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Canonical simulations of supersymmetric SU(N) Yang-Mills quantum mechanics

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The fermion loop formulation naturally separates partition functions into their canonical sectors. Here we discuss various strategies to make use of this for supersymmetric SU(N) Yang-Mills quantum mechanics obtained from dimensional reduction in various dimensions and present numerical results for the separate canonical sectors with fixed fermion numbers. We comment on potential problems due to the sign of the contributions from the fermions and due to flat directions.

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1. Introduction

Since long it is suspected that SU(N) gauge theories can be regarded as the low energy effective theory of N D-branes in specific parameter regimes. In this way, dimensionally reduced large-N supersymmetric Yang-Mills (SYM) gauge theories might provide a nonperturbative formulation of the string/M-theory which describes the dynamics of the D-branes. In particular, the connection between black p-branes and SYM gauge theories in $d = (p+1)$ dimensions allows to study black hole thermodynamics through the corresponding strongly coupled gauge theory. Apart from providing tests of the gauge/gravity duality, SYM quantum mechanics is interesting *per se*. There are various intriguing expectations concerning the behaviour of the theory in specific sectors of fixed fermion number [1, 2]. In particular, while certain canonical sectors only have a discrete energy spectrum, some sectors allow for the discrete spectrum to be immersed in a continuous one reaching down to zero energy. These continuous spectra can presumably be associated with the so-called flat directions of the potential which are present in the classical theory, but may or may not survive the quantisation of the theory.

In these proceedings we report on our ongoing effort to perform nonperturbative calculations in SYM quantum mechanics with gauge group SU(N). Here we concentrate on $\mathscr{N} = 4$ SYM quantum mechanics which is obtained from dimensional reduction of the $\mathscr{N} = 1$ SYM gauge theory in $d = 4$ dimensions by compactifying the three spatial dimensions. To define the theory we employ the lattice regularisation in Euclidean time proposed in [3, 4]. The bosonisation of the theory on the lattice by means of the fermion loop formulation [5, 6, 7] decomposes the fermion contributions into fermion sectors with fixed fermion number. Our recent progress in understanding the algebraic structure of the fermion loop formulation allows the explicit construction of transfer matrices [8]. The transfer matrices in turn provide the starting point for the construction of local fermion update algorithms which allow to directly simulate fixed canonical sectors of the theory.

In the following we summarise the derivation of the transfer matrices for generic fermion number sectors, recapitulating our results from [8], and show first results for some simple observables such as the Polyakov loop, the moduli of the bosonic fields and the fermion action from simulations in fixed canonical sectors of the theory.

2. Lattice regularisation and canonical sectors

We directly start with $\mathscr{N} = 4$ SYM quantum mechanics obtained from dimensional reduction of $\mathscr{N} = 1$ SYM in $d = 4$ dimensions down to $d = 1$ dimension. The action in Euclidean time can be written as

$$S = \frac{1}{g^2}\int_0^\beta dt\,\mathrm{Tr}\left\{(D_tX_i)^2 - \frac{1}{2}[X_i,X_j]^2 + \overline{\psi}D_t\psi - \overline{\psi}\sigma_i[X_i,\psi]\right\} \tag{2.1}$$

where $D_t = \partial_t - i[A(t),\cdot]$ denotes the covariant derivative using the time component of the SU(N) gauge field $A(t)$, while the spatial components become the bosonic fields $X_i(t)$ with $i = 1,2,3$. The anticommuting 2-component complex fermion fields $\overline{\psi}(t)$, $\psi(t)$ interact with the bosonic fields X_i through a Yukawa-type interaction involving the three Pauli-matrices σ_i. We note that all the fields are in the adjoint representation of SU(N). The discretisation of the Lagrangian on a time lattice

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with L_t points is straightforward and yields for the bosonic part

$$S_B = \frac{1}{g^2}\sum_{t=0}^{L_t-1} \mathrm{Tr}\left\{D_t X_i(t) D_t X_i(t) - \frac{1}{2}[X_i(t),X_j(t)]^2\right\} \tag{2.2}$$

where $D_t X_i(t) = U(t)X_i(t+1)U^\dagger(t) - X_i(t)$ is the covariant forward derivative and $U(t)$ is an element of the gauge group SU(N). For the fermionic part of the action we introduce a Wilson term with Wilson parameter $r=1$ in order to avoid fermion doubling. With this choice of r the massless Wilson Dirac operator in one dimension involves just the forward derivative and one obtains

$$S_F = \frac{1}{g^2}\sum_{t=0}^{L_t-1} \mathrm{Tr}\left\{\overline{\psi}(t) D_t \psi(t) - \overline{\psi}(t)\sigma_i [X_i(t),\psi(t)]\right\} \tag{2.3}$$

for the fermion action. More specifically, we have

$$S_F = \frac{1}{2g^2}\sum_{t=0}^{L_t-1}\left[-\overline{\psi}^a_\alpha(t) W^{ab}_{\alpha\beta}(t) e^{+\mu}\psi^b_\beta(t+1) + \overline{\psi}^a_\alpha(t)\Phi^{ac}_{\alpha\beta}(t)\psi^c_\beta(t)\right] \equiv \overline{\psi}\mathscr{D}_{p,a}[U,X_i;\mu]\psi \tag{2.4}$$

where $W(t)$ denote the real adjoint gauge link matrices

$$W^{ab}_{\alpha\beta}(t) = 2(\sigma_0)_{\alpha\beta}\otimes \mathrm{Tr}\{T^a U(t) T^b U(t)^\dagger\}\,. \tag{2.5}$$

Note that we have introduced a chemical potential μ in the standard way [9]. The subscripts p,a for the Dirac matrix $\mathscr{D}$ denote periodic or antiperiodic boundary conditions, respectively, for the fermionic fields. The Yukawa interaction matrices $\Phi(t)$ are $n_f^{\max}\times n_f^{\max}$ with $n_f^{\max} = 2(N^2-1)$ and read

$$\Phi^{ac}_{\alpha\beta}(t) = (\sigma_0)_{\alpha\beta}\otimes\delta^{ac} - 2\,(\sigma_i)_{\alpha\beta}\otimes \mathrm{Tr}\{T^a[X_i(t),T^c]\}\,. \tag{2.6}$$

Two remarks are in order. Firstly, all supersymmetry breaking terms apart from the ones introduced by the discretisation in eq. (2.2) and (2.4) are forbidden by the gauge symmetry. Hence, supersymmetry is expected to be automatically restored in the continuum limit without any fine tuning [3, 4]. Secondly, the Wilson term breaks the time reversal symmetry, or equivalently the particle-hole exchange symmetry. This reflects itself in the fact that the action in eq. (2.4) only allows forward propagating fermions. As a consequence, the exchange symmetry between the related fermion sectors with n_f and $n_f^{\max} - n_f$ becomes exact only in the continuum limit.

Let us now derive exact expressions for the fermionic contributions to the partition function of the theory for a given fixed gauge and boson field background – the canonical determinants. In quantum mechanics the lattice regulated determinant of the Dirac matrix can readily be calculated, and one obtains

$$\det\mathscr{D}_{p,a}[U,X_i;\mu] = \det\left[\mathscr{T}\mp e^{+\mu L_t}\right] \quad \text{with} \quad \mathscr{T}\equiv\prod_{t=0}^{L_t-1}\Phi(t)W(t)\,. \tag{2.7}$$

This essentially corresponds to the dimensionally reduced determinant for Wilson fermions derived in [10, 11] except that here the dimensional reduction is from the full matrix to a $n_f^{\max}\times n_f^{\max}$ 'flavour' matrix. It is now easy to get the canonical determinants from the fugacity expansion

$$\det\mathscr{D}_{p,a}[U,X_i;\mu] = \sum_{n_f=0}^{2(N^2-1)}(\mp e^{\mu L_t})^{n_f}\det\mathscr{D}_{n_f}[U,X_i] \tag{2.8}$$

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which identifies the canonical determinants as the coefficients of the characteristic polynomial. These coefficients can be expressed in terms of the elementary symmetric functions S_k of order k of the eigenvalues $\{\tau_i, i = 1,\ldots,n_f^{\max}\}$ of $\mathscr{T}$,

$$S_k(\mathscr{T}) \equiv S_k(\{\tau_i\}) = \sum_{1\leq i_1<\cdots<i_k\leq n_f^{\max}} \prod_{j=1}^{k} \tau_{i_j}\,, \tag{2.9}$$

and one eventually obtains

$$\det \mathscr{D}_{n_f}[U,X_i] = S_{n_f^{\max}-n_f}(\mathscr{T})\,. \tag{2.10}$$

So the crucial object for the calculation of the canonical determinants is the product $\mathscr{T}$ of the matrices $\Phi(t)$ and $W(t)$ which in fact is a product of transfer matrices [8], as was suspected already in [10].

3. Transfer matrices for the canonical sectors

The explicit construction of the fermion transfer matrices for each fermion sector is most easily done via the fermion loop formulation [8] which in essence is an exact (hopping) expansion of the fermionic Boltzmann factor to all orders. In this formulation, the contributions of the fermions to the partition function are obtained by summing over all possible closed oriented fermion loops which are forward propagating in time for any given gauge and boson field background. The loop configuration space naturally separates into subspaces characterised by the number of forward propagating fermions n_f. The transfer matrix elements are explicitly given in terms of the cofactors $C(\Phi)$ and the complementary minors $M(W)$,

$$\left(T^{\Phi}_{n_f}\right)_{AB} = C_{\not{B}\not{A}}(\Phi) \;= (-1)^{p(A,B)} \det \Phi^{\not{B}\not{A}}\,, \tag{3.1}$$

$$\left(T^{W}_{n_f}\right)_{AB} = M_{AB}(W) = \det W^{AB}\,, \tag{3.2}$$

where A,B are sets of indices $A,B \subseteq \{1,\ldots,n_f^{\max}\}$ of order n_f and $p(A,B) = \sum_{i\in A} i + \sum_{j\in B} j$. $\Phi^{\not{B}\not{A}}$ denotes the matrix obtained from Φ by deleting the rows with indices from B and the columns with indices from A, while W^{AB} denotes the matrix obtained from W by picking only the rows with indices from A and columns with indices from B. The size of the transfer matrices is given by the number of such sets for a given n_f and corresponds to the number of forward propagating fermion states $N_{\text{states}} = n_f^{\max}!/(n_f^{\max}-n_f)!\cdot n_f!$. The fermion contribution to the partition function in sector n_f is then simply given by

$$\det \mathscr{D}_{n_f}[U,X_i] = \text{Tr}\left[\prod_{t=0}^{L_t-1} T^{\Phi}_{n_f}(t)\cdot T^{W}_{n_f}(t)\right] \tag{3.3}$$

and one can use the Cauchy-Binet formula and some further algebra [8] to show that

$$\left[\prod_{t=0}^{L_t-1} T^{\Phi}_{n_f}(t)\cdot T^{W}_{n_f}(t)\right]_{AB} = (-1)^{p(A,B)} \det \mathscr{T}^{\not{A}\not{B}} = C_{\not{A}\not{B}}(\mathscr{T})\,, \tag{3.4}$$

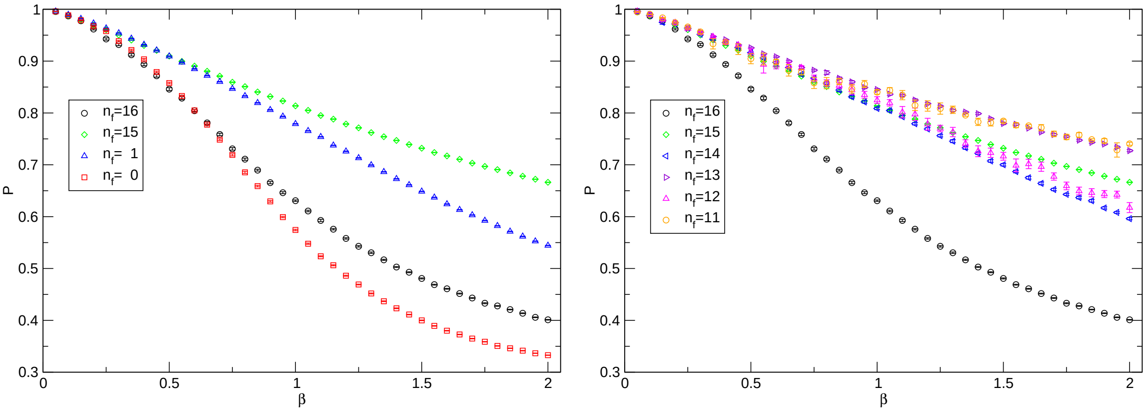

Figure 1: The modulus P of the Polyakov loop as a function of β for SU(3) on a $L_t = 5$ lattice in various canonical sectors.

hence the canonical determinant is simply given by the sum over the principal minors of order n_f of $\mathscr{T}$ denoted by $E_{n_f}(\mathscr{T})$,

$$\det \mathscr{D}_{n_f}[U,X_i] = \sum_B \det \mathscr{T}^{\backslash B \backslash B} \equiv E_{n_f}(\mathscr{T})\,. \tag{3.5}$$

Finally, it is easy to show that $E_{n_f}(\mathscr{T}) = S_{n_f^{\max}-n_f}(\mathscr{T})$ which proves the equivalence between the representation using the transfer matrices and the one in eq. (2.10).

Some remarks are in order. Firstly, we note that the matrix $\mathscr{T}$ describes the dimensionally reduced effective action for the Polyakov loop coupled to the bosonic fields X_i. Secondly, our result for the canonical determinants in principle allows for local fermion update algorithms, but in practice only the sectors with $n_f = 0$ and $n_f = n_f^{\max}$ can be implemented straightforwardly, while in other sectors algorithms along the lines in [12] can be employed. Thirdly, the construction of the transfer matrices and the calculation of the canonical determinants in terms of those is applicable to QCD, since the algebraic structures of the theories are the same.

4. Canonical simulations

Here we present our first results from simulations of the system employing the gauge group SU(N) with $N = 3$ and $n_f^{\max} = 16$ directly in the various canonical sectors[1]. First we note that the canonical determinants are real because the eigenvalues τ_i of $\mathscr{T}$ are real or come in complex conjugate pairs. Furthermore, for the sectors $n_f = 0$ and $n_f = n_f^{\max}$ (quenched) one can prove that the canonical determinants are positive. In these two sectors we update the bosonic degrees of freedom using a local Metropolis algorithm. In the other sectors we use Metropolis updates based on eq. (3.5) and currently simulate only in the configuration space with positive determinants.

In the following we show results on a lattice with temporal extent $L_t = 5$. We measure the moduli of the Polyakov loop and the scalar field defined by

$$P = \left| \mathrm{Tr} \prod_t U(t) \right|, \qquad R^2 \equiv |X|^2 = X_i^a X_i^a\,. \tag{4.1}$$

[1]Note also the recent effort using the RHMC algorithm for SU(2) in [13].

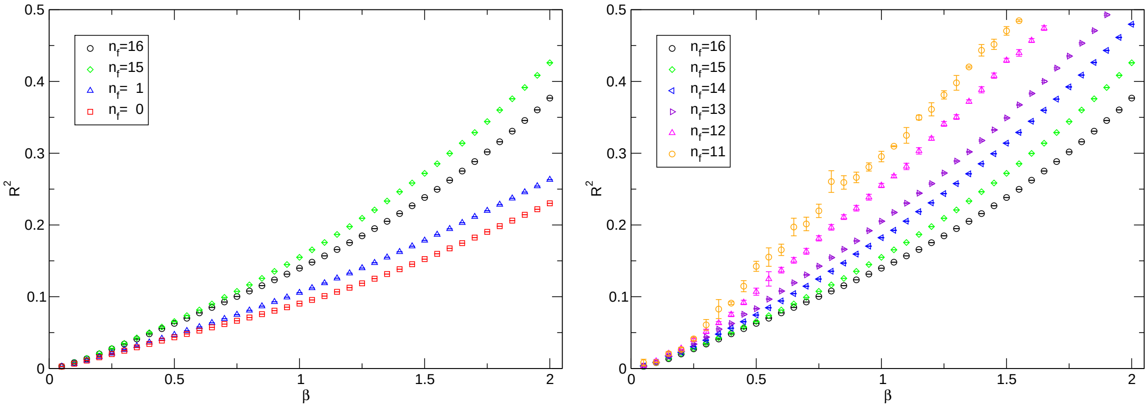

Figure 2: The square of the modulus of the scalar field R^2 as a function of β for SU(3) on a $L_t = 5$ lattice in various canonical sectors.

We note that in some sectors the simulations become unstable and R^2 grows without bound. We believe that this is because the flat directions become unstable due to lattice artefacts, and we expect the behaviour to disappear towards the continuum limit. In the left panel of Figure 1 we show P as a function of the temporal extent of the system parametrised by β for the sectors $n_f = 0$ and 16, and for $n_f = 1$ and 15. Each pair should be degenerate in the continuum and we see that this is indeed the case for $\beta \lesssim 0.6$, while for larger values of β lattice artefacts lift the degeneracy. The (physical) differences between the sectors is illustrated in the right panel of Figure 1 where we show P in the sectors $n_f = 16$ down to 11.

Next, in Figure 2 we show the square of the modulus of the scalar field R^2 as a function of β for the same combinations of sectors as in Figure 1. Again we find that the degeneracy between the $n_f = 0$ and 16, $n_f = 1$ and 15, and so on, is lifted by lattice artefacts towards large values of β.

Finally, in Figure 3 we show the fermionic action $S_F = \langle \ln \det \mathscr{D}_{n_f} \rangle_{n_f}$ as a function of β in various canonical sectors. We find that the degeneracy of this observable between the mirror sectors becomes better and better towards $\beta \to 0$, suggesting that reweighting between the mirror sectors could become feasible in that limit, or more generally towards the continuum limit.

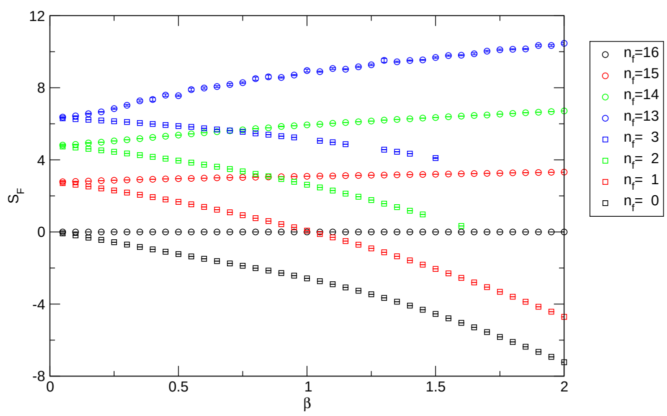

Figure 3: The fermionic action as a function of β for SU(3) on a $L_t = 5$ lattice in various canonical sectors.

5. Summary and outlook

In this contribution we summarise the derivation of explicit transfer matrices for $\mathcal{N} = 4$ SYM quantum mechanics with generic gauge group SU(N) discretised on a time lattice. The transfer matrices are defined separately in each canonical sector with fixed fermion number n_f and form the basis for canonical simulations of the theory. One caveat is that in those sectors where the canonical determinants are not positive definite, the local Metropolis algorithm is currently not very efficient and only samples configurations with positive determinants.

Several paths are now open for further investigation. From an algorithmic viewpoint, it is interesting to examine the systematics of reweighting ensembles of configurations from one fermion sector to another, or from simulations at finite (imaginary) chemical potential. Concerning the physics of the model, it is interesting to calculate correlation functions and energy spectra in the various canonical sectors. The investigation of the phase transition in the large-N limit of the $\mathcal{N} = 16$ SYM quantum mechanics is most useful for a further understanding of the thermodynamics of certain black holes. The results in these proceedings are a first step towards these goals.

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