\|_{L^2(K_j)}^2+\alpha_j\left\|f-\Pi_{K_j}f\right\|_{L^2(K_j)}^2\right)^{1/2}\left(2\sum_{j=1}^{N}|\!|\!|v|\!|\!|_{\widetilde{K}_j}^2\right)^{1/2}\\ &\quad+C_I\left(\sum_{j=1}^{N-1}\varepsilon^2\gamma_j\left|[\![u_{\mathrm{hp}}']\!](x_j)\right|^2\right)^{1/2}\left(\sum_{j=1}^{N-1}\left(|\!|\!|v|\!|\!|_{\widetilde{K}_j}^2+|\!|\!|v|\!|\!|_{\widetilde{K}_{j+1}}^2\right)\right)^{1/2}\end{aligned}$$

Observing that

$$\sum_{j=1}^{N}|\!|\!|v|\!|\!|_{\widetilde{K}_j}^2\leq 3\,|\!|\!|v|\!|\!|^2,\qquad \sum_{j=1}^{N-1}\left(|\!|\!|v|\!|\!|_{\widetilde{K}_j}^2+|\!|\!|v|\!|\!|_{\widetilde{K}_{j+1}}^2\right)\leq 6\,|\!|\!|v|\!|\!|^2,$$

we finally see that

$$|a(u,v)-a(u_{\mathrm{hp}},v)|\ \leq\sqrt{12}C_I\left(\sum_{j=1}^{N}\eta_{K_j}^2\right)^{1/2}|\!|\!|v|\!|\!|,$$

with η_{K_j} from (8). Dividing both sides of this inequality by $|\!|\!|v|\!|\!|$ and taking the supremum for all $v\in H_0^1(\Omega)$ shows Theorem 3.1.

Remark 3.5. In the case $d>0$, following along the lines of [27] and [22], and employing p-dependent norm equivalence estimates in order to be able to involve suitable cut-off functions locally, it is possible to prove ε-robust local lower bounds for the error in terms of the error indicators η_{K_j} and some data oscillation terms. Specifically, if d satisfies $0<d_0\leq\inf_{x\in\Omega}d(x)\leq\sup_{x\in\Omega}d(x)\leq d_1<\infty$ and $\beta\in(1/2,1]$ is fixed, then, the lower bounds

$$\alpha_j\|f-(-\varepsilon u_{\mathrm{hp}}''+du_{\mathrm{hp}})\|_{L^2(K_j)}^2\leq C\left[p_j^2\,|\!|\!|u-u_{\mathrm{hp}}|\!|\!|_{K_j}^2+\alpha_jR_{K_j}^2\right],\quad 1\leq j\leq N,$$

and

$$\gamma_j\varepsilon^2|[\![u_{\mathrm{hp}}]\!](x_j)|^2\leq C\left[p_j^2\,|\!|\!|u-u_{\mathrm{hp}}|\!|\!|_{K_j\cup K_{j+1}}^2+\alpha_jR_{K_j}^2+\alpha_{i+1}R_{K_{j+1}}^2\right],\quad 1\leq j\leq N-1,$$

can be proved. Here, for any element $K_j \in \mathcal{T}$, $1 \leq j \leq N$, the data oscillation term R_{K_j} is defined by

$$R_{K_j} = p_j^{\beta} \left[\left\| \Phi_{K_j}^{\beta/2}(f - \Pi_{K_j} f) \right\|_{L^2(K_j)} + \left\| \Phi_{K_j}^{\beta/2}(du_{\mathrm{hp}} - \Pi_{K_j}(du_{\mathrm{hp}})) \right\|_{L^2(K_j)} \right] + \|f - \Pi_{K_j} f\|_{L^2(K_j)} + \|du_{\mathrm{hp}} - \Pi_{K_j}(du_{\mathrm{hp}})\|_{L^2(K_j)}.$$

The constant $C > 0$ depends only on the ratio d_1/d_0, the choice of $\beta \in (1/2, 1]$, and the shape-regularity parameter μ from (4); see Appendix B (in particular, Theorem B.4) for details. It is worth stressing that the L^2-projector $\Pi_{K_j} : L^2(K_j) \to \mathbb{P}_{p_j}(K_j)$ can be replaced with a projection onto a space of polynomials of degree λp_j for a fixed $\lambda > 0$. While the constant C then additionally depends on λ, this allows to exploit smoothness of the coefficient function d in the treatment of the second term in R_{K_j}.

4. Numerical Experiments

The purpose of this section is to illustrate the *a posteriori* error estimates from Theorem 3.1 in the context of some specific numerical experiments. We will emphasize on the robustness of the error indicators with respect to ε as $\varepsilon \to 0$, and on the capability of hp-FEM to deliver exponential rates of convergence.

4.1. hp-Adaptive Procedure.

We shall apply an hp-adaptive algorithm which is based on the following ingredients:

(a) *Element marking:* The elementwise error indicators η_{K_j} from Theorem 3.1 are employed in order to mark elements for refinement. More precisely, we fix a parameter $\theta \in (0,1)$ (in the experiments below we choose $\theta = 0.5$) and select elements to be refined according to the *Dörfler marking* criterion:

$$\theta \sum_{j=1}^{N} \eta_{K_j}^2 \leq \sum_{j'=1}^{M} \eta_{K_{j'}}^2. \tag{D}$$

Here, the indices j' are chosen such that the error indicators $\eta_{K_{j'}}$ from (8) are sorted in descending order, and M is minimal.

(b) *hp-refinement criterion:* The decision of whether a marked element in step (a) is refined with respect to h (element bisection) or p (increasing the local polynomial order by 1) is based on a smoothness testing approach. Specifically, if the (numerical) solution is considered smooth on a marked element K_j, then the polynomial degree is increased by 1 on that particular element (no element bisection), otherwise the element is bisected (retaining the current polynomial degree p_j on both subelements). In order to evaluate the smoothness of the solution u_{hp} on a marked element K_j, we employ an elementwise smoothness indicator as introduced in [12, Eq. (3)]:

$$\mathcal{F}_j^{p_j}[u_{\mathrm{hp}}] := \begin{cases} \dfrac{\left\| \frac{\mathsf{d}^{p_j-1}}{\mathsf{d}x^{p_j-1}} u_{\mathrm{hp}} \right\|_{L^\infty(K_j)}}{h_j^{-1/2} \left\| \frac{\mathsf{d}^{p_j-1} u_{\mathrm{hp}}}{\mathsf{d}x^{p_j-1}} \right\|_{L^2(K_j)} + \frac{1}{\sqrt{2}} h_j^{1/2} \left\| \frac{\mathsf{d}^{p_j} u_{\mathrm{hp}}}{\mathsf{d}x^{p_j}} \right\|_{L^2(K_j)}} & \text{if } \frac{\mathsf{d}^{p_j-1}}{\mathsf{d}x^{p-1}} u_{\mathrm{hp}}|_{K_j} \not\equiv 0, \\ 1 & \text{if } \frac{\mathsf{d}^{p_j-1}}{\mathsf{d}x^{p_j-1}} u_{\mathrm{hp}}|_{K_j} \equiv 0. \end{cases} \tag{F}$$

Here, the basic idea is to consider the continuous Sobolev embedding $H^1(K_j) \hookrightarrow L^\infty(K_j)$, which implies that

$$\sup_{v \in H^1(K_j)} \frac{\|v\|_{L^\infty(K_j)}}{h_j^{-1/2} \|v\|_{L^2(K_j)} + \frac{1}{\sqrt{2}} h_j^{1/2} \|v'\|_{L^2(K_j)}} \leq 1;$$

see [12, Proposition 1]. In particular, it follows that $\mathcal{F}_j^{p_j}[u_{\mathrm{hp}}] \leq 1$. For ease of evaluation, note that, by taking the derivative of order $p_j - 1$ in the definition (F), the smoothness indicator $\mathcal{F}_j^{p_j}[u_{\mathrm{hp}}]$ is evaluated for linear functions only; in this case, it can be shown that

$$\frac{1}{2} \approx \frac{\sqrt{3}}{\sqrt{6}+1} \leq \mathcal{F}_j^{p_j}[u_{\mathrm{hp}}] \leq 1;$$

cf. [12, Section 2.2]. The numerical solution u_{hp} is classified smooth on K_j if $\mathcal{F}_j^{p_j}[u_{\mathrm{hp}}] \geq \tau$ and otherwise nonsmooth, for a prescribed smoothness testing parameter $\tau \in (\sqrt{3}/(\sqrt{6}+1), 1)$ (in our experiments we choose $\tau = 0.6$). Incidentally, representing the local solution $u_{\mathrm{hp}}|_{K_j}$ in terms of (local) Legendre polynomials (or more general Jacobi polynomials), any derivatives of u_{hp} can be evaluated exactly by means of appropriate recurrence relations. We refer to the papers [12] (see also [29, 30]) for more details on this smoothness testing strategy.

Combing the above ideas leads to the following *hp*-adaptive refinement algorithm:

Algorithm 1. *Choose prescribed parameters $\theta \in (0,1)$ and $\tau \in \left(\frac{\sqrt{3}}{\sqrt{6}+1}, 1\right)$ for the Dörfler marking as well as for the hp-decision process as described before, respectively. Furthermore, consider a (coarse) initial mesh $\mathcal{T}^0$, and an associated polynomial degree vector $\boldsymbol{p}^0$. Set $n = 0$. Then, perform the following iteration (until a given maximum iteration number is reached, or until the estimated error is sufficiently small):*

(1) Compute the numerical solution $u_{\mathrm{hp}}^n \in V_{\mathrm{hp}}(\mathcal{T}^n, \boldsymbol{p}^n)$ from (5)*, and evaluate the error indicators $\{\eta_{K_j}\}_{K_j \in \mathcal{T}^n}$ defined in* (8)*.*
(2) Mark the elements in $\mathcal{T}^n$ based on the Dörfler marking (D)*.*
(3) Create the mesh $\mathcal{T}^{n+1}$ with corresponding polynomial degree distribution $\boldsymbol{p}^{n+1}$: For each marked element $K_j \in \mathcal{T}^n$ evaluate the smoothness indicator $\mathcal{F}_j^{p_j}[u_{\mathrm{hp}}]$ from (F)*; if there holds $\mathcal{F}_j^{p_j}[u_{\mathrm{hp}}] \geq \tau$ then increase the polynomial degree p_j^n by 1, i.e., $p_j^n \leftarrow p_j^n + 1$, otherwise bisect K_j into two new elements (taking p_j^n for both elements). Increase n by 1, i.e., $n \leftarrow n + 1$.*

In the ensuing experiments, we will start Algorithm 1 based on a uniform initial mesh consisting of 10 elements, and a polynomial degree distribution $\boldsymbol{p}^0 = (1, \ldots 1)$.

4.2. **Example 1:** We begin by looking at the singularly perturbed reaction-diffusion problem

$$-\varepsilon u'' + u = 1 \quad \text{on } \Omega = (-1,1), \qquad u(-1) = u(1) = 0.$$

This problem is coercive and has exactly one (analytic) solution. For small $\varepsilon \ll 1$ the exact solution exhibits a boundary layer at $x = 0$ and $x = 1$ which needs to be resolved properly by the *hp*-adaptive FEM. In Figure 1 the *hp*-mesh after 24 adaptive refinement steps is displayed for $\varepsilon = 10^{-4}$. We observe that the boundary layer is resolved by some mild *h*-refinement and by increasing p in the same area. Moreover, the mesh remains unrefined in the center of the domain where the exact solution is nearly constant 1. In addition, in Figures 2 and 3 we show the errors measured with respect to the norm $|||\cdot|||$ from (6) as well as the estimated errors. The exponential decay of both quantities for different choices of ε becomes clearly visible in the semi-logarithmic plot. Finally, the efficiency indices, i.e., the ratio between the estimated and true errors, are depicted in Figure 4; they oscillate between 1 and 4, and do not deteriorate as $\varepsilon \to 0$, thereby clearly testifying to the robustness of the *a posteriori* error estimate from Theorem 3.1.

4.3. **Example 2:** In this experiment, we consider Airy's equation

$$-\varepsilon u'' + xu = 1 \quad \text{on } \Omega = (-1,1), \qquad u(-1) = u(1) = 0.$$

The particularity of this example is that, for $0 < \varepsilon \ll 1$, the corresponding differential operator is coercive for $x \geq 1$, however, it becomes hyperbolic near $x = -1$; this becomes evident in Figure 5, where the numerical solution is shown for $\varepsilon = 10^{-4}$. The oscillating regime for $x < 0$ requires a proper resolution by the *hp*-FEM as shown in the *hp*-mesh in Figure 6. The decay of the estimated error is plotted in Figure 7 for various choices of ε. In particular, for small ε, we see that, after a number of initial refinements resolving the oscillations, the algorithm provides exponentially converging results.

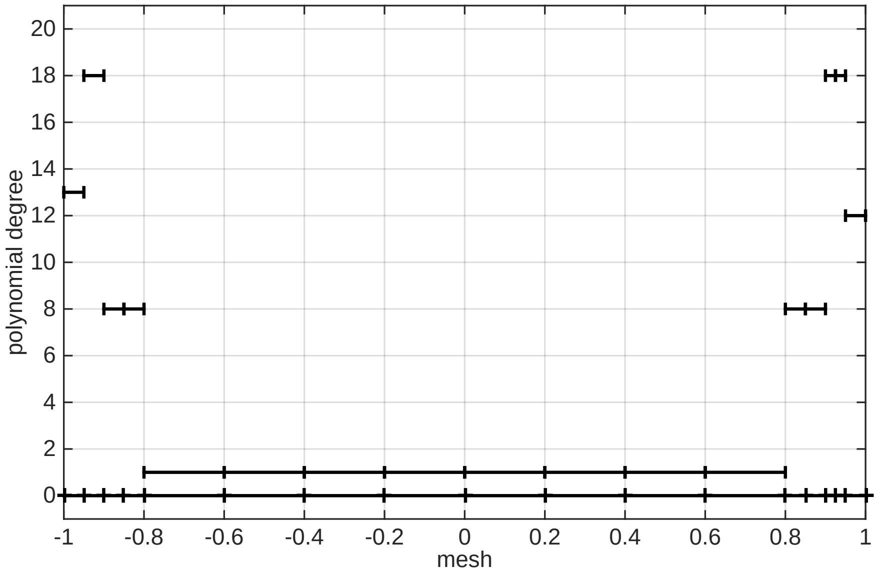

FIGURE 1. Example 1 for $\varepsilon = 10^{-4}$: Adaptively generated hp-mesh after 24 refinement steps (17 elements, maximal polynomial degree 18).

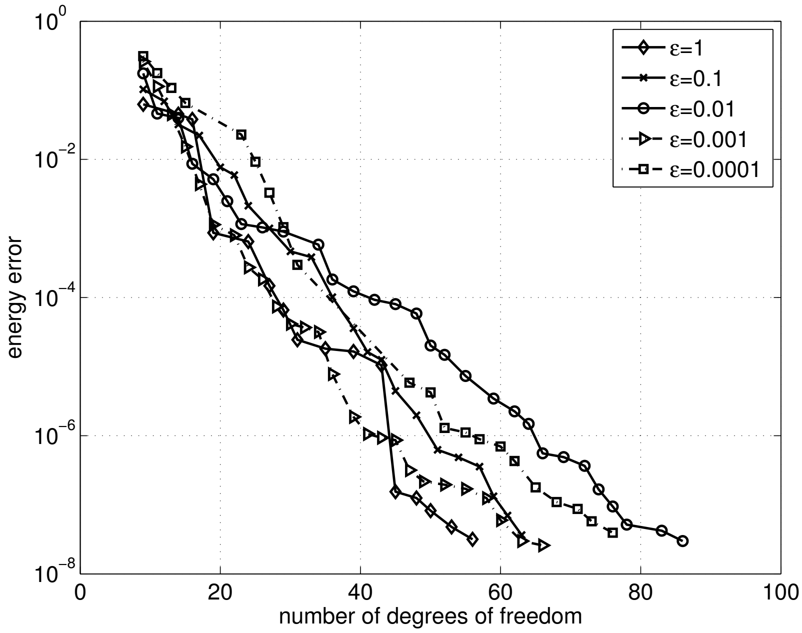

FIGURE 2. Example 1: Energy error for different choices of ε.

5. Conclusions

In this paper we have studied the numerical approximation of linear second-order boundary value problems (with possibly non-constant reaction coefficient) by the hp-FEM. In particular, we have derived an *a posteriori* error estimate for a natural residual-type norm that is robust with respect to the (possibly) small perturbation parameter and explicit with respect to the local mesh

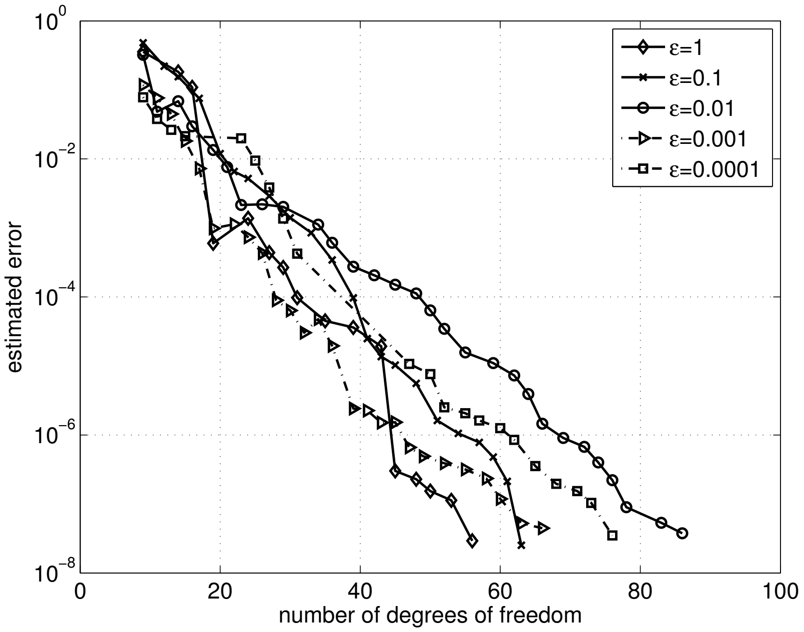

FIGURE 3. Example 1: Estimated error for different choices of ε.

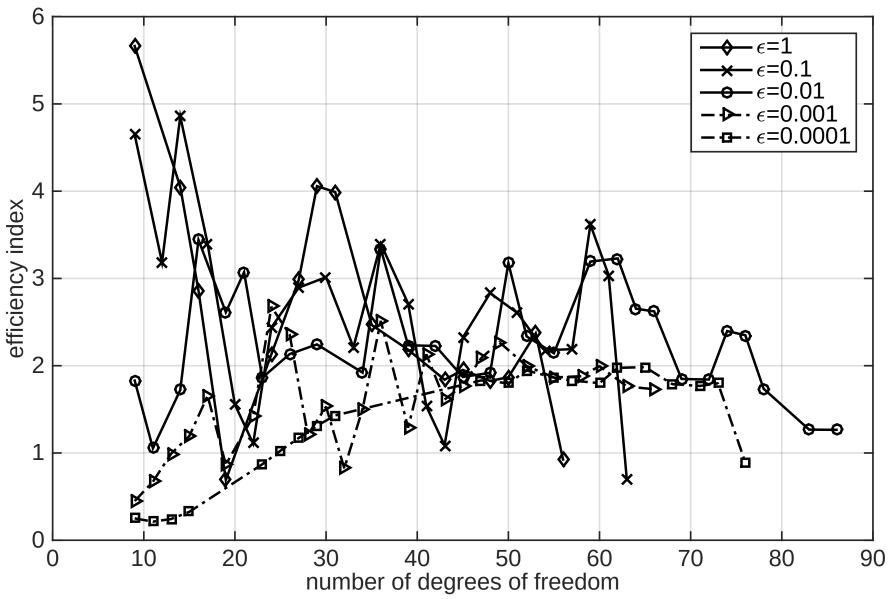

FIGURE 4. Example 1: Efficiency indices for different choices of ε.

size and polynomial degree. Numerical experiments for both coercive as well as partly coercive differential equations underline the robustness of the error bound. In addition, an appropriate combination of the error estimate with a smoothness testing procedure reveals that the method is able to achieve exponential rates of convergence.

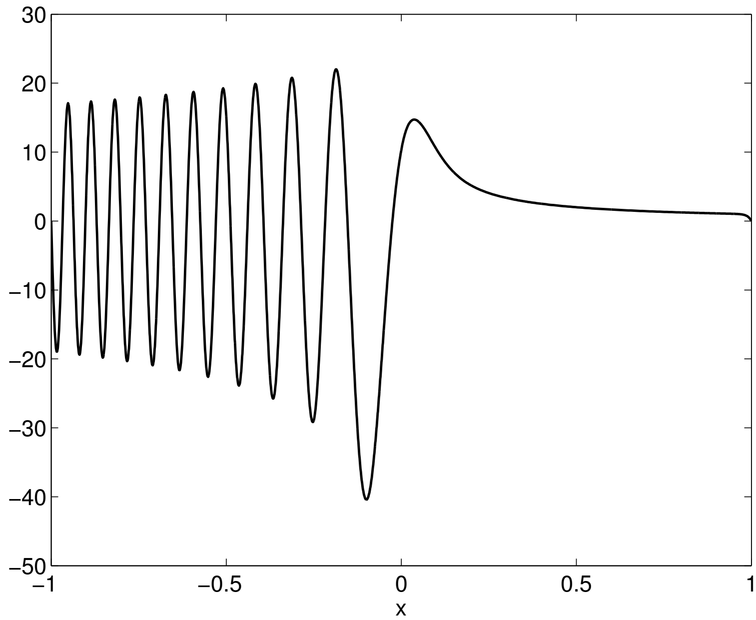

FIGURE 5. Example 2 for $\varepsilon = 10^{-4}$: Numerical solution with strong oscillations for $x < 0$.

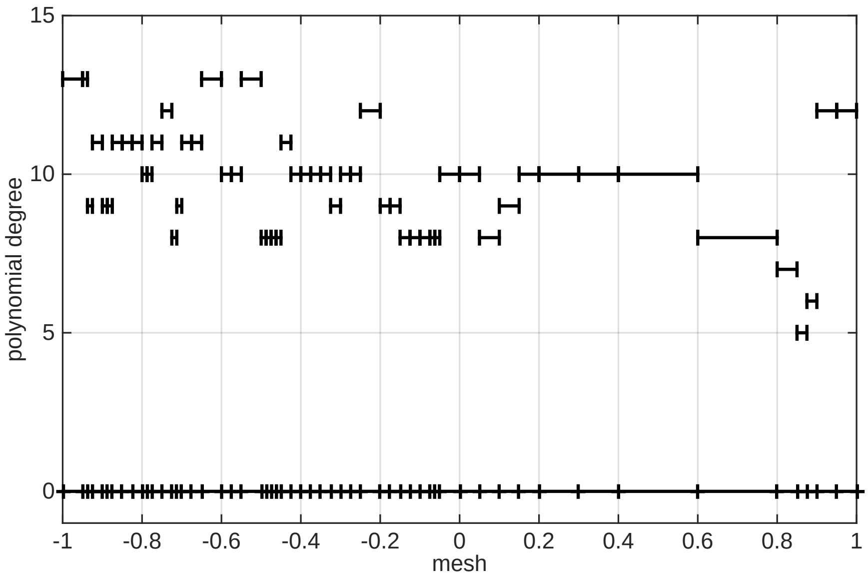

FIGURE 6. Example 2 for $\varepsilon = 10^{-4}$: Adaptively generated hp-mesh after 75 refinement steps (55 elements, maximal polynomial degree 13).

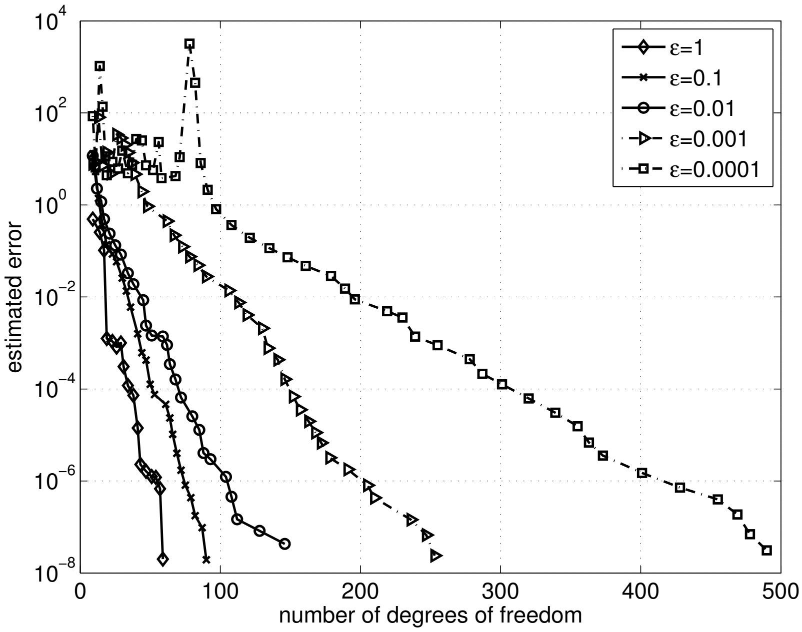

FIGURE 7. Example 2: Estimated errors for different choices of ε.

Appendix A. Multiplicative Trace Inequality

Lemma A.1. *Let $h > 0$ and $w \in H^1(0,h)$. Then, the multiplicative trace inequality*

$$\max\{|w(0)|, |w(h)|\}^2 \leq h^{-1}\|w\|^2_{L^2(0,h)} + 2\|w\|_{L^2(0,h)}\|w'\|_{L^2(0,h)}$$

holds true.

Proof. By density of $C^\infty([0,h])$ in $H^1(0,h)$, we may suppose that w is smooth. There holds

$$w(0)^2 = \int_0^h \frac{\mathsf{d}}{\mathsf{d}x}\left[\left(h^{-1}x - 1\right)w(x)^2\right]\,\mathsf{d}x = h^{-1}\int_0^h w(x)^2\,\mathsf{d}x + 2\int_0^h \left(h^{-1}x - 1\right)w(x)w'(x)\,\mathsf{d}x.$$

Then, applying the Cauchy-Schwarz inequality and noticing that $\left|1 - h^{-1}x\right| < 1$ for $x \in (0,h)$, results in

$$|w(0)|^2 \leq h^{-1}\|w\|^2_{L^2(0,h)} + 2\|w\|_{L^2(0,h)}\|w'\|_{L^2(0,h)}.$$

By symmetry, the same bound can be obtained for $|w(h)|^2$. This completes the proof. □

Appendix B. Details on the efficiency bound

We follow [22], taking care of the presence of the singular perturbation parameter ε as well as the fact that the coefficient d is possibly variable. For an element $K_i = (x_{i-1}, x_i)$, $1 \leq i \leq N$, let Φ_{K_i} be the scaled distance function from $\partial K_i = \{x_{i-1}, x_i\}$, i.e.,

$$\Phi_{K_i}(x) = h_i^{-1}\min(|x - x_{i-1}|, |x - x_i|), \qquad x \in K_i.$$

Lemma B.1. *Let u_{hp} be the hp-FEM solution of* (5)*, and $K_i \in \mathcal{T}$, $1 \leq i \leq N$. Then, for any $\beta \in (-1/2, 1]$ there exists a constant $C_\beta > 0$ such that*

$$\begin{aligned}
&\|f - (-\varepsilon u''_{\mathrm{hp}} + du_{\mathrm{hp}})\|_{L^2(K_i)} \\
&\quad\leq C_\beta\Bigg\{\left(\sqrt{\varepsilon}\frac{p_i^2}{h_i} + p_i^\beta\left\|\sqrt{|d|}\right\|_{L^\infty(K_i)}\right)|\!|\!|u - u_{\mathrm{hp}}|\!|\!|_{K_i} \\
&\qquad + p_i^\beta\left[\left\|\Phi_{K_i}^{\beta/2}(f - \Pi_{K_i}f)\right\|_{L^2(K_i)} + \left\|\Phi_{K_i}^{\beta/2}(du_{\mathrm{hp}} - \Pi_{K_i}(du_{\mathrm{hp}}))\right\|_{L^2(K_i)}\right] \\
&\qquad + \|f - \Pi_{K_i}f\|_{L^2(K_i)} + \|du_{\mathrm{hp}} - \Pi_{K_i}(du_{\mathrm{hp}})\|_{L^2(K_i)}\Bigg\}.
\end{aligned} \tag{13}$$

Remark B.2.

(i) As written in Lemma B.1, Π_{K_i} signifies the $L^2(K_i)$-projection. This is not essential and could be replaced with other approximation operators. It is also not necessary that Π_{K_i} maps into the space of polynomials of degree p_i—it could as well be the space of degree $2p_i$.

(ii) The right-hand side of (13) involves the hp-FEM solution u_{hp} from (5). If Π_{K_i} maps into the space of polynomials of degree $2p_i$, then the term $du_{\mathrm{hp}} - \Pi_i(du_{\mathrm{hp}})$ can be controlled provided some *a priori* control of $\|u_{\mathrm{hp}}\|_{L^2(\Omega)}$ or at least of $\|\sqrt{|d|}u_{\mathrm{hp}}\|_{L^2(\Omega)}$ is available.

(iii) For those elements where d is bounded away from 0, Lemma B.1 provides indeed a lower bound since then

$$\sqrt{\alpha_i} \sim \min\left\{\frac{h_i}{\sqrt{\varepsilon}p_i}, \left\|\frac{1}{\sqrt{|d|}}\right\|_{L^\infty(K_i)}\right\}$$

is of order $\mathcal{O}(1)$. In fact, if $\inf_{x\in K_i}|d(x)|$ and $\sup_{x\in K_i}|d(x)|$ are of comparable magnitude, then

$$\sqrt{\alpha_i}\left(\sqrt{\varepsilon}\frac{p_i^2}{h_i} + p_i^\beta\left\|\sqrt{|d|}\right\|_{L^\infty(K_i)}\right) \leq Cp_i, \tag{14}$$

where the constant $C > 0$ depends only on the ratio

$$\frac{\sup_{x\in K_i}|d(x)|}{\inf_{x\in K_i}|d(x)|}. \tag{15}$$

(iv) If the ratio (15) cannot be controlled well (e.g., if $|d|$ becomes arbitrarily small or even zero on K_i), then the efficiency bound breaks down unless the element is sufficiently small (relative to ε).

Proof of Lemma B.1. Let $\beta \in (0,1]$. On K_i define

$$v_{K_i} := \Phi_{K_i}^{\beta} \cdot (\Pi_{K_i}(f|_{K_i}) - (-\varepsilon(u_{\rm hp}|_{K_i})'' + \Pi_{K_i}(du_{\rm hp}|_{K_i}))).$$

We write

$$\begin{aligned}\left\|\Phi_K^{-\beta/2} v_{K_i}\right\|_{L^2(K_i)}^2 &= \int_{K_i} (\Pi_{K_i} f - (-\varepsilon u_{\rm hp}'' + \Pi_{K_i}(du_{\rm hp})) v_{K_i} \,\mathsf{d}x \\ &= \int_{K_i} (f - (-\varepsilon u_{\rm hp}'' + du_{\rm hp})) v_{K_i} \,\mathsf{d}x + \int_{K_i} (\Pi_{K_i} f - f) v_{K_i} \,\mathsf{d}x \\ &\quad - \int_{K_i} (\Pi_{K_i}(du_{\rm hp}) - du_{\rm hp}) v_{K_i} \,\mathsf{d}x \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

We first focus on the term I_1. Since the function v_{K_i} vanishes at the endpoints of K_i, we may view it, by extension by zero outside of K_i, as an element of $H_0^1(\Omega)$. We observe

$$I_1 = a(u, v_{K_i}) - a(u_{\rm hp}, v_{K_i}) = a(u - u_{\rm hp}, v_{K_i}) \leq \|u - u_{\rm hp}\|_{K_i} \|v_{K_i}\|_{K_i},$$

where the subscript K_i in the norms indicates that the defining integral is taken over K_i and not over Ω.

We now claim that, for $\beta \in (1/2, 1]$, we have

$$\|v_{K_i}\|_{K_i} \leq C \left[\sqrt{\varepsilon} p_i^{2-\beta} h_i^{-1} + \left\|\sqrt{|d|}\right\|_{L^\infty(K_i)}\right] \left\|\Phi^{-\beta/2} v_{K_i}\right\|_{L^2(K_i)}. \tag{16}$$

To see this, we compute with the product rule

$$\begin{aligned}\|v_{K_i}'\|_{L^2(K_i)} &\lesssim \left\|\Phi_{K_i}^{\beta}(\Pi_{K_i} f - (-\varepsilon u_{\rm hp}'' + \Pi_{K_i}(du_{\rm hp})))'\right\|_{L^2(K_i)} \\ &\quad + h_i^{-1} \left\|\Phi_{K_i}^{\beta-1}(\Pi_{K_i} f - (-\varepsilon u_{\rm hp}'' + \Pi_{K_i}(du_{\rm hp}))\right\|_{L^2(K_i)},\end{aligned}$$

and use the fact that $\Pi_{K_i} f - (-\varepsilon u_{\rm hp}'' + \Pi_{K_i}(du_{\rm hp}))$ is a polynomial: For the first term, we employ [22, Lemma 2.4, 3^{rd} estimate], and for the second term, we apply [22, Lemma 2.4, 2^{nd} estimate] (this is the point where we need $\beta > 1/2$ so that $2(\beta - 1) > -1$) to get

$$\begin{aligned}&\left\|\Phi_{K_i}^{\beta}(\Pi_{K_i} f - (-\varepsilon u_{\rm hp}'' + \Pi_{K_i}(du_{\rm hp})))'\right\|_{L^2(K_i)} \\ &\qquad \lesssim p_i^{2-\beta} h_i^{-1} \left\|\Phi_{K_i}^{\beta/2}(\Pi_{K_i} f - (-\varepsilon u_{\rm hp}'' + \Pi_{K_i}(du_{\rm hp})))\right\|_{L^2(K_i)} \\ &\qquad = p_i^{2-\beta} h_i^{-1} \left\|\Phi_{K_i}^{-\beta/2} v_{K_i}\right\|_{L^2(K_i)},\end{aligned}$$

and analogously,

$$h_i^{-1}\left\|\Phi_{K_i}^{\beta-1}(\Pi_{K_i} f - (-\varepsilon u_{\rm hp}'' + \Pi_{K_i}(du_{\rm hp})))\right\|_{L^2(K_i)} \lesssim p_i^{2-\beta} h_i^{-1} \left\|\Phi_{K_i}^{-\beta/2} v_{K_i}\right\|_{L^2(K_i)}.$$

Furthermore, we note the simple estimate

$$\left\|\sqrt{|d|} v_{K_i}\right\|_{L^2(K_i)} \leq \left\|\sqrt{|d|}\right\|_{L^\infty(K_i)} \left\|\Phi_{K_i}^{-\beta/2} v_{K_i}\right\|_{L^2(K_i)}.$$

It follows that

$$\|v_{K_i}\|_{K_i} \lesssim \left(\sqrt{\varepsilon} p_i^{2-\beta} h_i^{-1} + \left\|\sqrt{|d|}\right\|_{L^\infty(K_i)}\right) \left\|\Phi_{K_i}^{-\beta/2} v_{K_i}\right\|_{L^2(K_i)},$$

which is the claimed estimate (16).

The terms I_2 and I_3 are estimated straightforwardly by

$$|I_2| + |I_3| \leq \left(\left\| \Phi_{K_i}^{\beta/2}(f - \Pi_{K_i} f) \right\|_{L^2(K_i)} + \left\| \Phi_{K_i}^{\beta/2}(du_{\text{hp}} - \Pi_{K_i}(du_{\text{hp}})) \right\|_{L^2(K_i)} \right) \left\| \Phi_{K_i}^{-\beta/2} v_{K_i} \right\|_{L^2(K_i)}.$$

We conclude for any $\beta \in (1/2, 1]$ the existence of a constant $C > 0$ (depending only on β) such that

$$\begin{aligned} \left\| \Phi_{K_i}^{-\beta/2} v_{K_i} \right\|_{L^2(K_i)} \lesssim & \left(\sqrt{\varepsilon} p_i^{2-\beta} h_i^{-1} + \left\| \sqrt{|d|} \right\|_{L^\infty(K_i)} \right) \|\!| u - u_{\text{hp}} |\!\|_{K_i} \\ & + \left\| \Phi_{K_i}^{\beta/2}(f - \Pi_{K_i} f) \right\|_{L^2(K_i)} + \left\| \Phi_{K_i}^{\beta/2}(du_{\text{hp}} - \Pi_{K_i}(du_{\text{hp}})) \right\|_{L^2(K_i)}. \end{aligned} \tag{17}$$

We now turn to bounding the volume contribution of the *a posteriori* error estimator. We fix $\beta \in (1/2, 1]$, and estimate with the aid of [22, Lemma 2.4, 2^{nd} estimate]:

$$\begin{aligned} & \|f - (-\varepsilon u''_{\text{hp}} + du_{\text{hp}})\|_{L^2(K_i)} \\ & \quad \leq \left\| \Pi_{K_i} f - (-\varepsilon u''_{\text{hp}} + \Pi_{K_i}(du_{\text{hp}})) \right\|_{L^2(K_i)} + \|f - \Pi_{K_i} f\|_{L^2(K_i)} + \|du_{\text{hp}} - \Pi_{K_i}(du_{\text{hp}})\|_{L^2(K_i)} \\ & \quad \lesssim p_i^\beta \left\| \Phi_{K_i}^{\beta/2} \left(\Pi_{K_i} f - (-\varepsilon u''_{\text{hp}} + \Pi_{K_i}(du_{\text{hp}})) \right) \right\|_{L^2(K_i)} \\ & \qquad + \|f - \Pi_{K_i} f\|_{L^2(K_i)} + \|du_{\text{hp}} - \Pi_{K_i}(du_{\text{hp}})\|_{L^2(K_i)} \\ & \quad = p_i^\beta \|\Phi_{K_i}^{-\beta/2} v_{K_i}\|_{L^2(K_i)} + \|f - \Pi_{K_i} f\|_{L^2(K_i)} + \|du_{\text{hp}} - \Pi_{K_i}(du_{\text{hp}})\|_{L^2(K_i)}, \end{aligned}$$

which, recalling (17), results in (with implied constant depending on $\beta \in (1/2, 1]$)

$$\begin{aligned} \|f - (-\varepsilon u''_{\text{hp}} + du_{\text{hp}})\|_{L^2(K_i)} \lesssim & \left(\sqrt{\varepsilon} \frac{p_i^2}{h_i} + p_i^\beta \left\| \sqrt{|d|} \right\|_{L^\infty(K_i)} \right) \|\!| u - u_{\text{hp}} |\!\|_{K_i} \\ & + p_i^\beta \left[\left\| \Phi_{K_i}^{\beta/2}(f - \Pi_{K_i} f) \right\|_{L^2(K_i)} + \left\| \Phi_{K_i}^{\beta/2}(du_{\text{hp}} - \Pi_{K_i}(du_{\text{hp}})) \right\|_{L^2(K_i)} \right] \\ & + \|f - \Pi_{K_i} f\|_{L^2(K_i)} + \|du_{\text{hp}} - \Pi_{K_i}(du_{\text{hp}})\|_{L^2(K_i)}. \end{aligned}$$

This completes the proof. □

For every interior node x_i, $i = 1, \ldots, N-1$, let $\omega_i := (x_{i-1}, x_{i+1})$ be the node patch associated with node x_i.

Lemma B.3. *Let x_i be an interior node with node patch ω_i, and u_{hp} the hp-FEM solution from* (5)*. For any $\delta_i > 0$, there holds*

$$\varepsilon |[\![u'_{\text{hp}}]\!](x_i)| \leq \left(\left(\varepsilon \delta_i^{-1} \right)^{1/2} + \delta_i^{1/2} \left\| \sqrt{|d|} \right\|_{L^\infty(\omega_i)} \right) \|\!| u - u_{\text{hp}} |\!\|_{\omega_i} + \delta_i^{1/2} \|r_i\|_{L^2(\omega_i)}, \tag{18}$$

where $r_i := f - (-\varepsilon u''_{\text{hp}} + du_{\text{hp}})$ is a function defined on $\omega_i \setminus \{x_i\}$.

Proof. Let x_i be an interior node, and $\delta_i > 0$. Moreover, let $\psi_i \in H_0^1(\omega_i)$ be a cut-off function with $\psi_i(x_i) = 1$, and

$$\|\psi_i\|_{L^2(\omega_i)} \leq \delta_i^{1/2}, \qquad \|\psi'_i\|_{L^2(\omega_i)} \leq \delta_i^{-1/2}.$$

Then, the function $\widetilde{\psi}_i := [\![u'_{\text{hp}}]\!](x_i) \psi_i$ belongs to $H_0^1(\omega_i)$ (and is extended by zero to yield a function in $H_0^1(\Omega)$). An integration by parts gives

$$\begin{aligned} \varepsilon |[\![u'_{\text{hp}}]\!](x_i)|^2 = \varepsilon [\![u'_{\text{hp}}]\!](x_i) \widetilde{\psi}_i(x_i) &= -\int_{\omega_i} \varepsilon u'_{\text{hp}} \widetilde{\psi}'_i \, \mathsf{d}x - \int_{\omega_i} \varepsilon u''_{\text{hp}} \widetilde{\psi}_i \, \mathsf{d}x \\ &= a(u - u_{\text{hp}}, \widetilde{\psi}_i) - \int_{\omega_i} r_i \widetilde{\psi}_i \, \mathsf{d}x, \end{aligned}$$

and thus

$$\varepsilon |[\![u'_{\text{hp}}]\!](x_i)|^2 \leq |[\![u'_{\text{hp}}]\!](x_i)| \left[\|\!| u - u_{\text{hp}} |\!\|_{\omega_i} \|\!| \psi_i |\!\|_{\omega_i} + \|r_i\|_{L^2(\omega_i)} \|\psi_i\|_{L^2(\omega_i)} \right].$$

We conclude with the properties of ψ_i:

$$\varepsilon|[\![u_{\rm hp}']\!](x_i)| \le \left(\left(\varepsilon\delta_i^{-1}\right)^{1/2} + \delta_i^{1/2}\left\|\sqrt{|d|}\right\|_{L^\infty(\omega_i)}\right)|\!|\!|u-u_{\rm hp}|\!|\!|_{\omega_i} + \delta_i^{1/2}\|r_i\|_{L^2(\omega_i)},$$

which is the asserted estimate. □

As already mentioned in Remark B.2, a particularly good setting for efficiency estimates is that the coefficient function d is bounded from below.

Theorem B.4. *Suppose that there exist constants $0 < d_0 \le d_1 < \infty$ with*

$$d_0 \le \inf_{x\in\Omega} d(x) \le \sup_{x\in\Omega} d(x) \le d_1.$$

Fix $\beta \in (1/2, 1]$. Then there exists a constant $C > 0$ (depending only on β, the ratio d_1/d_0, and the shape-regularity parameter μ from (4)) such that the following is true:

(i) Let K_i, $1 \le i \le N$, be an element. Then,

$$\alpha_i\|f-(-\varepsilon u_{\rm hp}''+du_{\rm hp})\|^2_{L^2(K_i)} \le C\left[p_i^2\,|\!|\!|u-u_{\rm hp}|\!|\!|^2_{K_i} + \alpha_i R_{K_i}^2\right],$$

where we set

$$R_{K_i} = p_i^\beta\left[\left\|\Phi_{K_i}^{\beta/2}(f-\Pi_{K_i}f)\right\|_{L^2(K_i)} + \left\|\Phi_{K_i}^{\beta/2}(du_{\rm hp}-\Pi_{K_i}(du_{\rm hp}))\right\|_{L^2(K_i)}\right] + \|f-\Pi_{K_i}f\|_{L^2(K_i)} + \|du_{\rm hp}-\Pi_{K_i}(du_{\rm hp})\|_{L^2(K_i)}.$$

(ii) Let $\omega_i = K_i\cup K_{i+1}\cup\{x_i\}$ be the node patch associated with the interior node x_i, $1\le i\le N-1$. Then

$$\gamma_i\varepsilon^2|[\![u_{\rm hp}]\!](x_i)|^2 \le C\left[p_i^2\,|\!|\!|u-u_{\rm hp}|\!|\!|^2_{\omega_i} + \alpha_i R_{K_i}^2 + \alpha_{i+1}R_{K_{i+1}}^2\right].$$

Proof. The estimate in (i) follows directly from Lemma B.1 and the observation (14). For (ii) we employ Lemma B.3. Let K_i, K_{i+1} be the two elements sharing node x_i. By the shape regularity property (4), and recalling Remark 3.2 we see that

$$\gamma_i \sim \sqrt{\varepsilon^{-1}\alpha_i} \sim \sqrt{\varepsilon^{-1}\alpha_{i+1}}.$$

We will simply write α for α_i and γ for γ_i. We make use of the freedom to select $\delta_i := \alpha/\gamma$ in Lemma B.3. Then, employing (18) and involving $\gamma\sim\sqrt{\varepsilon^{-1}\alpha}$, we arrive at

$$\begin{aligned}\gamma\varepsilon^2|[\![u_{\rm hp}']\!](x_i)|^2 &\lesssim \gamma\left(\frac{\varepsilon}{\delta_i}+\delta_i\|d\|_{L^\infty(\omega_i)}\right)|\!|\!|u-u_{\rm hp}|\!|\!|^2_{\omega_i} + \gamma\delta_i\|r_i\|^2_{L^2(\omega_i)}\\ &= \left(\frac{\gamma^2}{\alpha}\varepsilon+\alpha\|d\|_{L^\infty(\omega_i)}\right)|\!|\!|u-u_{\rm hp}|\!|\!|^2_{\omega_i} + \alpha\|r_i\|^2_{L^2(\omega_i)}\\ &\lesssim (1+d_1/d_0)\,|\!|\!|u-u_{\rm hp}|\!|\!|^2_{\omega_i} + \alpha\|r_i\|^2_{L^2(\omega_i)}.\end{aligned}$$

We close the proof by remarking that the term $\alpha\|r_i\|^2_{L^2(\omega_i)}$ has been estimated earlier in (i). □

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