

# Intragroup transfers, intragroup diversification and their risk assessment

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Received: 15 December 2015 / Accepted: 5 November 2016 / Published online: 19 November 2016  
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**Abstract** When assessing group solvency, an important question is to what extent intragroup transfers may be taken into account, as this determines to which extent diversification can be achieved. We suggest a framework to explicitly describe the families of admissible transfers that range from the free movement of capital to excluding any transactions. The constraints on admissible transactions are described as random closed sets. The paper focuses on the corresponding solvency tests that amount to the existence of acceptable selections of the random sets of admissible transactions.

**Keywords** Fungibility risks · Intragroup diversification · Intragroup transactions · Set-valued risk measures · Risk assessments for groups · Solvency tests for groups

**JEL Classification** G32 · C65 · G20

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The authors are grateful to Philipp Keller for many interesting and important remarks on the misuse of IGTs. The comments of the referee have lead to several improvements of the manuscript. IM was partially supported by Swiss National Foundation Grant 200021\_153597.

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## 1 Introduction

Risk-based solvency frameworks (such as Solvency II or the Swiss Solvency Test (SST)) assess the financial health of insurance companies by quantifying the capital adequacy through calculating the solvency capital requirement. Roughly speaking, companies can use their own economic capital models (internal models) for calculating the available capital amounts (net assets) after one year provided the internal model is approved by the insurance supervisor. The random variable given by the capital amount after one year is required to be acceptable with respect to a prescribed risk measure.

Since key market players are organized in groups, the question of setting appropriate solvency requirements for groups becomes highly relevant, and the quantification of risks for a group of different legal entities (agents) is an essential aim of regulators. The main feature of the group setting is the possibility of intragroup transfers (IGT) that may alter financial positions of individual agents. Section 2 surveys fundamental ideas in this relation that have been already mentioned in [Filipović and Kunz \(2008\)](#), [Filipović and Kupper \(2007, 2008\)](#), [Keller \(2007\)](#), [Luder \(2007\)](#) and are intensively discussed in the financial industry. A key question is to what extent IGTs should be taken into account for the purpose of risk assessment.

The choice and admissibility of IGTs may influence the risk assessment. These admissible transfer instruments range from the free movement of capital between the agents (unconstrained approach) to the case when no transfers are allowed at all (strictly granular risk assessment). In between, we consider allowing transactions that prohibit transfers that render the giver bankrupt or those originating from a bankrupt agent. One might consider imposing some safety margins or taking into account fungibility issues, see Sect. 6.

Our aim is to provide a unified framework in order to explicitly describe such transfers and the relevant risk measures. So far the idea of using random closed sets (most importantly random cones) to describe multiasset portfolios is well established (see [Kabanov and Safarian 2009](#)). We show that a similar approach may be used to describe the sets of admissible IGTs. The key idea of our approach is to regard the group as acceptable if there exists an admissible transfer that renders acceptable the individual positions of all agents. The family of vectors representing the capital amounts added to (or released from) each agent that make the group acceptable serves as a risk measure for the group. This idea is similar to the risk assessment of multiasset portfolios from [Hamel et al. \(2011\)](#) and [Molchanov and Cascos \(2016\)](#), while the main difference is the non-conical dependence of the set of admissible transfer instruments from the current capital position. As a result we end up with the new possibility to explicitly translate realistic (non-conical) transfer constraints in a quantitative risk management framework that assesses genuine intragroup diversification benefits.

Section 3 recalls major concepts related to random closed sets, their selections, thereby following and extending some results from [Molchanov and Cascos \(2016\)](#). Section 4 introduces the random sets of admissible IGTs. Furthermore, it formulates and analyses the relevant capital requirements in terms of set-valued risk assessment of the relevant random sets of admissible positions. Since the family of admissible IGTs typically depends to a large extent on the level of distress a group is faced with, this dependence creates non-conical and non-linear effects.

Section 5 recasts the classical granular and consolidated approaches into our setting. We show that, under some canonical assumptions, the granular approach can be obtained by restricting the set of IGTs in the *unconstrained solvency test*, i.e. in the test without any constraints on IGTs. In the coherent case, the latter corresponds to the consolidated solvency approach, which is frequently used in practice. Furthermore, we discuss the unconstrained solvency test in important non-coherent cases. Various restrictions of transfers are discussed in Sect. 6.

Section 7 deals with the setting of unequally placed companies, where some companies possess the capital of others and so simple addition of capital amounts would lead to double counting. It is shown that this setting can be naturally incorporated in our set-valued framework.

Separating the financial outcome after a certain period from the (possibly restricted) transfers between the companies (or in other applications between portfolios) can lead to diversification effects that are *different* from the usual case of convex risk measures, since not all of them work in the same direction, see Sect. 8.

In Sect. 9 we discuss the computation of the relevant sets, give inner and outer bounds which can be easily implemented and discuss some numerical examples.

Our setting differs from studies of systemic risk, where the individual acceptability of each agent does not suffice for the acceptability of the whole financial network (see [Amini et al. 2013](#)). While, in common with the studies of systemic risks in [Feinstein et al. \(2015\)](#), our approach involves inverting a set-valued risk measure, it does not rely on considering an equilibrium in the system of agents.

## 2 Existing solvency tests for groups

### 2.1 Legal entity approaches

A very basic but key observation for market regulators is that it is not in the obligation of an insurance group but of individual legal entities to pay for claims of policy holders. This basic observation can become particularly relevant under stress. In view of that, it has often been emphasized in the literature that risk assessment and capital requirements for groups of companies should take place on an individual basis (see e.g. [Filipović and Kunz 2008](#); [Filipović and Kupper 2007, 2008](#); [Keller 2007](#); [Luder 2007](#)). This means that each legal entity is requested to set aside the capital necessary to make its risk acceptable. This approach is often called *legal entity approach*.

The legal entity approach appears in two basic variants: stand-alone and granular. In the *stand-alone* solvency assessment, the capital amounts (net assets) of each legal entity are modeled separately and regarded as random variables on probability spaces that might differ between the legal entities. All other group members are considered as third parties, i.e. are treated in the same way as non-members of the group.

The *granular* approach aims at developing a *joint* model for all legal entities in a group. The existence of the group has an impact on the legal entities, meaning that effects of the group on individual entities should constitute a part of the model. These effects result from certain IGTs and ownership relations within the group and should hence be taken into account in the model. Typical examples are reinsurance

agreements, financial guarantees, and intragroup loans.<sup>1</sup> While the whole collection of capital positions for all involved legal entities is modeled as a random vector, the granular approach assesses each of its components separately and these depend on the IGTs chosen by the group. In particular note that in this *existing* solvency test the capital requirement is *not* given by a *single figure*.

Before July 2015 this specific legal entity solvency approach with certain restrictions on the IGTs was the default case in the regulation of Swiss based insurance groups since the introduction of the SST. Note that the Swiss group supervision was declared to be “particularly strong” by International Monetary Funds (IMF) in 2014.<sup>2</sup>

As opposed to the stand-alone approach, the granular approach relies on joint modeling of the capital amounts and IGTs so that the information on other entities and the random variables describing their capital positions flows directly and consistently<sup>3</sup> into the modeling of any particular component. In other words, the granular approach relies on the modeling of each particular component on a richer probability space. This may lead to *different marginal distributions* between the stand-alone and the granular approach. Hence, it seems to be almost impossible to suitably model on the stand-alone basis the particularly dangerous situation where several group members simultaneously run into problems.

Mathematically speaking, consider  $d$  legal entities whose terminal capital positions are described by the vector  $C = (C_1, \dots, C_d)$ . Since the main point of this paper is to separate the financial positions after the relevant time period from the set of admissible IGTs we will later assume that  $C$  is the capital position of all agents *without* any IGTs.

It is assumed that the risks of the legal entities are evaluated using monetary risk measures, namely the risk measure  $r_i$  for the  $i$ th legal entity. The individual risks of each legal entity after IGTs build the vector

$$\mathbf{r}(\tilde{C}) = (r_1(\tilde{C}_1), \dots, r_d(\tilde{C}_d)),$$

where the position  $\tilde{C}$  represents the capital positions after IGTs. It is regarded acceptable if  $r_i(\tilde{C}_i) \leq 0$  for all  $i$ , meaning that  $\mathbf{r}(\tilde{C}) \leq 0$ , where the inequality between vectors is understood coordinatewisely and so means that  $\mathbf{r}(\tilde{C})$  has all non-positive components.

## 2.2 Consolidated approach versus granular with fixed IGTs

Intragroup transactions can be used for increasing the “diversification” of risks within the group, and so they *can* be in the interest of all policy holders (see e.g. Filipović and Kupper 2008), if they are applied in *reality*. In particular, IGTs may be used to

<sup>1</sup> For simplicity of representation we consider no hybrid instruments in this paper and we also do not enter in the important but non-mathematical debate about legal requirements IGTs should satisfy.

<sup>2</sup> <https://www.imf.org/external/pubs/ft/scr/2014/cr14265.pdf>, p. 6.

<sup>3</sup> E.g. the terminal position of the legal entity usually depends on some risk factors (equity, interest rates, mortality etc.). In the stand-alone case the risk factor models do not need to be the same. In the granular case the terminal positions are modeled based on the same risk factors.

offset the risks of some, e.g. poorly performing, legal entities, while not necessarily immediately diminishing the disposable assets of other legal entities.

Since the IGTs can reduce the solvency capital requirements, it is important that they are not of a purely hypothetical nature. Thus, they have to be *realistic* in situations when they are needed. The family of feasible transfers depends on whether or not the group is in a stressed situation, i.e. they are particularly exposed to *fungibility risks*, as is well-known in practice. Furthermore, note that not all policy holders (of different legal entities within the group) have the same interests. Hence, IGTs have to be “sufficiently balanced” with respect to the interests of all *policy holders*. It is important to note that IGTs have a general potential for being misused. Thus, even if transfers are based on legally binding and enforceable contracts, it is still possible that the corresponding transfers are not realistic, not sufficiently in line with the interests of some policy holders, etc., so that other/further restrictions may have to be taken into account. The decision about the restrictions, which are finally taken into account in a concrete solvency framework is a political and legal but not mathematical question, therefore it is not addressed in this paper. However, what we provide here is a tool to translate restrictions into the quantitative risk management framework.

It becomes increasingly popular in practice to regard the group as acceptable if the random variable

$$D = \sum_{j=1}^d C_j \tag{2.1}$$

is acceptable with respect to a prescribed risk measure.<sup>4</sup> This approach is called the *consolidated solvency test*. It is a frequently discussed interpretation that the classical consolidated approach *implicitly* assumes full fungibility of capital between all different legal entities of the group (see e.g. Keller 2007; Luder 2007).

Filipović and Kupper (2007, 2008) and Filipović and Kunz (2008) made an important step towards understanding intra group diversification and quantifying regulatory schemes being sandwiched between the granular approach without any admissible IGTs, in the following called *strictly granular*, and the consolidated solvency tests. Concrete examples are based on dividend payments and reinsurance contracts and are described by some random transfer instruments (linearly independent random variables)  $Z_0, \dots, Z_n$ , so that the terminal risk profile of each agent is given by

$$C_i + \sum_{j=0}^n x_i^j Z_j, \quad i = 1, \dots, d,$$

where the transferred amounts  $x_0, \dots, x_n$  belong to a specified (feasible) subset of  $\mathbb{R}^{n+1}$  and satisfy the clearing condition

$$\sum_{i=1}^d \sum_{j=0}^n x_i^j Z_j \leq 0 \quad \text{a.s.}$$

<sup>4</sup> For simplicity of the representation we ignore here subtleties regarding the so-called (a) symmetric valuation. For a discussion of participation in subsidiaries we refer to Sect. 7.

It is assumed in [Filipović and Kupper \(2008\)](#) that the group aims to minimise the aggregate required group capital

$$\sum_{i=1}^d r_i \left( C_i + \sum_{j=0}^n x_i^j Z_j \right),$$

subject to the feasibility and clearing conditions on the weights  $x_i^j$ .

Apart from the mentioned independency assumption, [Filipović and Kupper \(2008\)](#) assume a single currency setting, absence of transaction costs, that the weights are deterministic numbers and that the admissible transfers are restricted to be a linear combination of fixed instruments without explicit fungibility constraints that depends on the level of distress the group is exposed to.

Fungibility constraints on possible dividend payments are considered in the very concrete bottom-up group diversification analysis presented in [Filipović and Kunz \(2008\)](#). The natural link between admissible IGTs and the minimization problem of the total required capital of the group, which is emphasized in [Filipović and Kupper \(2008\)](#), relates this work to the extensive literature on *optimal risk sharing* (see e.g. [Acciaio 2007](#); [Acciaio and Svindland 2009](#); [Asimit et al. 2013](#); [Barrieu and El Karoui 2005](#); [Barrieu and Scandolo 2008](#); [Gerber 1979](#); [Ravanelli and Svindland 2014](#)) and the literature cited therein. For optimization under restriction to certain so-called cash invariant sets we refer to [Filipović and Svindland \(2008\)](#), partially based on [Filipović and Kupper \(2008\)](#), for portfolios of risk vectors (including the influence of dependence on the risk of a portfolio) see [Kiesel and Rüschemdorf \(2010\)](#).

In the interest of brevity, the main focus of this paper concerning the granular approach is on the task to directly include concrete admissibility constraints for IGTs in a solvency framework without restricting the analysis to concretely given contracts, and we leave to agents the task of choosing specific IGTs from the family of admissible ones.

From the practical perspective, groups may not necessarily aim to minimize the total capital requirement, since some legal entities may have different placements or roles within the group (like subsidiaries and head offices), some may be reluctant to commit own capital in order to compensate the losses of other legal entities or may do this only given certain conditions (that are random), etc. In view of this, the total required capital for the group may not serve as a right utility for the group, or at least the agents might seek to minimize the total required capital only under additional constraints that do not seem to be reflected in the literature so far.

### 2.3 Consolidated test with fungibility constraints

The consolidated solvency approach is based on the acceptability of the random variable in (2.1), where the group is simply considered as if it were one “legal entity”. It is the default case in Solvency II. Since July 2015 the default case of the SST is also based on this approach. However the regulator may impose additional requirements concerning availability and fungibility of capital within the group and also impose

capital add-ons in case of seriously restricted fungibility that is not reflected in the models. In view of that, it is essential to derive a proper and mathematically well-founded way of describing transfer possibilities within the group and to find and to derive a framework which allows to *adequately* include fungibility constraints into the consolidated framework.

By translating the consolidated test into the unconstrained solvency test in the subadditive case, we show that no fungibility constraints are taken into account in the risk-measurement. We also suggest a framework that represents fungibility constraints in a solvency test via simply and explicitly restricting the set of admissible transfers.

### 3 Set-valued portfolios and set-valued risks

A set  $A \subseteq \mathbb{R}^d$  is *lower* if  $y \leq x$  (coordinatewisely) for  $x \in A$  implies that  $y \in A$ . The family of upper sets coincides with the family of reflected lower sets, i.e.  $A$  is an upper set if and only if  $\{-x : x \in A\}$  is a lower set. The topological closure of  $A \subseteq \mathbb{R}^d$  is denoted by  $\text{cl } A$  and its boundary by  $\partial A$ . Denote  $A + a = \{x + a : x \in A\}$ .

Fix a complete probability space  $(\Omega, \mathfrak{F}, \mathbf{P})$ . Let  $X$  be a lower *random closed set* in  $\mathbb{R}^d$ , i.e.  $X$  is a random element taking values in the family of lower closed sets in  $\mathbb{R}^d$ . The measurability requirement on  $X$  is understood as  $\{\omega : X(\omega) \cap K \neq \emptyset\} \in \mathfrak{F}$  for all compact sets  $K$  in  $\mathbb{R}^d$  (see Molchanov 2005). The set  $X$  is called a *set-valued portfolio* in Molchanov and Cascos (2016). In our setting, the points of  $X$  describe the terminal capitals of  $d$  companies of a group after all admissible IGTs. In many cases,  $X$  is almost surely convex, meaning that almost all its realizations are convex sets.

Let  $L^p(\mathbb{R}^d)$  be the family of  $p$ -integrable random vectors in  $\mathbb{R}^d$ , where  $p \in \{0\} \cup [1, \infty]$  is fixed. The choice  $p = 0$  yields the family of all random vectors. Denote by  $\|\xi\|_p$  the  $L^p$ -norm of  $\xi$  for  $p \geq 1$ . A random vector  $\xi$  in  $\mathbb{R}^d$  is said to be a *selection* of  $X$  if  $\xi \in X$  almost surely. Such a random vector may be viewed as a particular terminal position achieved after a certain IGT. We assume throughout that  $X$  is *p-integrable*, i.e.  $X$  possesses at least one  $p$ -integrable selection. In other words, the family  $L^p(X)$  of all  $p$ -integrable selections of  $X$  is not empty.

We identify  $X$  with the family  $L^p(X)$  of all its  $p$ -integrable selections. This is justified by the fact that, for almost all  $\omega \in \Omega$ ,  $X(\omega)$  is the closure of  $\{\xi_n(\omega), n \geq 1\}$  for a sequence  $\{\xi_n, n \geq 1\} \subseteq L^p(X)$  (see Molchanov 2005, Prop. 2.1.2).

In the following  $\mathbf{r} = (r_1, \dots, r_d)$  denotes the vector composed of *monetary risk measures* with *finite* values defined on the space  $L^p(\mathbb{R})$  for  $p \in \{0\} \cup [1, \infty]$  (called  $L^p$ -risk measures). Canonical examples of such risk measures are the Value-at-Risk for all  $p$ , the Average Value-at-Risk for  $p \geq 1$  or the entropic risk measure for  $p = \infty$ .

Furthermore,  $\mathbf{r}$  is said to be *coherent* (resp. convex) if all its components are coherent (resp. convex) risk measures, and in this case we let  $p \in [1, \infty]$  (see Kaina and Rüschendorf 2009). The coherency or convexity assumptions are explicitly imposed whenever needed.

Being convex finite  $L^p$ -risk measures, the components of  $\mathbf{r}$  are Lipschitz in the  $L^p$ -norm (see Kaina and Rüschendorf 2009). In particular, they are strongly continuous. If  $p = \infty$ , additionally assume that the components of  $\mathbf{r}$  satisfy the *Fatou property* that

corresponds to the weak-star lower semicontinuity (that is with respect to the bounded a.s. convergence).

All convex (and coherent) risk measures are tacitly assumed to be *law invariant* and defined on a *non-atomic* probability space.

**Definition 3.1** A set-valued portfolio  $X$  is said to be *acceptable*, if it possesses a  $p$ -integrable selection with all individually acceptable marginals, i.e. there exists  $\xi \in L^p(X)$  such that  $\mathbf{r}(\xi) \leq 0$ . Then  $\xi$  is called an *acceptable selection* of  $X$ .

**Definition 3.2** The *selection risk measure*  $\mathbf{R}(X)$  is the closure of the set

$$\mathbf{R}_0(X) = \left\{ a \in \mathbb{R}^d : X + a \text{ is acceptable} \right\}.$$

Equivalently, the selection risk measure can be defined as

$$\mathbf{R}(X) = \text{cl} \bigcup_{\xi \in L^p(X)} \left( \mathbf{r}(\xi) + \mathbb{R}_+^d \right).$$

In (Hamel et al. 2013),  $\mathbf{R}(\xi + \mathbf{K})$  for  $\xi \in L^p(\mathbb{R}^d)$  and a cone  $\mathbf{K}$  is called a market extension of the regulator risk measure  $\mathbf{r}$ .

*Example 3.1* On the line all lower sets are half-lines, so that each set-valued portfolio in  $\mathbb{R}^1$  is given by  $X = (-\infty, \eta]$  for a random variable  $\eta$ . Then  $X$  is acceptable if and only if  $\eta$  is acceptable. Working with the half-line  $X$  instead of  $\eta$  does not alter financial realities, while being a useful tool in higher-dimensional situations. If  $X = \xi + \mathbb{R}_-^d$  for  $p$ -integrable random vector  $\xi$ , then  $\mathbf{R}(X) = \mathbf{r}(\xi) + \mathbb{R}_+^d$ .

*Example 3.2* Let

$$X = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_1 + \dots + x_d \leq \eta \right\}$$

for a random variable  $\eta$ . The family  $L^\infty(\partial X)$  of all essentially bounded selections of the boundary of  $X$  was studied in detail in Jouini et al. (2008) for  $d = 2$  and is called the set of attainable allocations, see also Gerber (1979).

In order to find acceptable selections of  $X$ , it is sensible to look only at those points of  $X$  that are not coordinatewisely dominated by any other selection of  $X$ . These points build a subset of  $\partial X$  denoted by  $\partial^+ X$  and are called *Pareto optimal* points of  $X$ .

**Lemma 3.1** *If  $X$  is convex, then  $\partial^+ X$  is a random closed set.*

*Proof* Assume  $x_n = (x_n^{(1)}, \dots, x_n^{(d)}) \in \partial^+ X$  is a sequence converging to  $x$ . By choosing subsequences, we can assume that all components converge monotonically. Let  $T$  be the set of components converging strongly decreasing. Assume  $x \notin \partial^+ X$ . Since  $X$  is closed there exists  $y \in \partial^+ X$  such that  $x \leq y$  and  $y^{(i)} > x^{(i)}$  for some  $i$ . Choose  $y$  such that the set  $S$  of all indices  $i$  for which this inequality holds is maximal.

If  $T \subseteq S$ , then for sufficiently large  $n$ ,  $y$  dominates  $x_n$ , contradicting the Pareto optimality of  $x_n$ . Assume that  $j \in T \setminus S$ . By convexity,  $\tilde{y}_n = \lambda y + (1 - \lambda)x_n \in X$  for  $\lambda \in [0, 1]$ . By taking  $\lambda$  sufficiently close to 1, we can achieve that  $\tilde{y}_n^{(i)} > x^{(i)}$  for  $i \in S$  and for sufficiently large  $n$ . Due to the strict monotonicity of  $x_n^{(j)}$ , we also have  $\tilde{y}_n^{(j)} > x^{(j)}$ , contradicting the maximality of  $S$ . Thus,  $\partial^+ X$  is closed.

For the measurability of  $\partial^+ X$ , it suffices to check that  $\Gamma = \{(\omega, x) : x \in \partial^+ X(\omega)\}$ , i.e. the graph of  $\partial^+ X$ , is a measurable set in the  $\sigma$ -algebra  $\mathfrak{F} \otimes \mathcal{B}(\mathbb{R}^d)$ , where  $\mathcal{B}(\mathbb{R}^d)$  is the Borel  $\sigma$ -algebra in  $\mathbb{R}^d$ . Indeed,

$$\Gamma = \bigcap_{q \in \mathbb{Q}_+^d} \{(\omega, x) : x \in X(\omega), x + q \notin X\},$$

where  $\mathbb{Q}_+$  is the family of positive rational numbers. This is justified, since a convex lower set is necessarily regular closed, i.e. coincides with the closure of its interior. □

**Lemma 3.2** *A convex set-valued portfolio  $X$  admits an acceptable selection if and only if  $\partial^+ X$  admits an acceptable selection.*

*Proof* Assume that  $X$  admits an acceptable selection  $\xi$ . If  $\xi$  is not Pareto optimal, consider the random closed set  $Y = X \cap (\xi + \mathbb{R}_+^d)$ . All selections of  $Y$  are acceptable,  $Y \cap \partial^+ X$  is almost surely non-empty and has measurable graph. By the measurable selection theorem (Kabanov and Safarian 2009, Th. 5.4.1),  $Y \cap \partial^+ X$  admits a measurable selection that is automatically acceptable. □

The scaling transformation of a set-valued portfolio is defined as  $tX = \{tx : x \in X\}$ . The sum of set-valued portfolios  $X + Y$  is the set-valued portfolio being the closure of all sums of selections of  $X$  and  $Y$ . It is known (Molchanov 2005) that such operations respect the measurability property, i.e.  $tX$  and  $X + Y$  are random closed sets.

The following result is proved by Molchanov and Cascos (2016) for  $\mathbf{r}$  composed of coherent risk measures, while obvious changes lead to its version for general monetary risk measures.

**Theorem 3.1** *The selection risk measure takes values being upper closed sets, and also*

- (i)  $R(X + a) = R(X) - a$  for all deterministic  $a \in \mathbb{R}^d$  (cash invariance);
- (ii) If  $X \subseteq Y$  a.s., then  $R(X) \subseteq R(Y)$  (monotonicity).

*If  $\mathbf{r}$  is convex and  $X, Y$  are almost surely convex set-valued portfolios, then  $R(X)$  takes convex values, is law invariant, and*

- (iii)  $R(\lambda X + (1 - \lambda)Y) \supseteq \lambda R(X) + (1 - \lambda)R(Y)$  for all deterministic  $\lambda \in [0, 1]$  (convexity).

*If, additionally, the components of  $\mathbf{r}$  are all homogeneous (i.e.  $\mathbf{r}$  is coherent), then*

- (iv)  $R(tX) = tR(X)$  for all  $t > 0$  (homogeneity);
- (v)  $R(X + Y) \supseteq R(X) + R(Y)$ ,

meaning that  $R$  is a set-valued coherent risk measure (see Hamel and Heyde 2010; Hamel et al. 2011).

The set  $\partial^+ X$  is said to be  $p$ -integrably bounded if

$$\|\partial^+ X\| = \sup \{ \|x\| : x \in \partial^+ X \} \in L^p(\mathbb{R}).$$

If  $p = \infty$ , this is the case if and only if  $\partial^+ X$  is almost surely a subset of a deterministic bounded set. For the sake of completeness we provide a proof of the closedness of  $R_0(X)$  for all  $p$  that does not use the coherency assumption as in Molchanov and Cascos (2016, Th. 3.6).

**Proposition 3.1** *If  $\partial^+ X$  is  $p$ -integrably bounded with  $p \in [1, \infty]$  and  $r$  is a convex  $L^p$ -risk measure, then  $R_0(X)$  is a closed set.*

*Proof* Let  $x_n \in R_0(X)$  and  $x_n \rightarrow x$ . By Lemma 3.2, there exists  $\xi_n \in L^p(\partial^+ X)$  such that  $r(\xi_n) \leq x_n$ .

Assume first that  $p \in [1, \infty)$ . Since  $\partial^+ X$  is  $p$ -integrably bounded, all its selections have uniformly bounded  $L^1$ -norms. By the Komlós theorem (see e.g. Kabanov and Safarian 2009, Th. 5.2.1) and passing to a subsequence,  $\bar{\xi}_n = n^{-1}(\xi_1 + \dots + \xi_n)$  converges a.s. to  $\xi$ . Then  $\bar{\xi}_n$  almost surely belongs to the convex hull of  $\partial^+ X$ , whence  $\|\bar{\xi}_n\| \leq \|\partial^+ X\|$  and  $\|\xi\| \leq \|\partial^+ X\|$ . Thus,  $\bar{\xi}_n \rightarrow \xi$  in  $L^p$ . The  $L^p$ -continuity of the components of  $r$  yields that

$$r(\xi) = \lim r(\bar{\xi}_n) \leq \lim n^{-1}(x_1 + \dots + x_n) = x,$$

so that  $x \in R_0(X)$ .

If  $p = \infty$ , the above inequality also applies in view of the assumed Fatou property and the fact that the norms of  $\bar{\xi}_n$  are bounded by the essential supremum of  $\|\partial^+ X\|$ .  $\square$

Denote by

$$h_X(u) = \sup \{ \langle u, x \rangle : x \in X \}$$

the support function of  $X$ , where  $\langle \cdot, \cdot \rangle$  is the scalar product in  $\mathbb{R}^d$ . Then  $h_X(u)$  is a random variable for each  $u$  that may take infinite values. The following result provides a simple outer bound for the selection risk measure of  $X$ .

**Theorem 3.2** (see Molchanov and Cascos 2016, Prop. 4.6) *Assume that  $r = (r, \dots, r)$  has all identical components for a coherent  $L^p$ -risk measure  $r$ . Then*

$$R(X) \subseteq \bigcap_{u \in \mathbb{R}_+^d} \{ x : \langle x, u \rangle \geq r(h_X(u)) \}, \tag{3.1}$$

where  $r(h_X(u)) = -\infty$  if  $h_X(u) = \infty$  with a positive probability.

## 4 Admissible IGTs, attainable positions and their risks

### 4.1 Admissible IGTs

Recall that  $C = (C_1, \dots, C_d)$  denotes the terminal positions of the legal entities evaluated on the granular basis, all expressed in the same currency. Assume that  $C$  is  $p$ -integrable.

A family of *admissible IGTs* is identified as the family  $L^p(I)$  of  $p$ -integrable selections of a random closed set  $I$  in  $\mathbb{R}^d$ . It is often the case that  $I$  depends on the terminal capital positions  $C$  and in this case  $I = I(C)$  is written as a function of  $C$  that might also depend on additional randomness, e.g. random exchange rates. This gives the possibility to model the important feature that realistic transfer possibilities depend on the level of distress of the economic environment.

The *attainable financial positions* at the terminal time after admissible transfers form the family of selections of the random closed set

$$X(C) = C + I(C).$$

It is natural to regard the set  $X = X(C)$  of attainable positions preferable over another set  $Y = Y(C)$  if, for each selection  $\eta \in Y$ , there is a selection  $\xi \in X$  such that  $\eta \leq \xi$  with probability one. This partial order can be realized as the inclusion order  $Y \subseteq X$  if the sets of attainable positions are lower sets in  $\mathbb{R}^d$ . For this, we assume that with each admissible IGT given by a random vector  $\zeta$ , the set  $I(C)$  also includes points that are less than or equal to  $\zeta$  in the coordinatewise order, so that  $I(C)$  and  $X(C)$  are lower sets. The lower set assumption is useful to formulate mathematical properties of risks. While initially it might not seem reasonable to consider IGTs that involve a disposal of some of the assets, the monotonicity property of risk measures implies that the agents or their group in no circumstances would opt for an IGT that involves uncompensated disposal of assets and even if they would pursue such IGT, then the position without such a disposal is also acceptable.

We assume throughout the rest of the paper that  $I$  is a lower set that almost surely contains the origin, and

$$\mathbb{R}_-^d \subseteq I(C) \subseteq H = \left\{ x \in \mathbb{R}^d : \sum x_i \leq 0 \right\}. \tag{4.1}$$

It means that the nil-transfer is admissible and that admissible intragroup transfers are financed by the group operating with the same currency. Sometimes it is useful to assume also that

$$I(C + y) \subseteq I(C) - y \tag{4.2}$$

for each  $y \in I(C)$ . Equivalently,  $X(C') \subseteq X(C)$  for all  $C' \in X(C)$ , meaning that the result cannot be improved by substituting one large transaction by several small ones.

It is essential to stress that  $I(C)$  is not necessarily a cone. In many examples, the set  $I(C)$  is convex, but it is not necessarily the case, e.g. for fixed transaction costs and indivisible assets.

*Example 4.1* If there is a fixed range of admissible IGTs given by  $x^{(1)}, \dots, x^{(k)}$ , then generally

$$I(C) = \bigcup_{i=1}^k (x^{(i)} + \mathbb{R}_-^d)$$

is a non-convex set that does not depend on  $C$ .

### 4.2 Risks of a group

The position  $C$  together with the corresponding admissible IGTs given by  $I(C)$  (or the corresponding set  $X(C)$  of attainable positions) is acceptable if  $0 \in R(X(C))$ . The conventional definition of risk measures in its set-valued variant by Hamel and Heyde (2010) and Cascos and Molchanov (2007) suggests passing from the acceptability criterion to the risk measure by considering the set of all  $x \in \mathbb{R}^d$  such that  $X(C + x)$  is acceptable.

**Definition 4.1** The *group risk* associated with the attainability set  $X(C)$  is

$$\mathcal{R}(X(\cdot), C) = \left\{ x \in \mathbb{R}^d : 0 \in R(X(C + x)) \right\}. \tag{4.3}$$

*Remark 4.1* The group risk can be regarded as the inverse of the set-valued function  $x \mapsto R(X(C + x))$  (see Aubin and Frankowska 1990). A similar inverse appears in Feinstein et al. (2015) as an approach sensitive to the capital levels, where  $C$  denotes the set of capital amounts for agents,  $X(C)$  is the set of equilibrium prices, and the inverse of the selection risk measure of  $X(C + x)$  (in our notation) determines the systemic risk associated with the system of agents. Definition 4.1 can be applied to determine the risks of some multiasset portfolios from Molchanov and Cascos (2016, Sec. 2.3) that depend non-linearly on the financial position. Note that the first argument of  $\mathcal{R}$  in (4.3) is a function.

*Remark 4.2* If  $I(C) = I$  is a convex cone that does not depend on  $C$ , like it is the case for the conical model of proportional transaction costs (see Hamel et al. 2011; Kabanov and Safarian 2009; Molchanov and Cascos 2016), then  $X(C + x) = X(C) + x$ , so that  $R(X(C + x)) = R(X(C)) - x$ , whence  $\mathcal{R}(X(\cdot), C) = R(X(C))$  is a convex set. As we see later on, in many cases of assessing the group risk, the set  $I(C)$  depends on  $C$ , so that  $X(C + x)$  may substantially differ from  $X(C) + x$ . Then  $\mathcal{R}(X(\cdot), C)$  may become non-convex and so considerably more complicated to compute.

- Proposition 4.1** (i) If  $X(x) \subseteq Y(x)$  for all  $x \in \mathbb{R}^d$ , then  $\mathcal{R}(X(\cdot), C) \subseteq \mathcal{R}(Y(\cdot), C)$ .  
 (ii)  $\mathcal{R}(X(\cdot), C)$  contains  $r(C) + \mathbb{R}_+^d$ .  
 (iii) If (4.2) holds, then  $\mathcal{R}(X(\cdot), C)$  is an upper set and is non-decreasing as function of  $C$ , that is  $\mathcal{R}(X(\cdot), C) \subseteq \mathcal{R}(X(\cdot), C')$  if  $C \leq C'$  a.s.

*Proof* (i) In this case  $R(X(C + x)) \subseteq R(Y(C + x))$  and so the inverse function given by (4.3) is also monotone.

(ii) Since  $\mathbb{R}_-^d \subseteq \mathbf{I}(C)$ , we have

$$\mathbf{R}(X(C)) \supseteq \mathbf{R}(C + \mathbb{R}_-^d) = \mathbf{r}(C) + \mathbb{R}_+^d,$$

so that  $\mathcal{R}(X(\cdot), C) \supseteq (\mathbf{r}(C) + \mathbb{R}_+^d)$  by (i).

(iii) Assume that  $C \leq C'$ , so that  $y = C - C' \in \mathbb{R}_-^d$ . By (4.1),  $z \in \mathbf{I}(C)$ , whence (4.2) yields that  $\mathbf{I}(C) \subseteq \mathbf{I}(C') - C + C'$ . Thus,  $X(C) \subseteq X(C')$  and  $\mathbf{R}(X(C + x)) \subseteq \mathbf{R}(X(C' + x))$ , and the total risk is monotonic by (i).

If  $x \leq y$ , then the above applies to  $C + x$  instead of  $C$  and  $C + y$  instead of  $C'$ , so that  $\mathbf{R}(X(C + x)) \subseteq \mathbf{R}(X(C + y))$ , whence if  $0 \in \mathbf{R}(X(C + x))$ , then also  $0 \in \mathbf{R}(X(C + y))$ . □

If all agents operate with the same currency, it is possible to quantify the risk using a single real number by considering the minimal total capital requirement for the group.

**Definition 4.2** The *total risk* associated with  $X(C)$  (also called the total group risk) is defined by

$$\mathcal{R}_\Sigma(X(\cdot), C) = \inf \left\{ \sum_{i=1}^d x_i : 0 \in \mathbf{R}(X(C + (x_1, \dots, x_d))) \right\}. \tag{4.4}$$

The total risk does not change if  $C$  is replaced by  $C + z$  for a deterministic vector  $z$  with  $\sum z_i = 0$ . By Proposition 4.1,

$$\mathcal{R}_\Sigma(X(\cdot), C) \leq \sum r_i(C_i).$$

It is easy to see that the total risk is the support function of  $\mathcal{R}(X(\cdot), C)$  in direction  $(-1, \dots, -1)$ . In particular, the acceptability of  $X(C)$  yields that  $\mathcal{R}_\Sigma(X(\cdot), C) \leq 0$ , but the opposite conclusion is not necessarily true. The non-positivity of the total risk yields only the existence of transfers  $(x_1, \dots, x_d)$  with the total capital requirement being zero that make  $X(C + x)$  acceptable. If the infimum in (4.4) is attained, then the vectors  $x = (x_1, \dots, x_d)$  that provide the infimum give possible allocations of the total risk between the legal entities. Then there is an acceptable selection  $\xi$  of  $X(C + x)$ , and the regulator could possibly request *conclusion of legally binding contracts* for transfers in order to arrive from  $C$  to  $\xi$ .

The set-valued map  $X(C + x)$  is said to be *upper semicontinuous* as function of  $x$  if, for all  $\varepsilon > 0$ ,  $x \in \mathbb{R}^d$ , and any sequence  $x_n$  that converges to  $x$ ,

$$X(C + x_n) \subseteq X(C + x) + B_{\zeta_n}, \tag{4.5}$$

where  $B_{\zeta_n}$  is the closed ball of radius  $\zeta_n$  centred at the origin and  $\|\zeta_n\|_p \rightarrow 0$ . This property can be equivalently formulated for  $\mathbf{I}(C + x)$ .

**Proposition 4.2** Let  $\mathbf{r}$  be a coherent  $L^p$ -risk measure with  $p \in [1, \infty]$ . If  $\partial^+ X(C + x)$  is  $p$ -integrably bounded for all  $x \in \mathbb{R}^d$  and  $X(C + x)$  is upper semicontinuous as function of  $x$ , then the set  $\mathcal{R}(X(\cdot), C)$  is closed. If also  $X(C + x) \subseteq \xi + x + \mathbb{R}_-^d$  for at least one  $\xi \in L^p(\mathbb{R}^d)$  and all  $x \in \mathbb{R}^d$ , then the infimum in (4.4) is attained.

*Proof* For each  $x \in \mathbb{R}^d$ , the set  $M(x) = \mathbf{R}(X(C + x))$  is closed by Proposition 3.1. Assume that  $0 \in M(x_n), n \geq 1$ , and  $x_n \rightarrow x$ . By Theorem 3.1(ii),

$$M(x_n) \subseteq \mathbf{R}(X(C + x) + B_{\zeta_n}).$$

For each selection  $\xi$  of  $X(C + x) + B_{\zeta_n}$ , there exists a selection  $\xi'$  of  $X(C + x)$  such that  $\|\xi - \xi'\| \leq \zeta_n$ . Since the components of  $\mathbf{r}$  are Lipschitz in the  $L^p$ -norm (see Kaina and Rüschemdorf 2009),  $\|\mathbf{r}(\xi) - \mathbf{r}(\xi')\| \leq c\|\zeta_n\|_p$  for a constant  $c$ , so that

$$M(x_n) \subseteq M(x) + B_{\varepsilon_n}$$

for  $\varepsilon_n = c\|\zeta_n\|_p \rightarrow 0$ . Thus,  $0 \in M(x) + B_{\varepsilon_n}$  for all  $n$ . In view of the closedness of  $M(x)$ , it contains the origin, so that  $x \in \mathcal{R}(X(\cdot), C)$ .

The monotonicity of the group risk (see Proposition 4.1) yields that

$$\mathcal{R}(X(\cdot), C) \subseteq \mathbf{r}(\xi) + \mathbb{R}_+^d,$$

and so the attainability of the infimum follows. □

The following basic properties of the introduced risks are easy to prove.

**Proposition 4.3** (i) *The group risk and the total risk are cash invariant, i.e.*

$$\begin{aligned} \mathcal{R}(X(\cdot + a), C + a) &= \mathcal{R}(X(\cdot), C) - a, \\ \mathcal{R}_\Sigma(X(\cdot + a), C + a) &= \mathcal{R}_\Sigma(X(\cdot), C) - \sum_{i=1}^d a_i. \end{aligned}$$

- (ii) *If  $\mathbf{r}$  used to construct the selection risk measure is a homogeneous risk measure and  $\mathbf{I}(tC) = t\mathbf{I}(C)$  for all  $t > 0$ , then  $\mathcal{R}(X(t\cdot), tC) = t\mathcal{R}(X(\cdot), C)$  and  $\mathcal{R}_\Sigma(X(t\cdot), tC) = t\mathcal{R}_\Sigma(X(\cdot), C)$  for all  $t > 0$ .*
- (iii) *If (4.2) holds, then  $\mathcal{R}_\Sigma(X(\cdot), C) \leq \mathcal{R}_\Sigma(X(\cdot), C')$  for  $C' \leq C$ .*

*Remark 4.3* Despite the group risk has natural properties of a monetary set-valued risk measure and the total risk is similar to monetary risk measures, we avoid calling them risk measures, since they depend on two arguments: a random closed set  $X(C)$  and a specific random point  $C$  inside this set.

The risk assessment for multiasset models may be subject to the so-called *risk arbitrage* meaning that it is possible to find a sequence of selections that can be made acceptable by adding capital that tends to minus infinity, so that it is possible to release an infinite capital maintaining the acceptability of the position. This is the case if and only if the total risk attains the value  $-\infty$ . We will show that this is impossible for convex risk measures due to condition (4.1). However, it may become relevant in the multi-currency setting with random exchange rates, see Sect. 6.5, and for general monetary risk measures. Recall that inf-convolution of coherent risk measures is defined by taking the closed convex hull of their acceptance sets, see Delbaen (2012).

**Proposition 4.4** *If  $\mathbf{r}$  is a convex risk measure with all identical components or a coherent risk measure with a non-trivial inf-convolution of its components, then the corresponding total risk is different from  $-\infty$  for all sets of IGTs that satisfy (4.1).*

*Proof* Condition (4.1) yields that  $\mathbf{X}(C) \subseteq C + \mathbf{H}$ , so that

$$\mathcal{R}(\mathbf{X}(\cdot), C) \subseteq \mathcal{R}(\cdot + \mathbf{H}, C) = \mathbf{R}(C + \mathbf{H}).$$

Thus, it suffices to consider the latter set. Assume first that  $\mathbf{r}$  is coherent and there exists a sequence  $\{x^{(k)}, k \geq 1\}$  in  $\mathbb{R}^d$  such that  $\sum_{i=1}^d x_i^{(k)} \rightarrow -\infty$  as  $k \rightarrow \infty$  and  $0 \in \mathbf{R}(C + \mathbf{H} + x^{(k)})$  for all  $k$ . Therefore, there exist  $\xi^{(k)} \in L^p(\mathbb{R}^d), k \geq 1$ , such that  $\sum_{i=1}^d \xi_i^{(k)} \leq 0$  a.s. and  $\mathbf{r}(C + \xi^{(k)} + x^{(k)}) \leq 0$ . Denote by  $r_*$  the inf-convolution of the components of  $\mathbf{r}$ . Then  $r_*(C_i + \xi_i^{(k)} + x_i^{(k)}) \leq 0$  for all  $i = 1, \dots, d$ . By subadditivity,

$$r_*\left(\sum_{i=1}^d C_i + \sum_{i=1}^d x_i^{(k)}\right) \leq r_*\left(\sum_{i=1}^d C_i + \sum_{i=1}^d \xi_i^{(k)} + \sum_{i=1}^d x_i^{(k)}\right) \leq 0.$$

Thus,  $r_*(\sum C_i) = -\infty$ , which is excluded by the condition. In the convex case the proof is similar with  $r_* = r$  and the sums replaced by convex combinations with weights  $1/d$ . □

The following example shows that using non-convex risk measures (like the Value-at-Risk) may lead to risk arbitrage in the high-dimensional setting. This kind of arbitrage is intimately related to the notions of divisibility of risk measures elaborated by R. Wang (University of Waterloo). Throughout the paper we use the following definition of the Value-at-Risk

$$\text{VaR}_\alpha(\eta) = -\inf \{x : \mathbf{P}(\eta \leq x) \geq \alpha\} \tag{4.6}$$

for a random variable  $\eta$ .

*Example 4.2* Assume that  $\mathbf{r}$  has all identical components  $r = \text{VaR}_\alpha$ . Furthermore, assume that  $C = 0$  almost surely on a non-atomic probability space, and  $\mathbf{I}(C) = \mathbf{H}$  in  $\mathbb{R}^d$ , where dimension  $d$  satisfies  $d > \alpha^{-1}$ . Partition  $\Omega$  into subsets  $A_1, \dots, A_d$  of probability  $d^{-1}$  each. Let  $\eta_i^{(n)}(\omega) = -(d-1)n$  for  $\omega \in A_i$  and  $\eta_i^{(n)}(\omega) = n$  otherwise,  $i = 1, \dots, d$ . Then  $\sum \eta_i^{(n)} = 0$ , so that  $\eta^{(n)} \in \mathbf{H}$ . Further,  $\lim_{n \rightarrow \infty} r(\eta_i^{(n)}) = -\infty$ , since  $\eta_i^{(n)} = n$  outside a set of probability  $d^{-1} < \alpha$ .

Such a construction is not possible if  $d < \alpha^{-1}$ . In this case, the limit property of  $r(\eta_i^{(n)})$  yields that for any large  $a$  and all  $i = 1, \dots, d, \eta_i^{(n)} > a$  outside a set  $A_i$  of probability at most  $\alpha$ . Since  $d\alpha < 1$ , the union of all these sets does not cover  $\Omega$ . This means that all components of  $\eta^{(n)}$  exceed  $a$  simultaneously with positive probability, so that  $\eta^{(n)}$  is not a selection of  $\mathbf{H}$ .

**Remark 4.4** In the two-dimensional setting, the existence of the risk arbitrage for  $I(C) = H$  and the risk measure  $\mathbf{r} = (r, r)$  means that

$$r(\zeta_n) + r(-\zeta_n) \rightarrow -\infty$$

for a sequence  $\zeta_n \in L^p(\mathbb{R})$ ,  $n \geq 1$ . This is clearly impossible if  $r$  is convex or if  $r = \text{VaR}_\alpha$  with  $\alpha < 1/2$ , while it might be the case if  $\alpha > 1/2$ .

### 4.3 Absolute acceptability

The main setting in the theory of multivariate risk measures concerns the case of a single agent operating with several currencies or on various markets. In this case, the existence of an acceptable selection from the set  $X$  of attainable positions is a natural acceptability requirement.

In contrary, the interests of agents, and in particular of policy holders of different legal entities in a group may differ. When trying to balance policy holder interests across the whole group, one may particularly appreciate transfers that do not worsen the situation of any policy holder, in other words satisfy the *individual rationality constraints* (see Jouini et al. 2008).

**Definition 4.3** The pair  $(X(\cdot), C)$  is called *absolutely acceptable* if  $X(C)$  admits a selection  $\xi \in L^p(X(C))$  such that  $\mathbf{r}(\xi) \leq 0$  and  $\mathbf{r}(\xi) \leq \mathbf{r}(C)$ .

In other words, such selection  $\xi$  has all individually acceptable components and each of its components has lower risk than the corresponding component of  $C$ . Financially, this may be interpreted as an admissible IGT that leads to acceptable positions of all agents without worsening the individual risk assessment of each individual agent.

The condition  $\mathbf{r}(\xi) \leq \mathbf{r}(C)$  may be relaxed by requiring that  $\mathbf{r}(\xi) \in \mathbf{r}(C) + \mathbf{K}$  for a cone  $\mathbf{K} \supseteq \mathbb{R}_-^d$  that describes the set of individual risks that are considered acceptable by all agents within the group. For a discussion of generalized individual rationality constraints we refer to Ravanelli and Svindland (2014).

Clearly,  $(X(C), C)$  is absolutely acceptable if  $C$  is acceptable, while the following example shows that the converse is not necessarily the case.

**Example 4.3** Let  $X(C) = C + H$  for  $C = (C_1, C_2)$ , and let  $\mathbf{r} = (r, r)$  have two identical components being the Expected Shortfall ( $\text{ES}_{0.01}$ ) at level 0.01. Assume that  $r(C_1) < 0$ , say with  $C_1$  having the standard normal distribution, and let  $C_2 = \min(a - C_1, 0)$  for some  $a \geq \mathbf{E}(C_1)$ , say  $a = \mathbf{E}(C_1)$ . Then  $C_2$  is clearly not acceptable, and so  $(C_1, C_2)$  is not acceptable. But  $(X(C), C)$  is absolutely acceptable. It is possible to reduce the risk of  $C_2$  by a transfer without worsening the risk of  $C_1$ , simply because the non-acceptability of  $C_2$  stems from its behaviour on a set on which  $C_1$  takes rather high values. More specifically, if  $\eta = C_2$ , then  $(C_1 + \eta, C_2 - \eta)$  is acceptable. Because  $C_2 - \eta = 0$ , we have  $0 = r(C_2 - \eta) < r(C_2)$ . Furthermore,  $r(C_1 + \eta) = r(C_1)$ , since  $\eta = 0$  on the event  $\{\omega : C_1(\omega) < a\}$  that has probability at least 1/2. Hence, the 0.01-quantiles (and all lower quantiles) of  $C_1$  and  $C_1 + \eta$  coincide, and we conclude  $r(C_1) = r(C_1 + \eta)$ . Hence  $(C_1 + \eta, C_2 - \eta)$  is acceptable and has a componentwisely lower risk than  $(C_1, C_2)$ .

In many cases, it is impossible to ensure that none of the agents suffers a deterioration of risk after the optimal risk allocation. The following result shows that it is possible to achieve individual rationality after an initial capital transfer.

**Proposition 4.5** *Assume that there exists  $\xi \in L^p(X(C))$  such that  $\sum r_i(\xi_i) = \mathcal{R}_\Sigma(X(\cdot), C)$ . Then there exists  $p = (p_1, \dots, p_d) \in \mathbb{R}^d$  such that  $\sum p_i = 0$  and  $\mathbf{r}(\xi + p) \leq \mathbf{r}(C)$ .*

*Proof* The set  $\mathcal{R}(X(\cdot), C)$  contains  $\mathbf{r}(C)$ . Furthermore,

$$M = (\mathbf{r}(C) + \mathbb{R}_-^d) \cap \left\{x : \sum x_i = \mathcal{R}_\Sigma(X(\cdot), C)\right\} \neq \emptyset,$$

since otherwise  $\sum r_i(C_i)$  would be lower than the total risk. For any  $a \in M$ ,  $p = -a + \mathbf{r}(\xi)$  satisfies the requirements. □

The vector  $p$  from Proposition 4.5 determines the prices of risk that the agents pay (or receive) at time zero in order that the resulting positions do not worsen the risk of any agent and that the total group risk is the smallest. This result was obtained in [Jouini et al. \(2008, Th. 3.3\)](#) for two agents.

## 5 Granular and consolidated tests

### 5.1 Strictly granular test

The *strictly granular* approach presumes that no non-trivial IGTs are allowed, so that  $I(C) = \mathbb{R}_-^d$ , and

$$X(C) = C + \mathbb{R}_-^d = (-\infty, C_1] \times \dots \times (-\infty, C_d].$$

Then all selections from  $X(C)$  have risks that are not better than  $\mathbf{r}(C)$ , so that  $\mathcal{R}(X(\cdot), C) = [\mathbf{r}(C), \infty)$  and  $\mathcal{R}_\Sigma(X(\cdot), C) = \sum r_i(C_i)$  for all monetary risk measures.

### 5.2 Consolidated and unconstrained tests

Recall that the increasingly popular *consolidated* approach requires the random variable  $D$  defined in (2.1) to be acceptable with respect to a prescribed risk measure. It turns out that, in the coherent case, this setting corresponds to the largest set of admissible IGTs and would imply unrestricted fungibility for all assets, i.e. at the end of the considered time period assets can be freely used to settle any liabilities within the group. In this case,

$$I(C) = H = \left\{x = (x_1, \dots, x_d) : \sum x_i \leq 0\right\}$$

is a half-space, and

$$X(C) = \left\{ x = (x_1, \dots, x_d) : \sum x_i \leq D \right\} = C + H. \tag{5.1}$$

Note that  $\max(0, -D)$  is  $p$ -integrable if and only if  $X(C)$  has a  $p$ -integrable selection. This may be the case even if  $C$  itself is not  $p$ -integrable.

We call the solvency test that requires  $X(C)$  from (5.1) to be acceptable the *unconstrained solvency test*. In this case,  $\mathcal{R}(X(\cdot), C) = \mathbf{R}(X(C)) = \mathbf{R}(C + H)$  and

$$\mathcal{R}_\Sigma(X(\cdot), C) = \inf \left\{ \sum_{i=1}^d r(\xi_i) : \xi = (\xi_1, \dots, \xi_d) \in L^p(C + H) \right\}. \tag{5.2}$$

In the case of two agents, this situation is studied in depth by [Jouini et al. \(2008\)](#). The following result provides an independent analysis of this setting for  $\mathbf{r}$  with all identical components and extends it for the convex case and any number of agents.

**Theorem 5.1** *Let  $C = (C_1, \dots, C_d)$  be a  $p$ -integrable random vector and let  $\mathbf{r} = (r, \dots, r)$  for a monetary  $L^p$ -risk measure  $r$ . Furthermore, let  $X(C)$  be given by (5.1).*

- i) If  $r$  is coherent, then  $X(C)$  admits an acceptable selection if and only if  $r(D) \leq 0$ . The infimum in (5.2) is attained at  $\xi = d^{-1}(D, \dots, D)$  and  $\mathcal{R}_\Sigma(X(\cdot), C) = r(D)$ .*
- ii) Assume that, for any  $p$ -integrable random vector  $C$ ,  $X(C)$  admits an acceptable selection if and only if  $r(D) \leq 0$ . Then  $r$  is subadditive.*
- iii) If  $r$  is convex, then  $X(C)$  admits an acceptable selection if and only if  $r(\frac{D}{d}) \leq 0$ . The infimum in (5.2) is attained at  $\xi = d^{-1}(D, \dots, D)$  and  $\mathcal{R}_\Sigma(X(\cdot), C) = dr(D/d)$ .*
- iv) Assume that, for any  $d$  and for any  $(C_1, \dots, C_d)$ , the acceptability of the random vector  $(C_1, \dots, C_d)$  yields the acceptability of  $d^{-1}(D, \dots, D)$ . Then  $r$  is convex.*

*Proof* *i)* Assume that  $r(D) \leq 0$  and define  $\xi = d^{-1}(D, \dots, D)$ , which is a selection of  $X(C)$ . Since  $r$  is positive homogeneous, all components of  $\xi$  are acceptable, and  $\eta = \xi - C$  yields the corresponding IGT. Note that the sum of coordinates of  $\eta$  vanishes almost surely.

Conversely, assume that there exists a selection  $\xi$  of  $X(C)$  such that  $\mathbf{r}(\xi) \leq 0$ . Let  $\hat{\xi}_i$  be the projection of  $\xi$  onto the boundary of  $X(C)$ . Note that  $\sum \hat{\xi}_i \geq \sum \xi_i$ . Then  $\hat{\xi} = C + \eta$ , with  $\eta = \hat{\xi} - C$  such that  $\sum \eta_i = 0$  a.s. Hence,

$$r(D) = r\left(D + \sum \eta_i\right) = r\left(\sum \hat{\xi}_i\right) \leq r\left(\sum \xi_i\right) \leq \sum r(\xi_i) \leq 0.$$

The infimum in (4.4) equals the infimum of all  $a = \sum x_i$  such that  $r(D + a) \leq 0$  and so is attained.

*ii)* For any  $p$ -integrable random vector  $C$ , define  $C' = C + \mathbf{r}(C)$ . By the monetary property,  $\mathbf{r}(C') = 0$ , so every  $C'_1, \dots, C'_d$  is acceptable. By the assumption,  $D' = \sum C'_i$  is acceptable, while the monetary property yields that

$$0 \geq r(D') = r\left(\sum C_i + \sum r(C_i)\right) = r\left(\sum C_i\right) - \sum r(C_i)$$

as desired.

iii) If  $r(D/d) \leq 0$ , then  $\xi = d^{-1}(D, \dots, D)$  is an acceptable selection. Conversely, assume  $\xi = (\xi_1, \dots, \xi_d)$  is an acceptable selection. Then  $\sum \xi_i \leq \sum C_i$ , hence  $\frac{1}{d} \sum \xi_i \leq \frac{1}{d} \sum C_i$ . By convexity,

$$r(D/d) \leq r\left(\frac{1}{d} \sum \xi_i\right) \leq \frac{1}{d} \sum r(\xi_i) \leq 0.$$

iv) By the cash invariance, the assumption is equivalent to

$$r(D/d) \leq \frac{1}{d} \sum r(C_i).$$

We have to show that

$$r(\lambda C_1 + (1 - \lambda)C_2) \leq \lambda r(C_1) + (1 - \lambda)r(C_2)$$

for any  $C_1, C_2$  and  $0 \leq \lambda \leq 1$ . Due to the strong continuity of the components of  $\mathbf{r}$ , it suffices to show this for rational  $\lambda = \frac{m}{n}$ . Applying the assumption to the random vector  $(C_1, \dots, C_1, C_2, \dots, C_2)$  consisting of  $m$  copies of  $C_1$  and  $n - m$  copies of  $C_2$  yields that

$$r\left(\frac{mC_1 + (n - m)C_2}{n}\right) \leq \frac{m}{n}r(C_1) + \frac{n - m}{n}r(C_2),$$

hence, the assertion is proved. □

If the components of  $\mathbf{r}$  are not necessarily coherent, then the classical consolidated approach and the unconstrained approach do not necessarily result in the same risks. If  $D = \sum C_i$  is acceptable, then  $C + \mathbf{H}$  admits an acceptable selection, which is given, e.g., by  $(D, 0, \dots, 0)$ . However, neither the acceptability of  $D$  nor of  $\frac{1}{d}D$  can be concluded from the existence of an acceptable selection in the non-convex case, see Proposition 5.1. Thus, the group risk calculated on the basis of the unconstrained solvency test may be too optimistic in comparison with the classical consolidated test in the non-convex setting.

**Proposition 5.1** *Assume that  $d \geq 2$  and  $\mathbf{r}$  has identical components being the Value-at-Risk.*

- (i) *If  $C_1, \dots, C_d$  are acceptable for  $\text{VaR}_\alpha$ , then  $D = \sum C_i$  is acceptable for  $\text{VaR}_{d\alpha}$ .*
- (ii) *If  $\beta < d\alpha$ , then there are random variables  $C_1, \dots, C_d$  such that  $C_i$  is acceptable for  $\text{VaR}_\alpha$ , but  $D$  is not acceptable for  $\text{VaR}_\beta$ . In particular, this is the case for  $\beta = \alpha$ .*

*Proof* (i) The assumption yields  $\mathbf{P}(C_i < -\varepsilon) < \alpha$ , for each  $\varepsilon > 0$ . Then

$$\mathbf{P}(D < -d\varepsilon) \leq \sum \mathbf{P}(C_i < -\varepsilon) < d\alpha,$$

implying the acceptability of  $D$  under  $\text{VaR}_{d\alpha}$ .

- (ii) Choose a non-atomic probability space and let  $\Omega_1, \dots, \Omega_d$  be mutually disjoint events of probability  $\tilde{\alpha} = \alpha - (d\alpha - \beta)/(2d)$ . Define  $C_i(\omega) = -1$  for  $\omega \in \Omega_i$ , and  $C_i(\omega) = 0$  otherwise. Then  $\text{VaR}_\alpha(C_i) \leq 0$  for all  $i$ , while

$$\mathbf{P}(D < -1/2) = \mathbf{P}\left(\bigcup_{i=1}^d \Omega_i\right) = \sum_{i=1}^d \mathbf{P}(\Omega_i) = d\tilde{\alpha} = \frac{1}{2}(d\alpha + \beta) > \beta,$$

so that  $D$  is not acceptable for  $\text{VaR}_\beta$ . □

## 6 Restrictions of transfers

### 6.1 Restrictions of transfers and the total risk

In the general single currency setting, the sets of admissible terminal portfolios are sandwiched between the strictly granular approach and unconstrained approach. Which IGTs are accepted as admissible is not primarily a mathematical question. In the sequel we show how considerations regarding solvency assessment, transaction costs, and some sources of risks might be reflected in the design of the set  $\mathbf{I}(C)$  of admissible IGTs.

The choice  $\mathbf{I} = \mathbf{H}$  corresponds to the unconstrained risk assessment and so yields the largest possible set of admissible IGTs in the single currency setting.

The monotonicity property from Proposition 4.1 in the coherent case yields that, under any restrictions, the group will never achieve better risks comparing to those obtained by performing the unconstrained solvency test approach. This is formalized in the following proposition.

**Proposition 6.1** *Let  $C = (C_1, \dots, C_d)$  be a  $p$ -integrable random vector and let  $\mathbf{r} = (r, \dots, r)$  for a convex  $L^p$ -risk measure  $r$ . If  $\mathbf{X}(C)$  is any family of admissible IGTs, then the corresponding total risk is at least  $d r(d^{-1} \sum C_i)$ , which is the total risk of the unconstrained solvency test.*

A restriction of  $\mathbf{H}$  increases the complexity of the set-valued solvency tests considerably. In view of that it is natural to ask whether the impact on the resulting capital requirements is material or not compared to the unconstrained and strictly granular approaches.

**Proposition 6.2** *Assume that  $\mathbf{r} = (r, \dots, r)$  for a coherent  $L^p$ -risk measure  $r$ . If the unconstrained total group risk does not increase by restricting the transfers to  $\mathbf{I}(C)$ , then, for some  $x = (x_1, \dots, x_d)$  with  $\sum x_i = r(\sum C_i)$ , the set  $\mathbf{I}(C + x)$  contains a  $p$ -integrable selection  $\eta$  such that*

$$r\left(\sum C_i\right) = \sum r(C_i + \eta_i). \tag{6.1}$$

*Proof* If the total risk in the restricted setting does not increase in comparison with the unrestricted one, then there exists  $x$  with  $\sum x_i \leq r\left(\sum C_i\right)$ , such that

$$\sum r(C_i + \eta_i + x_i) \leq 0$$

for a selection  $\eta$  of  $I(C + x)$ . The monetary property yields that

$$\sum r(C_i + \eta_i) \leq \sum x_i \leq r\left(\sum C_i\right),$$

and the subadditivity property together with  $\sum \eta_i \leq 0$  yield the equality. □

### 6.2 No transfers causing or worsening bankruptcy

For any policy holder, one of the most important events to be avoided is bankruptcy of their counter-party. Hence, in order to balance the interests of all policy holders of all legal entities within the group, a natural restriction for admissible IGTs could be to exclude transfers that exceed the capital of a legal entity, if this capital is positive. Furthermore, it could also be argued that it would not be in line with policy holder interests of the bankrupt company if their bankruptcy’s dividend were reduced by intragroup transactions. It is also clear that fungibility is dramatically restricted in case of bankruptcy, e.g. to a certain bankruptcy’s dividend if legally binding and enforceable intragroup contract exist. Hence, in order to at least partially protect policy holder interests of a bankrupt subsidiary and in order to include some fungibility aspects it seems reasonable to prohibit transfers out of a company with negative capital.

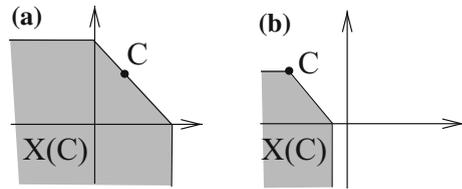
To sum up, there are several reasons to exclude transfers turning a non-bankrupt legal entity into a bankrupt one and also transfers out of a bankrupt one to another one (from the same group). We use the abbreviation NTB (*No Transfers causing or worsening Bankruptcy*) for this kind of IGTs. Here we make the simplifying assumption that bankruptcy is defined with respect to the terminal capital position  $C$  and not with respect to a different balance sheet, i.e. transfers may not turn a non-negative component of  $C$  into a negative one.

The corresponding set of attainable positions  $X(C) = C + I_{NTB}(C)$  is given by

$$I_{NTB}(C) = \left\{ (x_1, \dots, x_d) : \sum x_i \leq 0, x_i \geq -C_i^+, i = 1, \dots, d \right\},$$

where  $a^+ = \max(a, 0)$  for  $a \in \mathbb{R}$ . Note that (4.2) holds in this case and  $X(C + x)$  non-linearly depends on  $x$ . Furthermore,  $\partial^+ X(C)$  is  $p$ -integrably bounded and upper semicontinuous, so that  $\mathbf{R}(X(C))$  and the group risk  $\mathcal{R}(X(\cdot), C)$  are closed sets in the coherent case, see Proposition 4.2.

**Fig. 1** The set of attainable positions in case of both agents are solvent (a) and in case second agent is solvent (b)



For a group consisting of two agents, the set  $X(C)$  of terminal positions has vertices at  $(C_1 + C_2^+, C_2 - C_2^+)$  and  $(C_1 - C_1^+, C_2 + C_1^+)$ , see Fig. 1. Therefore,

$$h_{X(C)}(u) = \langle C, u \rangle + \begin{cases} C_2^+(u_1 - u_2), & u_1 \geq u_2, \\ C_1^+(u_2 - u_1), & u_1 < u_2, \end{cases} \tag{6.2}$$

for  $u = (u_1, u_2) \in \mathbb{R}_+^2$ .

### 6.3 Safety margin

Allowing transfers to vanishing capital for some agents (as it is possible under NTB) may still be considered too progressive, since the agents might end up with no capital buffer after IGTs, i.e. they are almost bankrupt. In view of that it is worth noticing that requirements on admissible IGTs can be made more stringent if the set  $I(C)$  is replaced by  $I(C - a)$  for a fixed vector  $a = (a_1, \dots, a_d)$  with non-negative components that set *safety margins* for terminal capital amounts. If  $X(C) = C + I(C - a)$ , then

$$\begin{aligned} \mathcal{R}(X(\cdot), C) &= \{x : \mathbf{R}(C + x + I(C + x - a)) \ni 0\} \\ &= a + \{x : \mathbf{R}(X(C + x)) \ni a\}, \end{aligned}$$

so that the group risk is obtained by inverting the selection risk measure at point  $a$ , cf. (4.3).

In the case of proportional safety margins,  $I(C)$  is replaced by  $I(C - \lambda C^+)$ , where  $\lambda C^+ = (\lambda_1 C_1^+, \dots, \lambda_d C_d^+)$  for  $\lambda \in [0, 1]^d$ .

### 6.4 Fungibility costs

So far we have assumed that capital can be transferred either in full or not at all. Due to fungibility constraints, the capital may only flow via complicated constructs that involve taking loans and cause serious future fungibility costs, in particular for transitional funding. This can be included in our framework by adapting the set  $X(C)$  of attainable terminal positions.

In the interest of brevity, we illustrate the construction with the help of a bivariate example. In the single currency setting  $\partial^+ X(C)$  is a curve that passes through  $C = (C_1, C_2)$  and has slope  $-1$  at that point if  $C_1, C_2 > 0$ . The fungibility difficulties may be modeled by changing the slope of this line according to increasing fungibility

costs for agents with low capital. If the capital amounts of the firms are large, and the transfers are small (so that the capital amounts after transfer exceed thresholds  $\bar{c}_1$  and  $\bar{c}_2$ ), it is well possible that fungibility costs from one company to another one vanish. However, if the capital of a company is small and the transfer is relatively large, then this can cause serious fungibility costs, which are getting larger the closer the company comes to bankruptcy after a transfer. Then e.g. for a modified NTB setting the boundary  $\{x = (x_1, x_2) : x \in \partial^+ X(C)\}$  can be modeled using two differential equations

$$\frac{\partial x_2}{\partial x_1} = - \left( \frac{\bar{c}_2}{x_2} \right)^p, \quad \frac{\partial x_1}{\partial x_2} = - \left( \frac{\bar{c}_1}{x_1} \right)^p \tag{6.3}$$

depending on whether we want to transfer money from the second to the first company (first equation,  $0 < x_2 < \bar{c}_2$ , in the considered modified NTB setting only relevant if  $C_2 > 0$ ), or from the first to the second company (second equation,  $0 < x_1 < \bar{c}_1$ , in the considered modified NTB setting only relevant if  $C_1 > 0$ ). Close to bankruptcy, the fungibility costs become immense and for all non-positive values of capital of the donor company no transfer is possible anymore. The parameter  $p \geq 1$  can be used for calibration to the company specific situations. The solution of these two equations together with the initial condition  $(x_1, x_2) = (C_1, C_2)$  produces a curve on the plane that yields  $\partial^+ X(C)$ . Note that  $X(C) = \partial^+ X(C) + \mathbb{R}_-^2$  satisfies (4.2).

### 6.5 Transaction costs

Consider the case, where transfers are subject to transaction costs, however unrestricted otherwise. In case of proportional transaction costs, each recipient surrenders a proportion of the amount determined by factor  $\pi \in [0, 1]$ . For simplicity, we assume  $\pi$  to be deterministic and the same for all legal entities. For two legal entities, the set of admissible IGTs for proportional transaction costs is

$$I = \{x = (x_1, x_2) : x_1 + \pi x_2 \leq 0, \pi x_1 + x_2 \leq 0\},$$

which is a convex cone that does not depend on the capital position  $C$ . The corresponding risks have been studied in depth by Hamel and Heyde (2010), Hamel et al. (2011) and Molchanov and Cascos (2016). The results of these papers for deterministic exchange cones apply in our setting. In particular,  $\mathcal{R}(X(\cdot), C) = \mathbf{R}(C + I)$  is the selection risk measure of the set-valued portfolio  $C + I$ .

In case of fixed transaction costs, each recipient surrenders a fixed amount  $a \geq 0$ , so that

$$I = \mathbb{R}_-^2 \cup \{x = (x_1, x_2) : x_1 + x_2 \leq -a\}.$$

This provides an example of a non-convex set of IGTs that also does not depend on  $C$ .

A particularly important case of transaction costs relates to the case where agents operate in different currencies. Then transfers between the currencies are subject to transaction costs and may also involve random exchange rates. In this case, it is also

problematic to consider the total risk, since there is no natural reference currency to express the risks of all agents. We leave this setting for future work.

## 7 Unequally placed agents

Most financial groups exhibit some hierarchical structures, namely there are parent and subsidiary agents, and even cross-holdings between different members of the same group often exist. For simplicity, consider the case of two agents: the first agent with capital  $S$  is a subsidiary and the second one with capital  $C_2$  is a holding (or parent). The parent company owns an option on the (full) available capital of the subsidiary, e.g. via liquidation, i.e. the parent has a long position in the derivative with payoff  $S^+ = \max(0, S)$ . Assume that the parent has no other assets or liabilities, so that her capital is  $S^+$ .

In this case, adding capital amounts would incur double counting of the positive part of  $S$ . Such double counting is typically excluded in previous works on optimal risk sharing (see Filipović and Kunz 2008). In our framework, it is possible to avoid double counting by adjusting the random set of attainable positions.

In view of the fact that a parent company ultimately has the legal right to get the (full) capital from the subsidiary, without an additional contract the parent can add the positive part of the capital of the subsidiary to its assets, and hence, to its capital. Consequently the subsidiary acquires the corresponding short position, which has to be considered in the risk calculation of the subsidiary, so that its consistent capital becomes  $-S^- \leq 0$ , where  $S^- = (-S)^+$  is the so-called limited liability put option. Thus, the net capital of the subsidiary is never strictly positive, for instance, the subsidiary would never be acceptable, except in cases where the capital of the subsidiary after taking into account the participations is concentrated at zero, which might happen, if the subsidiary were “long only” in assets while being completely financed by equity.

For the capital position  $C = (C_1, C_2) = (-S^-, S^+)$ , in the strictly granular setting the random set of attainable positions is given by

$$Y(C) = \begin{cases} (-\infty, 0] \times (-\infty, S] & \text{if } S \geq 0, \\ (-\infty, S] \times (-\infty, 0] & \text{otherwise,} \end{cases}$$

so that  $\partial^+ Y(C) = (-S^-, S^+)$ . Assume that the risks of the both group members are assessed using the same coherent risk measure  $r$ . Thus, in the strictly granular setting, the total risk amounts to

$$\mathcal{R}_{\Sigma}(Y(\cdot), C) = r(S^+) + r(-S^-) \geq r(S^+ - S^-) = r(S).$$

Allowing for IGTs, for any state of the world one has to fix the transfer from the parent to the subsidiary, which is the only feasible transfer direction. For this, the parent obtains a loan  $a$  secured upon  $S^+$ , and we do not take into account any fungibility difficulties that might occur in this relation. From this loan a random amount  $\eta$  is due to be transferred to the subsidiary at the terminal time. If  $S \geq 0$ , then the transferred

amount is immediately recovered and the loan is repaid, while if  $S < 0$ , then the parent recovers  $(S + \eta)^+$ . The set  $X(C)$  of attainable positions is characterized by  $\partial^+ X(C)$  given by  $-(S + \eta)^-, (S + \eta)^+ - \eta$  for all possible transfers  $\eta$ , which are random variables taking values in  $[0, a]$ . The corresponding “total risk” becomes

$$r(-(S + \eta)^-) + r((S + \eta)^+ - \eta) \geq r(S)$$

in view of the subadditivity of the risk measure. In the hypothetical case of an unlimited credit line, the equality, and thus, the optimal transfer is achieved if  $\eta = S^-$ .

The policy holders of a parent could benefit from the investment in the subsidiary without directly affecting reserves of the policy holders of the subsidiary, if the parent sells the subsidiary at the terminal time to a third party. Since this strategy needs a buyer in a potentially stressed market it is possibly, from the initial time point of view, recommendable not to fully take into account this possibility in a prudent regulation.

### 8 Diversification effects

Filipović and Kunz (2008) present a bottom-up approach to analyse intragroup diversification in a very concrete setting with given distributions and choice of predefined IGTs. For a rather recent similar analysis, see also Masayasu (2013).

The key property of coherent risk measures is their subadditivity that corresponds to the fact that diversification decreases risk. The non-linear feature of the IGTs brings new features to the diversification effects. For example, in many cases, e.g. in the NTB case,

$$I(C' + C'') \subseteq I(C') + I(C''), \tag{8.1}$$

where both  $C'$  and  $C''$  are  $d$ -dimensional vectors of capital values. Then the diversification of assets and liabilities narrows the range of admissible IGTs. On the other side, there exists a classical benefit from diversification effects. A similar situation arises in the diversification effects for systemic risks (see Feinstein et al. 2015).

*Example 8.1 (Univariate case)* In order to understand the diversification effects in the group setting, consider the one-dimensional case from the point of view of the group. Let  $C'$  and  $C''$  be two  $p$ -integrable random variables and let  $r$  be a univariate coherent  $L^p$ -risk measure. Define  $C = (C', C'')$  to be a random vector in  $\mathbb{R}^2$  and let  $\mathbf{r} = (r, r)$ . In view of the subadditivity of  $r$  the total risk of  $C$  with the IGTs given by  $I(C) = \mathbf{H}$  equals  $r(C' + C'')$ , while if  $I(C) = \mathbb{R}_-^2$ , then the total risk becomes  $r(C') + r(C'')$ . Thus, the classical diversification benefit can be phrased as the advantage that corresponds to increasing the set of admissible IGTs from  $\mathbb{R}_-^2$  to  $\mathbf{H}$ , in other words, from altering the strictly granular approach to the unconstrained one. Related to that, it should be stressed that a solvency test should never include unrealistic transfers, since otherwise the solvency test tends to underestimate the real risks a group is faced with. Furthermore, it should be noted that classical diversification compares  $r(C') + r(C'')$  with the risk of a particular selection of  $X(C) = C + \mathbf{H}$ , namely that of  $(C' + C'', 0)$ .

The classical concept of diversification is inherent for a single agent, who might have several business units with unrestricted capital flows between them. In case of groups, we see two basic effects:

- *consolidation* that amounts to increasing the set of admissible IGTs;
- *granularization* that corresponds to restricting the family of admissible IGTs.

In particular, a merger of two legal entities removes all fungibility barriers between them, as in the case of unconstrained approach, and so is a simple example of consolidation. On the contrary, a split may lead to some additional restrictions in capital transfers that can be viewed as granularization.

*Example 8.2* Consider the group  $C = (C_1, \dots, C_d)$  and assume that the first agent splits its operation into two subsidiaries so that  $C_1 = C_{11} + C_{12}$ . The effect of such granularization on the total risk depends on the set of admissible transfers between the two created subsidiaries and between them and the rest of the group. For instance, the total risk is retained if the two subsidiaries are considered on the unconstrained basis.

*Example 8.3* Consider a single agent with terminal capital  $\tilde{C}$  that is split into  $C = (C_1, C_2)$ , so that  $\tilde{C} = C_1 + C_2$ . The total risk after such granularization under a coherent risk measure  $r$  lies between  $r(C_1 + C_2) = r(\tilde{C})$  in the unconstrained case and  $r(C_1) + r(C_2)$  in the strictly granular setting.

*Example 8.4* If two groups are merged, a classical question is, whether or not also some of the legal entities (like two life- or non-life companies) should also be merged. Consider two groups  $C' = (C'_1, \dots, C'_{d'})$  and  $C'' = (C''_1, \dots, C''_{d''})$ . Their merge creates a new group  $C$  with  $d' + d''$  legal entities, so that one has to specify the family  $I(C) \subseteq \mathbb{R}^{d'+d''}$  of admissible IGTs. It is natural to assume that  $I(C) \supseteq I'(C') \times \{0\}$  and  $I(C) \supseteq \{0\} \times I''(C'')$ , meaning that the families  $I'(C')$  and  $I''(C'')$  of admissible IGTs within each of two primary groups are admissible after the merge. Still the larger group may be allowed to transfer capital between the components of  $C'$  and  $C''$ . In view of this,

$$\mathcal{R}_\Sigma(X(\cdot), C) \leq \mathcal{R}_\Sigma(X'(\cdot), C') + \mathcal{R}_\Sigma(X''(\cdot), C'').$$

Note that the coherence of components of  $\mathbf{r}$  is not needed for this. Hence, if two groups are merged without any merger of the legal entities we can expect (e.g. if the above assumptions are satisfied) the total risk to be subadditive. However, if we start to merge also legal entities, the situation is less clear due to reverse effects from (8.1).

Thus, in the context of risk assessment for groups, the diversification advantage can be formulated as follows.

Granularization does not diminish the risk, while consolidation does not increase the risk.

This fact is the sole monotonicity property of the set of admissible IGTs and does not rely on the subadditivity property of the involved risk measures.

### 9 Calculating the group risk

The calculation of the group risk requires finding  $x$  such that the selection risk measure of  $X(C + x)$  contains the origin. This is a serious computational problem, that can be solved by means of multicriterial optimization algorithms (see e.g. Hamel et al. 2013). However, bounds on the group risk can be obtained as follows, see Theorem 3.2.

**Proposition 9.1** *Let  $\mathbf{r} = (r, \dots, r)$  be a coherent  $L^p$ -risk measure with identical components. Let  $\xi(x)$  be any selection of  $X(C + x)$  for  $x \in \mathbb{R}^d$ . Then*

$$\begin{aligned} \{x \in \mathbb{R}^d : \mathbf{r}(\xi(x)) \leq 0\} &\subseteq \mathcal{R}(X(\cdot), C) \\ &\subseteq \bigcap_{u \in \mathbb{R}_+^d} \{x \in \mathbb{R}^d : r(h_{X(C+x)}(u)) \leq 0\}. \end{aligned} \tag{9.1}$$

In the following, assume that  $\mathbf{r}$  has all identical coherent components. In case of two agents, the calculation of the superset for the group risk can often be simplified by the following proposition.

**Proposition 9.2** *Assume that  $X(C)$  is almost surely convex in  $\mathbb{R}^2$  with*

$$\partial^+ I(C) \subseteq \{x = (x_1, x_2) : x_1 + x_2 = 0\}.$$

*Then the superset in (9.1) does not change if the intersection is reduced to  $u \in \{(1, 0), (0, 1), (1, 1)\}$ .*

*Proof* It suffices to show that if  $r(h_{X(C+x)}(u)) \leq 0$  for the three above mentioned  $u$ , then the inequality holds for all  $u \in \mathbb{R}_+^2$ . Without loss of generality assume that  $x = 0$ . The condition means that  $\partial^+ I(C)$  is the segment with two end-points  $(\zeta_1, -\zeta_1)$  and  $(-\zeta_2, \zeta_2)$ , where  $\zeta_1, \zeta_2$  are two non-negative random variables that might depend on  $C$ . Then

$$h_{X(C)}(u) = \langle C, u \rangle + \begin{cases} \zeta_1(u_1 - u_2) & \text{if } u_1 > u_2, \\ \zeta_2(u_2 - u_1) & \text{if } u_2 \geq u_1. \end{cases}$$

If  $u_1 > u_2$ , then

$$h_{X(C)}(u) = (u_1 - u_2)h_{X(C)}((1, 0)) + u_2h_{X(C)}((1, 1)).$$

The coherency of the risk measure yields that the risk of  $h_{X(C)}(u)$  is acceptable if both  $h_{X(C)}((1, 0))$  and  $h_{X(C)}((1, 1))$  are. The case of  $u_2 > u_1$  is similar. □

In the following we choose the NTB restrictions for two agents, possibly with safety margins, where Proposition 9.2 clearly applies. In case of a fixed safety margin  $(a_1, a_2)$ , the group risk is a subset of

$$\left\{ x = (x_1, x_2) : \begin{aligned} &r(C_1 + x_1 + (C_2 + x_2 - a_2)^+) \leq 0, \\ &r(C_2 + x_2 + (C_1 + x_1 - a_1)^+) \leq 0, r(C_1 + C_2) \leq x_1 + x_2 \end{aligned} \right\}. \tag{9.2}$$

Note that the last inequality defines the half-plane corresponding to the unconstrained setting. In the case of zero safety margin, the first two inequalities are superfluous, and so the superset from Proposition 9.1 does not differ from the group risk in the unconstrained setting. Indeed, if  $x_1 + x_2 \geq r(C_1 + C_2)$ , then without loss of generality we can assume the equality, so that  $x_1 = r(C_1 + C_2) - x_2$ . Then the first inequality in (9.2) requires

$$r(C_1 + (C_2 + x_2)^+) \leq r(C_1 + C_2) - x_2,$$

or, equivalently,

$$r(C_1 + C_2 + (C_2 + x_2)^+ - (C_2 + x_2)) \leq r(C_1 + C_2),$$

which always holds, since  $(C_2 + x_2)^+ \geq (C_2 + x_2)$ .

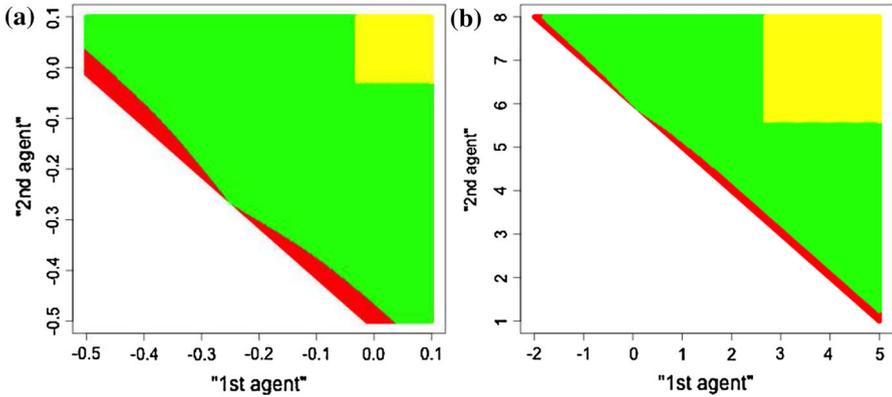
In order to obtain a subset of  $\mathcal{R}(X(\cdot), C)$ , choose the selection  $\xi(x)$  as the point of  $\partial^+ X(C + x)$  that is nearest to the diagonal line  $\{(x_1, x_2) : x_1 = x_2\}$ . Then  $\xi(0) = \frac{1}{2}(D, D)$  if  $D = C_1 + C_2 \geq 0$ . If  $D < 0$ , then  $\xi(0) = (D, 0)$  in case  $C_2 \geq 0$ ,  $\xi(0) = (0, D)$  if  $C_1 \geq 0$ , and  $\xi(0) = (C_1, C_2)$  if  $C_1, C_2 < 0$ . Thus,  $\xi(0)$  is acceptable if

$$\begin{aligned} &r\left(\frac{1}{2}D\mathbf{1}_{D \geq 0} + D\mathbf{1}_{D < 0, C_2 \geq 0} + C_1\mathbf{1}_{C_1 < 0, C_2 < 0}\right) \leq 0, \\ &r\left(\frac{1}{2}D\mathbf{1}_{D \geq 0} + D\mathbf{1}_{D < 0, C_1 \geq 0} + C_2\mathbf{1}_{C_1 < 0, C_2 < 0}\right) \leq 0. \end{aligned}$$

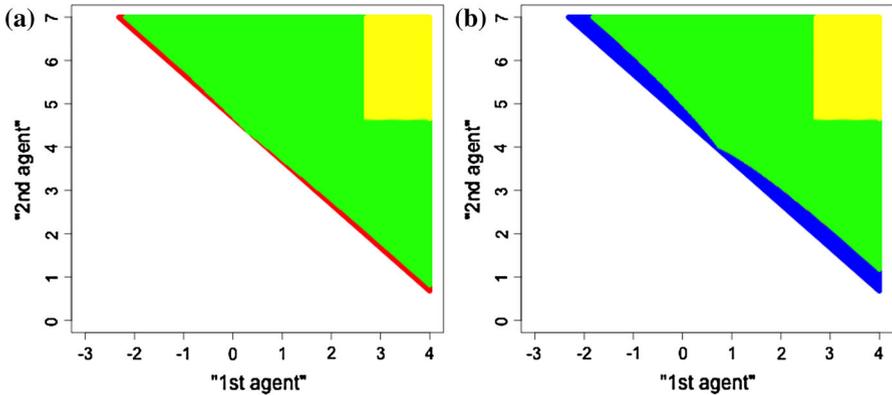
The inner approximation in (9.1) is obtained as the set of all  $x = (x_1, x_2)$  such that the above inequalities hold with  $C = (C_1, C_2)$  replaced by  $(C_1 + x_1 - a_1, C_2 + x_2 - a_2)$ , where  $(a_1, a_2)$  denotes the fixed safety margin. An upper bound for the total risk is obtained by finding the minimum of  $x_1 + x_2$  that satisfy these inequalities.

*Example 9.1* Consider the NTB setting without safety margin and with  $\mathbf{r} = (ES_{0.01}, ES_{0.01})$  as the underlying risk measure. Figure 2a shows the superset and the subset of the group risk for i.i.d.  $C_1, C_2$  having the uniform distribution on  $[0, 5]$ . The outer bound equals the group risk in the unconstrained setting. The upper right corner shows the group risk in the strictly granular case. While the bounds for  $\mathcal{R}(X(\cdot), C)$  given by Proposition 9.1 clearly differ, they both yield the same value of the total risk. This means that the total risk in the NTB setting for this example coincides with the unconstrained group risk. While the same phenomenon appears in all simulated examples with exchangeable  $(C_1, C_2)$ , we do not have a theoretical confirmation of this observation.

Figure 2b shows the bounds for the group risk in case  $C_1$  has the standard normal distribution independent of  $-C_2$  having the exponential distribution of mean one. In



**Fig. 2** Bounds for the group risk for the uniform distribution (a) and for combination of normal and exponential distributions (b). The strictly granular group risk is shown in yellow (light grey), the inner approximation in green (grey), the outer approximation in red (dark grey) (it coincides with the unconstrained group risk) (colour figure online)



**Fig. 3** Bounds for the group risk for the normally distributed assets without safety margin (a) and with fixed safety margin of 0.5 for both agents (b). The strictly granular group risk is shown in yellow (light grey), the inner approximation in green (grey), the outer approximation in red (dark grey), and the unconstrained group risk in blue (dark grey) (colour figure online)

this case the inner approximation to the group risk does not touch the outer approximation, however close they are.

*Example 9.2* Take  $\mathbf{r} = (ES_{0.01}, ES_{0.01})$  and let  $(C_1, C_2)$  be normally distributed with mean zero and the covariance matrix

$$\begin{pmatrix} 1 & -0.5 \\ -0.5 & 3 \end{pmatrix}.$$

Figure 3a shows approximations to the group risk without safety margin. Here the outer approximation coincides with the group risk in the unconstrained setting, while the inner approximation touches it and so shows that the total risk in the NTB setting coincides with the unconstrained total risk. Figure 3b shows the results for the fixed

safety margin set to 0.5 for the both agents. In this case the outer approximation in (9.1) coincides with the inner approximation and so yields the group risk. The outer set corresponds to the unconstrained setting. An indication for the high potential of IGTs for intragroup diversification is seen by comparing the strictly granular group risk with other ones.

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