

MAX-STABLE RANDOM SUP-MEASURES WITH COMONOTONIC TAIL DEPENDENCE

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ABSTRACT. Several objects in the Extremes literature are special instances of max-stable random sup-measures. This perspective opens connections to the theory of random sets and the theory of risk measures and makes it possible to extend corresponding notions and results from the literature with streamlined proofs. In particular, it clarifies the role of Choquet random sup-measures and their stochastic dominance property. Key tools are the LePage representation of a max-stable random sup-measure and the dual representation of its tail dependence functional. Properties such as complete randomness, continuity, separability, coupling, continuous choice, invariance and transformations are also analysed.

1. INTRODUCTION

Random sup-measures provide a unified framework for dealing with a number of objects that naturally appear in the Extremes literature including temporal extremal processes [20, 28], continuous choice models [29] or extremal loss in portfolios [41]; α -Fréchet sup-measures are the building blocks of max-stable processes [37]. In general, any stochastic process with upper semicontinuous paths can be viewed as a random sup-measure [27, 28, 39]. That is, the suprema of the process over sets yield a random sup-measure, while the values of the random sup-measure at singletons yield the upper semicontinuous process. The max-stability property of the process immediately translates into the same property of the random sup-measure.

Surprisingly, the notion of capacities and sup-measures has almost vanished from the theoretical developments on extremal processes over the past 20 years. This paper aims to clarify, extend and complement a number of results from the unifying perspective of sup-measures and capacities with streamlined proofs and connections to the theory of random sets and utility functions (or risk measures). The necessary preliminaries concerning capacities, random closed sets, random sup-measures, Choquet and extremal integrals are presented in Section 2.

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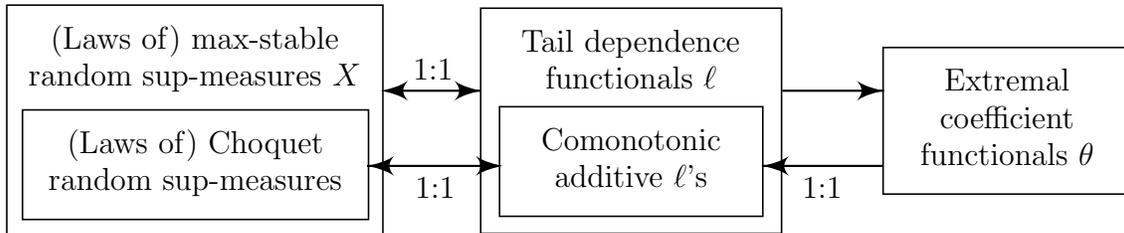
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Section 3 introduces *max-stable random sup-measures* X on a carrier space \mathbb{E} and their *tail dependence functional* ℓ , which are the central objects in this paper. They are natural generalisations of max-stable random vectors and their (stable) tail dependence functions. For a given function f on \mathbb{E} , the tail dependence functional $\ell(f)$ characterises the distribution of the extremal integral of f with respect to X and so uniquely determines the distribution of the random sup-measure X . The values $\ell(\mathbb{1}_K)$ of ℓ on indicator functions $\mathbb{1}_K$ are called *extremal coefficients* and denoted by $\theta(K)$.

Generalising [25, 30, 31] we give a *complete characterisation* of the tail dependence functional as an upper semicontinuous homogeneous max-completely alternating functional and of the extremal coefficient functional as an upper semicontinuous union-completely alternating functional. Motivated by the family of stochastic processes studied in [38] and characterised by the fact that their distributions are in a one-to-one correspondence with extremal coefficient functionals, we identify the family of max-stable random sup-measures that have the same property. While in [38] such processes were called Tawn–Molchanov processes (TM processes), here we call their sup-measure analogues *Choquet random sup-measures (CRSMs)*. The key argument relies on the fact that the *comonotonic additivity* property of the tail dependence functional ℓ ensures that ℓ equals the Choquet integral with respect to θ , and so the distribution of the random sup-measure is uniquely determined by θ . This observation clarifies a number of properties of TM processes from [38] and establishes connections with the studies of coherent risk measures that also appear as such Choquet integrals, see [8, 12]. The following graph illustrates the one-to-one correspondence between extremal coefficient functionals and CRSMs, cf. [38].



The classical *LePage series* representation [22] asserts that a symmetric stable random vector equals in distribution the sum of i.i.d. integrable random vectors scaled by successive points of the unit intensity Poisson process on the positive half-line. Its variant for max-stable processes is derived in [7]. In Section 4, we derive such a representation of a general max-stable random sup-measure as the maximum of i.i.d. copies of a random sup-measure scaled by successive Poisson points. The difficulty lies in the absence of a norm and a reference sphere in the space of (locally finite) sup-measures. Subsequently, CRSMs are characterised by the fact that the i.i.d. summands become scaled indicator random sup-measures.

Section 5 provides the *dual representation* of the general and CRSM tail dependence functionals as supremum over the Lebesgue integrals with respect a certain family of Radon measures. In the CRSM case, this family has an interpretation as distributions for selections of a random closed set. Such dual representations are related to dual representations of coherent risk measures in mathematical finance. For random vectors

(when the carrier space \mathbb{E} is finite), these families of measures are convex bodies that were called dependency sets or max-zonoids in [25] and [38]. Among all tail dependence functionals with fixed values on indicator functions (that is, with fixed extremal coefficients), the CRSM tail dependence functional is the largest one.

Random sup-measures with independent values on disjoint sets are called *completely random* or having independent peaks [27, 37]. They are now well understood including the corresponding integration theory that relies on the concept of the *extremal integral* [37]. The distribution of a max-stable completely random sup-measure is completely identified by its *control measure*, similarly to the situation with conventional α -stable completely random measures studied in details in [32]. In the completely random case, the tail dependence functional $\ell(f)$ equals the Lebesgue integral $\int f d\mu$ with respect to the control measure μ and so it is comonotonic additive. Therefore, max-stable completely random sup-measures belong to the family of CRSMs. In Section 6 it is shown that, conversely, a CRSM can always be realised as a max-stable completely random sup-measure if uplifted to the (much richer) space of all closed sets.

Section 7 addresses max-stable processes that appear by taking the values of max-stable random sup-measures at singletons and their *separability properties*. It also characterises the stochastic continuity of a CRSM and the corresponding TM process. Section 8 deals with *coupling* of general max-stable random sup-measures with CRSMs. In particular, by means of an appropriate coupling, it is possible to recover the independence of the argmax-set of a max-stable random sup-measure from its maximal value (and their distributions) in a streamlined proof and in a broader setup compared to the separable *continuous choice* models on compact spaces in [29].

Finally, Section 9 elaborates on further properties of both general max-stable random sup-measures and CRSMs related to *transformations* of their distributions *using Bernstein functions, rearrangement invariance* that corresponds to the law invariance property of risk measures in finance and is related to exchangeability properties, *stationarity* and *self-similarity*. Several *examples* of CRSMs are presented in Section 10, in particular, related to the recent study of random sup-measures in [20].

2. CAPACITIES, RANDOM SETS AND RANDOM SUP-MEASURES

Let \mathbb{E} be a locally compact Hausdorff second countable space, that we often assume to be the line \mathbb{R} or the Euclidean space \mathbb{R}^d . Denote by \mathcal{K} , \mathcal{F} , \mathcal{G} , and \mathcal{B} respectively the families of compact, closed, open, and Borel sets in \mathbb{E} .

A (*Choquet*) *capacity* φ is a non-decreasing function φ on the family of all subsets of \mathbb{E} with values in $[0, \infty]$, such that $\varphi(\emptyset) = 0$, $\varphi(A_n) \uparrow \varphi(A)$ if $A_n \uparrow A$ (inner regularity), and $\varphi(K_n) \downarrow \varphi(K)$ for compact sets $K_n \downarrow K$ (upper semicontinuity on compact sets), see [10, Appendix A.II] and [24, Appendix E]. It is assumed throughout that all capacities take finite values on compact sets. The capacity φ is said to be *finite* if $\varphi(\mathbb{E}) < \infty$ and *normalised* if $\varphi(\mathbb{E}) = 1$.

Complete alternation. A set-function $\varphi : \mathcal{K} \rightarrow [0, \infty]$ is said to be *completely alternating* on \mathcal{K} if the following recursively defined functionals

$$\begin{aligned}\Delta_{K_1}\varphi(K) &= \varphi(K) - \varphi(K \cup K_1), \\ \Delta_{K_n} \cdots \Delta_{K_1}\varphi(K) &= \Delta_{K_{n-1}} \cdots \Delta_{K_1}\varphi(K) - \Delta_{K_{n-1}} \cdots \Delta_{K_1}\varphi(K \cup K_n)\end{aligned}$$

are non-positive for all $n \geq 1$ and all $K, K_1, \dots, K_n \in \mathcal{K}$, see [24, Def. 1.1.8]. This definition corresponds to the complete alternation of the set-function φ on \mathcal{K} considered a semigroup with the union operation, see [4]. In particular, the non-positivity of $\Delta_{K_1}\varphi(K)$ is equivalent to the monotonicity of φ ; together with the non-positivity of $\Delta_{K_2}\Delta_{K_1}\varphi(K)$ they identify strongly subadditive (or concave) set-functions. If a set-function φ on \mathcal{K} is strongly subadditive and upper semicontinuous, then it can be consistently extended to a Choquet capacity on the family of all subsets of \mathbb{E} , see [10, Th. A.II.7].

Random closed sets. A *random closed set* Ξ in \mathbb{E} is a measurable map from a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ to \mathcal{F} endowed with the σ -algebra generated by the family

$$\mathcal{F}_K = \{F \in \mathcal{F} : F \cap K \neq \emptyset\}, \quad K \in \mathcal{K}.$$

The *Choquet theorem* from the theory of random sets (see [24, Sec. 1.2] and [4, Th. 6.6.19]) modified for not necessarily finite capacities as in [36, Th. 2.3.2] states that φ is a completely alternating upper semicontinuous capacity if and only if there exists a unique locally finite measure ν_φ on the family $\mathcal{F}' = \mathcal{F} \setminus \{\emptyset\}$ of non-empty closed sets such that

$$(1) \quad \nu_\varphi(\mathcal{F}_K) = \varphi(K), \quad K \in \mathcal{K}.$$

If φ is normalised, then ν_φ is a probability measure on \mathcal{F}' . This probability measure is the distribution of a random closed set Ξ in \mathbb{E} such that

$$\mathbf{P} \{\Xi \cap K \neq \emptyset\} = \varphi(K), \quad K \in \mathcal{K},$$

and φ is then called the *capacity functional* of Ξ .

Maxitive capacities. A capacity φ is called a *sup-measure* if

$$\varphi(\cup_{j \in J} G_j) = \sup_{j \in J} \varphi(G_j)$$

for any family $\{G_j, j \in J\}$ of open sets. This is the case if and only if φ is obtained as the extension of an upper semicontinuous set-function on \mathcal{K} such that

$$(2) \quad \varphi(K_1 \cup K_2) = \varphi(K_1) \vee \varphi(K_2), \quad K_1, K_2 \in \mathcal{K},$$

and so φ is called *maxitive* on \mathcal{K} , see [27]. Note that \vee denotes the maximum operation, for random vectors it denotes the coordinatewise maximum, and for functions their pointwise maximum. Each maxitive capacity φ is completely alternating, see [24, Th. 1.1.17], and

$$(3) \quad \varphi(K) = \sup\{g(x) : x \in K\}, \quad K \in \mathcal{K},$$

for an upper semicontinuous function g , see [24, Prop. 1.1.16] and [28, Th. 2.5]. The right-hand side of (3) is denoted by $g^\vee(K)$ and is called the *sup-integral* of g , while the

function $g(x) = \varphi(\{x\})$, $x \in \mathbb{E}$, is the *sup-derivative* of φ , see e.g. [28]. A particularly important maxitive capacity is the *indicator capacity* $\varphi(K) = \mathbb{1}_{F \cap K \neq \emptyset}$ for any fixed $F \in \mathcal{F}$.

Choquet and extremal integrals. The *Choquet integral* of a function $f : \mathbb{E} \mapsto \mathbb{R}_+ = [0, \infty)$ with respect to a capacity φ is defined by

$$(4) \quad \int f d\varphi = \int_0^\infty \varphi(\{f \geq t\}) dt,$$

where $\{f \geq t\} = \{x \in \mathbb{E} : f(x) \geq t\}$, see [9] and [24, Sec. 5.1]. If φ is a measure and f is a measurable non-negative function, this integral coincides with the Lebesgue integral. Furthermore,

$$(5) \quad \int f d\varphi = \int_0^\infty \varphi(\{f > t\}) dt,$$

since the function $\varphi(\{f \geq t\})$ is monotone in t , and so has at most a countable number of discontinuities if $\varphi(\{f \geq t\})$ is finite for all t , while otherwise the both sides are infinite, see [14, Eq. (6)] and [8, Th. 42, p. 123].

The *extremal integral*

$$(6) \quad \int^e f d\varphi = \sup\{\varphi(K) \inf_{x \in K} f(x) : K \in \mathcal{K}\}$$

was introduced in [14] in view of applications to the theory of large deviations. It is shown in [14, Prop. 3] that

$$(7) \quad \int^e f d\varphi = \sup_{t>0} t \varphi(\{f \geq t\}) = \sup_{t>0} t \varphi(\{f > t\}).$$

If $\varphi = g^\vee$ is the sup-integral of an upper semicontinuous function g , then

$$\int^e f dg^\vee = \sup_{x \in \mathbb{E}} f(x)g(x),$$

which justifies calling this integral the extremal one. In particular, if φ is a sup-measure and $f = \bigvee_{i=1}^n a_i \mathbb{1}_{A_i}$, then

$$(8) \quad \int^e f d\varphi = \max_{i=1, \dots, n} a_i \varphi(A_i).$$

By USC_0 (respectively C_0) we denote the family of all non-negative bounded upper semicontinuous (respectively continuous) functions on \mathbb{E} with relatively compact support $\{x \in \mathbb{E} : f(x) \neq 0\}$. Both the Choquet integral and extremal integral are finite if the integrand belongs to USC_0 or if the integrand is bounded and $\varphi(\mathbb{E})$ is finite.

Lemma 2.1. *If φ and ν_φ are related by (1), then, for each $f \in \text{USC}_0$,*

$$\int f d\varphi = \int f^\vee d\nu_\varphi \quad \text{and} \quad \int^e f d\varphi = \int^e f^\vee d\nu_\varphi.$$

Proof. It suffices to note that

$$\varphi(\{f \geq t\}) = \nu_\varphi(\{F : F \cap \{f \geq t\} \neq \emptyset\}) = \nu_\varphi(\{F : f^\vee(F) \geq t\})$$

and to apply the definitions of the Choquet and extremal integrals. \square

The following result is known for the Choquet integral from [8, Th. 43, p. 124] and [9, Ch. 8] and for the extremal integral it follows from the upper semicontinuity of φ .

Lemma 2.2. *If $f_n(x) \downarrow f(x)$ for all $x \in \mathbb{E}$, and $f, f_1, f_2, \dots \in \text{USC}_0$, then*

$$\int f_n d\varphi \downarrow \int f d\varphi \quad \text{and} \quad \int^e f_n d\varphi \downarrow \int^e f d\varphi \quad \text{as } n \rightarrow \infty.$$

In particular, the values of the integrals on any $f \in \text{USC}_0$ can be approximated by their values on step-functions that approximate f from above.

By approximating $f \in \text{USC}_0$ with step-functions, it is easy to see that, for all $f \in \text{USC}_0$ and any sup-measure φ , the integral given by (6) coincides with the extremal integral introduced in [37].

Comonotonic additivity. Both the Choquet integral and the extremal integral are homogeneous, e.g.

$$\int (cf) d\varphi = c \int f d\varphi, \quad c \geq 0.$$

While the Choquet integral is not a linear functional of f , it is *comonotonic additive* meaning that

$$\int (f + g) d\varphi = \int f d\varphi + \int g d\varphi$$

for two comonotonic functions f and g . Recall that f and g are *comonotonic* if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for all $x, y \in \mathbb{E}$, see [9, Prop. 5.1] and [34], where it is also shown that each normalised comonotonic additive monotone functional can be represented as a Choquet integral. The subadditivity property of the Choquet integral

$$\int (f + g) d\varphi \leq \int f d\varphi + \int g d\varphi$$

is equivalent to the concavity property of φ , see [9, Th. 6.3].

Random sup-measures. A sequence $\{\varphi_n, n \geq 1\}$ of sup-measures converges *sup-vaguely* to φ if $\limsup_{n \rightarrow \infty} \varphi_n(K) \leq \varphi(K)$ and $\liminf_{n \rightarrow \infty} \varphi_n(G) \geq \varphi(G)$ for all $K \in \mathcal{K}$ and $G \in \mathcal{G}$, see e.g. [28, Def. 2.6] and [39]. The sup-vague topology generates the Borel σ -algebra on the family of sup-measures and so makes it possible to define a *random sup-measure* X . Its distribution is determined by the joint distributions of random vectors $X(K_1), \dots, X(K_m)$ for all finite collections of compact sets K_1, \dots, K_m . These distributions form the system of *finite-dimensional distributions* of X . A random sup-measure X is said to be *integrable* if $X(K)$ is integrable for all $K \in \mathcal{K}$.

By approximating $f \in \text{USC}_0$ from above using step-functions it is easy to see that the Choquet and extremal integrals of f with respect to a random sup-measure are random variables.

Lemma 2.3. *The distribution of a random sup-measure X is uniquely determined by the distributions of $\int^e f dX$ for all $f \in \text{USC}_0$.*

Proof. If $f = \bigvee u_i \mathbb{1}_{K_i}$, then $\int^e f dX = \bigvee u_i X(K_i)$ by (8). Thus, it is possible to obtain the joint distribution of $X(K_1), \dots, X(K_m)$, i.e. the finite dimensional distribution of X , from the distribution of $\int^e f dX$ for varying coefficients $u_1, \dots, u_m \in \mathbb{R}_+$. \square

A random sup-measure is said to be *completely random* if its values on disjoint sets are jointly independent, see [37]; it is said to have independent peaks in [27].

3. MAX-STABLE RANDOM SUP-MEASURES AND THEIR TAIL DEPENDENCE FUNCTIONALS

Max-stable random vectors. A random variable has a unit Fréchet distribution if its cumulative distribution function is $\exp\{-at^{-1}\}$, $t > 0$, where $a > 0$ is called the scale parameter. A random vector $\xi = (\xi_1, \dots, \xi_d)$ is called *semi-simple max-stable* if, for all $u \in \mathbb{R}_+^d = [0, \infty)^d$, the max-linear combination $\bigvee_{j=1}^d u_j \xi_j$ is a unit Fréchet variable with scale parameter denoted by $\ell(u)$, see [6]. The function $\ell : \mathbb{R}_+^d \mapsto \mathbb{R}_+$ is called (*stable*) *tail dependence function* and has the following properties, see [3, 25, 31].

- (i) ℓ is *homogeneous*, i.e. $\ell(cu) = c\ell(u)$ for all $u \in \mathbb{R}_+^d$ and all $c > 0$.
- (ii) ℓ is *subadditive*, i.e. $\ell(u + v) \leq \ell(u) + \ell(v)$ for all $u, v \in \mathbb{R}_+^d$.
- (iii) ℓ is *max-completely alternating*, i.e. the successive differences

$$\begin{aligned} \Delta_{u_1}^\vee \ell(u) &= \ell(u) - \ell(u \vee u_1), \\ \Delta_{u_n}^\vee \cdots \Delta_{u_1}^\vee \ell(u) &= \Delta_{u_{n-1}}^\vee \cdots \Delta_{u_1}^\vee \ell(u) - \Delta_{u_{n-1}}^\vee \cdots \Delta_{u_1}^\vee \ell(u \vee u_n) \end{aligned}$$

are all non-positive for all $n \geq 1$ and all $u, u_1, \dots, u_n \in \mathbb{R}_+^d$.

Since ℓ is a sublinear (homogeneous and subadditive) function, it defines a norm on \mathbb{R}_+^d called a D -norm in [2, 11]. In fact, the homogeneity and max-complete alternation suffice to characterise the tail dependence function as can be seen from a slight modification of [31, Th. 6] and [30, Th. 4], see also [25, Th. 7].

Theorem 3.1. *A function $\ell : \mathbb{R}_+^d \mapsto \mathbb{R}_+$ is a tail dependence function of a semi-simple max-stable random vector ξ in \mathbb{R}^d if and only if ℓ is homogeneous and max-completely alternating.*

Max-stable random sup-measures. A random sup-measure X is called *semi-simple max-stable* (in the sequel we say that X is a *max-stable random sup-measure*) if its finite-dimensional distributions $X(K_1), \dots, X(K_m)$ are semi-simple max-stable random vectors for all $K_1, \dots, K_m \in \mathcal{K}$, $m \geq 1$.

Lemma 3.2. *A random sup-measure X is semi-simple max-stable if and only if $\int^e f dX$ is a unit Fréchet random variable for all $f \in \text{USC}_0$.*

Proof. Necessity. The statement is true for functions taking a finite number of values and then by approximation for $f \in \text{USC}_0$ using Lemma 2.2.

Sufficiency. For any $u_1, \dots, u_m \geq 0$ and $K_1, \dots, K_m \in \mathcal{K}$, the random variable $\bigvee u_i X(K_i)$ is equal to $\int^e f dX$ for $f = \bigvee u_i \mathbb{1}_{K_i} \in \text{USC}_0$, which is unit Fréchet distributed. Hence $(X(K_1), \dots, X(K_m))$ is a semi-simple max-stable random vector. \square

Tail dependence functional. In the sequel the scale parameter of the unit Fréchet variable $\int^e f dX$ will be denoted by $\ell(f)$. The function $\ell : \text{USC}_0 \mapsto \mathbb{R}_+$ is called the *tail dependence functional* of X . By Lemma 2.3, the tail dependence functional uniquely determines the law of X .

Theorem 3.3. *A functional $\ell : \text{USC}_0 \mapsto \mathbb{R}_+$ is the tail dependence functional of a (necessarily unique) max-stable random sup-measure if and only if ℓ is homogeneous, completely alternating on USC_0 equipped with the maximum operation, and upper semi-continuous on USC_0 meaning that $\ell(f_n) \downarrow \ell(f)$ for $f_n \downarrow f$.*

Proof. Necessity. The homogeneity property is trivial. The values of ℓ on $f_1, \dots, f_m \in \text{USC}_0$ and their partial maxima are the extremal coefficients of the semi-simple max-stable random vector $(\int^e f_1 dX, \dots, \int^e f_m dX)$ and their complete alternation property follows from Theorem 3.1. If $f_n \downarrow f$, then $\int^e f_n dX \downarrow \int^e f dX$ a.s. by Lemma 2.2, so that $\ell(f_n) \downarrow \ell(f)$, see also [37, Lemma 2.1 (iv)].

Sufficiency. Let $X(K_1), \dots, X(K_m)$ for $K_1, \dots, K_m \in \mathcal{K}$ be the semi-simple random vector with the tail dependence function $\ell_{K_1, \dots, K_m}(u) = \ell(\bigvee_{i=1}^m u_i \mathbb{1}_{K_i})$, $u \in \mathbb{R}_+^m$. The function ℓ_{K_1, \dots, K_m} is indeed a tail dependence function, since it inherits max-completely alternation and homogeneity from ℓ . This system of finite-dimensional distributions is consistent, since

$$\ell\left(\bigvee_{i=1}^{m+1} u_i \mathbb{1}_{K_i}\right) \downarrow \ell\left(\bigvee_{i=1}^m u_i \mathbb{1}_{K_i}\right) \quad \text{as } u_{m+1} \downarrow 0.$$

Thus, there exists a unique max-stable random sup-measure X with the specified finite-dimensional distributions. By Lemma 2.2 and the upper semi-continuity of ℓ , the tail dependence functional of X coincides with ℓ . \square

Proposition 3.4. *The tail dependence functional ℓ is subadditive, i.e. $\ell(f + g) \leq \ell(f) + \ell(g)$ for all $f, g \in \text{USC}_0$.*

Proof. By approximation from below, it suffices to derive the result for step-functions f and g that, without loss of generality, can be taken as $f = \sum a_i \mathbb{1}_{K_i}$ and $g = \sum b_i \mathbb{1}_{K_i}$ for disjoint $K_1, \dots, K_m \in \mathcal{K}$. Then $\ell(f)$ equals the tail dependence function of the random vector $(X(K_1), \dots, X(K_m))$ in direction (a_1, \dots, a_m) and similar interpretations hold for $\ell(g)$ and $\ell(f + g)$. It suffices to refer to the subadditivity property of the tail dependence function. \square

Extremal coefficient functional. Let X be a max-stable random sup-measure with tail dependence functional ℓ . The set-function $\theta(K) = \ell(\mathbb{1}_K)$, $K \in \mathcal{K}$, will be termed the *extremal coefficient functional* of X . It is necessarily a capacity on \mathcal{K} as the following lemma shows. Note that $\theta(K)$ is the scale parameter of the unit Fréchet law of $X(K)$.

Lemma 3.5. *A functional $\theta : \mathcal{K} \mapsto \mathbb{R}_+$ is the extremal coefficient functional of a stable sup-measure if and only if it is completely alternating, upper semicontinuous and satisfies $\theta(\emptyset) = 0$.*

Proof. Necessity. The (union-)complete alternation property of θ follows from the max-complete alternation property of the tail dependence functional ℓ , cf. Theorem 3.3 for functions $f_i(x) = \mathbb{1}_{K_i}(x)$, $i = 1, \dots, m$, taking into account that $\mathbb{1}_{K_i} \vee \mathbb{1}_{K_j} = \mathbb{1}_{K_i \cup K_j}$. The upper semicontinuity of θ follows from the upper semicontinuity of ℓ noticing that $\mathbb{1}_{K_n} \downarrow \mathbb{1}_K$ as $K_n \downarrow K$. Since ℓ is homogeneous, $\theta(\emptyset) = 0$.

Sufficiency. Setting $\ell(f) = \int f d\theta$ the Choquet integral with respect to θ , we see that the functional ℓ satisfies the properties of a tail dependence functional, cf. Theorem 3.3. \square

Choquet random sup-measures (CRSMs). In general, the information contained in the extremal coefficient functional θ is not sufficient to recover the tail dependence function ℓ and so the distribution of the corresponding max-stable random sup-measure. Now we single out particular max-stable random sup-measures, whose distributions are completely characterised by the extremal coefficient functional.

Definition 3.6. A stable sup-measure X is said to be a *Choquet random sup-measure* (CRSM) if its tail dependence functional ℓ is comonotonic additive.

Theorem 3.7. *A stable sup-measure X is a CRSM if and only if its tail dependence functional ℓ is given by the Choquet integral*

$$(9) \quad \ell(f) = \int f d\theta$$

with respect to its extremal coefficient functional θ . The functional θ uniquely determines the distribution of X .

Proof. Sufficiency follows from the comonotonic additivity of the Choquet integral. For the proof of necessity, consider a sequence $\{K_n, n \geq 1\}$ of compact sets that grows to \mathbb{E} . For each $n \geq 1$, the tail dependence functional ℓ is comonotonic additive on functions f supported by K_n if and only if it can be represented as the Choquet integral with respect to a capacity θ_n (see [34]), i.e. $\ell(f) = \int f d\theta_n$ for f supported by K_n , where $\theta_n(K) = \ell(\mathbb{1}_K)$, $K \in \mathcal{K}$, $K \subset K_n$. Thus, θ_n is the extremal coefficient functional of $X(K)$, $K \subset K_n$. Noticing that $\theta_n(K) = \theta_m(K)$ for $m > n$ and $K \subset K_n$, (9) holds for $\theta(K) = \theta_n(K)$ with $K \subset K_n$. \square

Remark 3.8. CRSMs appear as weak limits for the scaled maxima of indicator random sup-measures. This also relates to their series representation derived in the following section.

4. SERIES REPRESENTATIONS

Series representation of max-stable random sup-measures. A useful tool for the study of stable random elements is their series representation in terms of the sum (or maximum) of i.i.d. random elements scaled by the successive points of the unit intensity Poisson process, see [7] and [22] for the max and sum-stable cases, and [5] for general semigroups. The following result provides a series decomposition for max-stable random sup-measures.

Denote by λ the Lebesgue measure on \mathbb{R}_+ and let $\{\Gamma_i, i \geq 1\}$ be the sequence of successive points of the unit intensity Poisson process on \mathbb{R}_+ . Denote by \mathfrak{S} the family of all sup-measures on \mathbb{E} , and by $\mathfrak{S}_{\text{ind}}$ the family of scaled indicator sup-measures $c\mathbf{1}_{F \cap K \neq \emptyset}$ for $c > 0$ and $F \in \mathcal{F}$. Their non-trivial subsets will be denoted by $\mathfrak{S}' = \mathfrak{S} \setminus \{0\}$ and $\mathfrak{S}'_{\text{ind}} = \mathfrak{S}_{\text{ind}} \setminus \{0\}$, respectively.

Theorem 4.1. *A random sup-measure X is max-stable if and only if it can be decomposed as a max-series*

$$(10) \quad X \stackrel{d}{\sim} \bigvee_{i \geq 1} \Gamma_i^{-1} Y_i,$$

where $\{Y_i, i \geq 1\}$ is a sequence of i.i.d. copies of an integrable random sup-measure Y and independent of the sequence $\{\Gamma_i, i \geq 1\}$. The tail dependence functional X is then given by

$$(11) \quad \ell(f) = \mathbf{E} \int^e f dY, \quad f \in \text{USC}_0.$$

The random sup-measure X is a.s. non-trivial if and only if Y is a.s. non-trivial.

Proof. Sufficiency. If X is given by the right-hand side of (10), then

$$\int^e f dX = \bigvee_{i \geq 1} \Gamma_i^{-1} \int^e f dY_i$$

is a unit Fréchet random variable with the scale parameter $\ell(f)$ given by (11), which is finite if $f \in \text{USC}_0$ and Y has integrable values on compact sets.

Necessity. It suffices to consider the case of an a.s. non-trivial random sup-measure X . Note that a max-stable random sup-measure is necessarily max-infinitely divisible. By [27, Th. 5.1] and noticing that the support of the distribution of $X(K)$ is the whole \mathbb{R}_+ , the sup-measure X can be represented as

$$(12) \quad X \stackrel{d}{\sim} \bigvee_{i \geq 1} \eta_i,$$

where $\{\eta_i, i \geq 1\}$ form a Poisson process with the unique intensity measure Λ on \mathfrak{S}' , that is called the Lévy measure.

At this point it is useful to view the space \mathfrak{S} as a convex cone which is the abelian semigroup with the semigroup operation being maximum and the scaling given by scaling the values of sup-measures, see [4] and [5]. A separating family of semicharacters on (\mathfrak{S}, \vee) is given by $\chi_{K,a}(\varphi) = \mathbf{1}_{\varphi(K) \leq a}$, $\varphi \in \mathfrak{S}$, $K \in \mathcal{K}$, and $a > 0$. This means that

two different sup-measures yield different values for a semicharacter from this family. It is easy to see that condition **(C)** of [5] is satisfied, while (12) means that the Lévy measure of X is supported by \mathfrak{S} in the terminology of [5]. By [5, Th. 6.1], Λ is 1-homogeneous with respect to scaling, i.e.

$$\Lambda(\{c\varphi : \varphi \in B\}) = c^{-1}\Lambda(B), \quad c > 0,$$

for all Borel $B \subset \mathfrak{S}'$. Let $\{K_i, i \geq 1\}$ be the closures of relatively compact sets that form a countable base for the topology of \mathbb{E} . Then

$$-\log \mathbf{P} \{X(K_i) \leq a_i, i = 1, \dots, m\} = \Lambda(\{\varphi : \max_{i=1, \dots, m} \varphi(K_i) > a_i\}),$$

for $a_1, \dots, a_m > 0$ and $m \geq 1$. By repeating an argument from the proof of [7, Th. 1], there exist $b_i > 0, i \geq 1$, such that Λ is supported by $\mathfrak{S}_b = \{\varphi \in \mathfrak{S}' : r(\varphi) < \infty\}$, where $r(\varphi) = \sup_{i \geq 1} b_i X(K_i)$. Denote $S = \{\varphi \in \mathfrak{S}_b : r(\varphi) = 1\}$ and define the map $T : \mathfrak{S}_b \mapsto (0, \infty) \times S$, by letting $T(\varphi) = (r(\varphi), \varphi/r(\varphi))$, whose inverse is simply $T^{-1}(r, \varphi) = r\varphi$. By the homogeneity property of Λ on \mathfrak{S}_b and the homogeneity of r , it is easily seen that the push-forward of Λ under T is the product measure $u^{-2}du \otimes \pi(d\varphi)$ for a finite measure π on S given by

$$\pi(B) = \Lambda(\{\varphi : r(\varphi) > 1, \varphi/r(\varphi) \in B\}).$$

If the b_i are scaled by the same constant, π can be adjusted to become a probability measure on \mathfrak{S}_b . Conversely, Λ is fully determined by π through

$$\Lambda(\{\varphi : \max_{i=1, \dots, m} \varphi(K_i) > a_i\}) = \int_S \bigvee_{i=1}^m \frac{\varphi(K_i)}{a_i} \pi(d\varphi).$$

Finally, (10) follows by letting Y_i be i.i.d. with distribution π . \square

Remark 4.2. The intensity measure Λ of the Poisson process $\{\eta_i, i \geq 1\}$ from (12) is a homogeneous measure on \mathfrak{S}' . Sometimes, Λ is decomposed as the push-forward of the product of the measure with density t^{-2} on $(0, \infty)$ and a not necessarily finite measure ν on \mathfrak{S}' . Then, instead of (10), one obtains the representation $\bigvee_{i \geq 1} t_i^{-1} Y_i$, where $\{(t_i, Y_i), i \geq 1\}$ is the Poisson process on $\mathbb{R}_+ \times \mathfrak{S}'$ with intensity measure $\lambda \otimes \nu$. The special feature of (10) is the fact that such a Poisson process can be viewed as the Poisson process on the positive half-line marked by i.i.d. copies of a random sup-measure.

Remark 4.3. The distribution of Y in Theorem 4.1, i.e. the probability measure π on \mathfrak{S}' that was constructed in the proof, is said to be the *spectral measure* of X . The spectral measure is not unique, e.g. it is possible to replace Y with ζY , where ζ is any non-negative random variable independent of Y with the unit expectation. Two random sup-measures, Y and Y' , yield the same max-stable random sup-measure if $\mathbf{E} \int^e f dY = \mathbf{E} \int^e f dY'$ for all $f \in \text{USC}_0$.

In case of a countable carrier space $\mathbb{E} = \{x_i, i \geq 1\}$, it means that the sequences $\{Y(\{x_i\}), i \geq 1\}$ and $\{Y'(\{x_i\}), i \geq 1\}$ are zonoid equivalent, see [26]. The uniqueness of π (and Y) is achieved if the values of Y are normalised, e.g. by assuming that $Y \in S$ as introduced in the proof of Theorem 4.1.

If $X(\mathbb{E})$ is a.s. finite, the uniqueness can be achieved by requiring that $Y(\mathbb{E}) = c$ for a constant $c > 0$. In this case, the proof of Theorem 4.1 simplifies using $\varphi(\mathbb{E})$ instead of $r(\varphi)$.

Series representation of CRSMs. The following result characterises CRSMs in terms of their series representations. Recall that $\mathcal{F}' = \mathcal{F} \setminus \{\emptyset\}$.

Theorem 4.4. *A random sup-measure X is a CRSM with the extremal coefficient functional θ if and only if*

$$(13) \quad X(K) \stackrel{d}{\sim} \bigvee_{i \geq 1} t_i^{-1} \mathbf{1}_{F_i \cap K \neq \emptyset}, \quad K \in \mathcal{K},$$

where $\{(t_i, F_i), i \geq 1\}$ is the Poisson process on $\mathbb{R}_+ \times \mathcal{F}'$ with intensity $\lambda \otimes \nu$ for a locally finite measure ν on \mathcal{F}' such that

$$(14) \quad \nu(\mathcal{F}_K) = \theta(K), \quad K \in \mathcal{K}.$$

Proof. Sufficiency. A random sup-measure given by (13) is necessarily semi-simple max-stable. The local finiteness of ν implies that at most a finite number of pairs (t_i, F_i) satisfy $F_i \cap K \neq \emptyset$ and $t_i \leq s$ for any $K \in \mathcal{K}$ and $s \geq 0$, so that $X(K)$ is almost surely finite. For any $f \in \text{USC}_0$,

$$\int^e f dX \stackrel{d}{\sim} \bigvee_{i \geq 1} t_i^{-1} f^\vee(F_i)$$

is the series representation of the unit Fréchet random variable. In order to find its scale parameter we calculate the void probability of the Poisson process $\{(t_i, F_i)\}$ as follows

$$\begin{aligned} \mathbf{P} \left\{ \int^e f dX < s \right\} &= \exp\{-(\lambda \otimes \nu)(\{(t, F) : f^\vee(F)t^{-1} \geq s\})\} \\ &= \exp \left\{ - \int_0^\infty \nu(\{F : f^\vee(F) \geq ts\}) dt \right\} \\ &= \exp \left\{ - s^{-1} \int_0^\infty \nu(\{F : f^\vee(F) \geq t\}) dt \right\} \\ &= \exp \left\{ - s^{-1} \int f^\vee d\nu \right\}. \end{aligned}$$

By Lemma 2.1, $\ell(f) = \int f d\theta$ for θ given by (14), and so ℓ is comonotonic.

Necessity. By Lemma 3.5 and the Choquet theorem, there exists a unique measure ν on \mathcal{F}' that satisfies (14). The random sup-measure constructed by (13) has the tail dependence functional $\int f d\theta$, which equals $\ell(f)$ by Theorem 3.7. \square

Corollary 4.5. *A CRSM X such that $X(\mathbb{E})$ is almost surely positive and finite, can be represented as*

$$(15) \quad X(K) \stackrel{d}{\sim} \theta(\mathbb{E}) \bigvee_{i \geq 1} \Gamma_i^{-1} \mathbf{1}_{\Xi_i \cap K \neq \emptyset},$$

where $\{\Xi_i, i \geq 1\}$ is a sequence of i.i.d. a.s. non-empty random closed sets in \mathbb{E} with the capacity functional $\mathbf{P}\{\Xi_1 \cap K \neq \emptyset\} = \theta(K)/\theta(\mathbb{E})$ and independent of the sequence $\{\Gamma_i, i \geq 1\}$.

Proof. Since the measure ν on \mathcal{F}' related to θ by (14) is finite and non-vanishing, the Poisson process $\{(t_i, F_i)\}$ with the intensity $\lambda \otimes \nu$ can be viewed as the unit intensity Poisson process $\{\Gamma_i, i \geq 1\}$ on \mathbb{R}_+ scaled by $\theta(\mathbb{E})^{-1}$ and independently marked by a sequence of random elements in \mathcal{F}' that are distributed according to the normalised ν . \square

The following result characterises CRSMs as those having the spectral measure supported by the family $\mathfrak{S}'_{\text{ind}}$ of non-trivial scaled indicator sup-measures.

Theorem 4.6. *A non-trivial random sup-measure X is a CRSM if and only if (10) holds with $\{Y_i, i \geq 1\}$ being i.i.d. copies of an integrable random sup-measure Y with distribution supported by $\mathfrak{S}'_{\text{ind}}$ and independent of the sequence $\{\Gamma_i, i \geq 1\}$. The extremal coefficient functional of X is*

$$(16) \quad \theta(K) = \mathbf{E}Y(K), \quad K \in \mathcal{K}.$$

Proof. Sufficiency is easy to see noticing that if $Y(K) = \tau \mathbf{1}_{\Xi \cap K \neq \emptyset}$, then $\ell(f) = \mathbf{E}[\tau f^\vee(\Xi)]$ is comonotonic additive. By (11),

$$\theta(K) = \mathbf{E} \int^e \mathbf{1}_K dY = \mathbf{E}Y(K).$$

Necessity. First, X admits the representation given by (10). Since $X(K_0)$ is a.s. finite, Corollary 4.5 applies. Therefore, $X(K)$, $K \subset K_0$, admits the representation as the max-series built from scaled indicator random sup-measures. Since the Lévy measure of X , i.e. the intensity of the Poisson process that appears in (12) is unique, the corresponding spectral measure π is supported by $\mathfrak{S}'_{\text{ind}}$. Thus, $Y(K)$, $K \subset K_0$, almost surely belongs to the family $\mathfrak{S}'_{\text{ind}}$. The conclusion follows from the fact that K_0 is arbitrary. \square

Remark 4.7. The random sup-measure Y in Theorem 4.6 can be represented as $Y(K) = \tau \mathbf{1}_{\Xi \cap K \neq \emptyset}$. If $Y(\mathbb{E}) = \tau$ is integrable, then θ is finite and the LePage series (15) yields a version of X . Thus, the most interesting case of Theorem 4.6 corresponds to non-integrable τ , where the dependency between τ and Ξ ensures that $Y(K)$ is integrable for all $K \in \mathcal{K}$. For example, if $\mathbb{E} = \mathbb{R}_+$ and $\Xi = [\tau, \infty)$, then $\mathbf{E}Y(K) = \mathbf{E}[\tau \mathbf{1}_{\tau \leq \sup K}] < \infty$ for $K \in \mathcal{K}$, no matter if τ is integrable or not.

Example 4.8. Consider a sup-measure $\varphi(K) = \sup\{g(x) : x \in K\}$ for an upper semicontinuous function $g : \mathbb{E} \mapsto [0, 1]$ and let $Y(K) = \mathbf{1}_{\Xi \cap K \neq \emptyset}$ with random closed set Ξ that has the capacity functional φ , that is $\Xi = \{x : g(x) \geq U\}$ for the uniform random variable U in $[0, 1]$. Then (10) with Y_i being i.i.d. copies of Y yields the CRSM X with the extremal coefficient functional $\varphi(K)$. If Y_i are chosen to be deterministic and equal φ , then (10) yields the max-stable random sup-measure $\tilde{X}(K) = \zeta \varphi(K)$, where ζ is the unit Fréchet random variable with scale parameter one. Thus, X and \tilde{X} share the same extremal coefficient functional, while \tilde{X} has the tail dependence

functional $\int^e f d\varphi$, which is not comonotonic additive and so it is not a CRSM, and the CRSM X has the tail dependence functional $\int f d\varphi$. Their extremal coefficients coincide, since the Choquet and extremal integrals return the same value on indicator functions.

Corollary 4.9. *Let Y be an integrable random sup-measure. Then $Y \in \mathfrak{S}_{\text{ind}}$ a.s. if and only if*

$$(17) \quad \mathbf{E} \int^e f dY = \mathbf{E} \int f dY, \quad f \in \text{USC}_0.$$

Proof. Necessity. Since the left-hand side of (17) is the tail dependence function $\ell(f)$ of a CRSM constructed by (10), it is comonotonic additive. The statement follows from (16) and (9), so that

$$\ell(f) = \int f d\theta = \int_0^\infty \mathbf{E}Y(\{f \geq t\}) dt = \mathbf{E} \int f dY.$$

Sufficiency. If (17) holds, the left-hand side of (17) is the tail dependence functional of a random sup-measure X constructed by (10). Since the right-hand side of (17) is comonotonic additive, X is a CRSM. It follows from Theorem 4.6 that $Y \in \mathfrak{S}_{\text{ind}}$ a.s. \square

5. DUAL REPRESENTATIONS

The following result provides a dual representation for tail dependence functionals of max-stable random sup-measures. Denote by \mathbf{M} the family of Radon measures on the Borel σ -algebra \mathcal{B} in \mathbb{E} .

Theorem 5.1. *Let X be a max-stable random sup-measure. Then*

$$(18) \quad \ell(f) = \sup_{\mu \in \mathbf{M}} \int f d\mu, \quad f \in \text{USC}_0,$$

for a convex family

$$(19) \quad \mathbf{M} = \left\{ \mu \in \mathbf{M} : \int f d\mu \leq \ell(f), f \in \text{USC}_0 \right\}.$$

Proof. The tail dependence functional restricted to the family \mathbf{C}_0 of continuous functions on \mathbb{E} with compact support is a capacity in the sense of [13, Def. 4.1]. By [13, Th. 5.3], (18) holds for all $f \in \mathbf{C}_0$ with \mathbf{M} replaced by

$$\mathbf{M}_c = \left\{ \mu \in \mathbf{M} : \int f d\mu \leq \ell(f), f \in \mathbf{C}_0 \right\}.$$

It follows from [1, Th. 3.13] and Urysohn's lemma that, for all $f \in \text{USC}_0$, there exists a sequence of functions $\{f_n, n \geq 1\}$ from \mathbf{C}_0 approximating f from above. Then the upper semicontinuity and Fatou's lemma yield that $\mathbf{M}_c = \mathbf{M}$. Hence, (18) holds for all $f \in \mathbf{C}_0$.

In [13, Def. 4.2], the functional on \mathbf{C}_0 is extended to \mathbf{USC}_0 by approximation from above. In view of the existence of a sequence of continuous functions approximating $f \in \mathbf{USC}_0$ from above, and the upper semicontinuity of ℓ , we deduce that this extension of ℓ from \mathbf{C}_0 to \mathbf{USC}_0 coincides with the originally defined ℓ . By [13, Th. 5.5], $\ell(f)$ is given by (18). \square

Remark 5.2. The functional (18) constructed for an arbitrary convex family \mathbf{M} may fail to satisfy the complete alternation property, and so is not necessarily the tail dependence functional of a max-stable random sup-measure.

Proposition 5.3. *The functional (18) is the tail dependence functional of a CRSM X with extremal coefficient functional θ if and only if $\mathbf{M} = \mathbf{M}_\theta$, where*

$$(20) \quad \mathbf{M}_\theta = \{\mu \in \mathbb{M} : \mu(K) \leq \theta(K), K \in \mathcal{K}\}.$$

Proof. Necessity. By letting $f = \mathbf{1}_K$ in (19), it is easily seen that $\mathbf{M} \subset \mathbf{M}_\theta$. If X is a CRSM, then its tail dependence functional has the dual representation (18) with the family \mathbf{M} given by (19). By Lemma 2.2 and Fatou's lemma, \mathbf{M} is the family of all $\mu \in \mathbb{M}$ such that $\int f d\mu \leq \ell(f)$ for all step-functions $f = \sum a_i \mathbf{1}_{K_i}$ with $a_1, \dots, a_n > 0$ and $K_1 \supset K_2 \supset \dots \supset K_n$. The comonotonic additivity of ℓ yields that $\int f d\mu \leq \ell(f)$ for such functions f if and only if $\mu(K) \leq \ell(\mathbf{1}_K) = \theta(K)$ for all $K \in \mathcal{K}$, i.e. $\mu \in \mathbf{M}_\theta$ whenever $\mu \in \mathbf{M}$.

Sufficiency. If $\mathbf{M} = \mathbf{M}_\theta$, then [15, Prop. 2.3] yields that $\ell(f) = \int f d\theta$, which is the tail dependence functional of the CRSM with extremal coefficient functional θ , cf. Theorem 3.7. \square

Corollary 5.4. *Among all laws of max-stable random sup-measures sharing the same extremal coefficient functional θ , the (necessarily unique) CRSM law has the largest tail dependence functional.*

Proof. The assertion follows from Theorem 5.1 and Proposition 5.3, since \mathbf{M}_θ from (20) includes the family \mathbf{M} given by (19) if $\ell(\mathbf{1}_K) = \theta(K)$ for $K \in \mathcal{K}$. \square

Remark 5.5. If $\theta(\mathbb{E}) = 1$, then \mathbf{M}_θ can be further restricted to consist of probability distributions of all selections of the random closed set Ξ with the capacity functional θ , that is random elements ξ in \mathbb{E} such that ξ and Ξ can be realised on the same probability space to ensure that $\xi \in \Xi$ a.s.

Remark 5.6. The value $X(\mathbb{E})$ is a.s. finite if and only if the total mass of all measures from \mathbf{M} in (18) is uniformly bounded. However, even in this case, $\ell(f + a)$ is not necessarily equal to $\ell(f) + a\ell(1)$ for $a \in \mathbb{R}_+$, since the measures μ in (18) may have varying total masses. Max-stable random sup-measures satisfying $\ell(f+a) = \ell(f) + a\ell(1)$ for all $a \in \mathbb{R}_+$ form a family sandwiched between the CRSM and general max-stable random sup-measures. If $\ell(1) = 1$, then the functional $\ell(-f)$ has the properties of a coherent risk measure, see [8, 12]. In particular, the subadditivity property shows that diversification reduces risks, and $\ell(-(f + a)) = \ell(-f) - a$ is called the cash-invariance property. This property makes it possible to extend ℓ onto the family of all bounded measurable functions.

6. COMPLETE RANDOMNESS

Recall that a random sup-measure is said to be *completely random* if it assumes jointly independent values on disjoint sets. Hence, the tail dependence functional ℓ of a max-stable completely random sup-measure is finitely additive on linear combinations of indicator functions of disjoint sets, and, by approximation, is finitely additive on USC_0 . The upper semicontinuity property yields that $\ell(f) = \int f d\mu$ for a Radon measure μ (called *control measure*) that necessarily coincides with the extremal coefficient functional θ . Conversely, if θ is finitely additive, then it corresponds to a max-stable completely random sup-measure. This yields the following result.

Proposition 6.1. *Let X be a max-stable random sup-measure with extremal coefficient functional θ . Then the following are equivalent:*

- (i) X is completely random.
- (ii) θ is finitely additive.
- (iii) θ is a Radon measure μ .
- (iv) The tail dependence functional of X admits the representation (18) with \mathbf{M} being a singleton $\mathbf{M} = \{\mu\}$.

Each max-stable completely random sup-measure X is a CRSM, and each CRSM becomes completely random if uplifted to the space \mathcal{F}' of non-empty closed sets.

Proposition 6.2. *A max-stable random sup-measure X on \mathbb{E} is a CRSM if and only if $X(K) = Z(\mathcal{F}_K)$, $K \in \mathcal{K}$, for a max-stable completely random sup-measure Z on \mathcal{F}' .*

Proof. It follows from (13) that $X(K)$ is obtained as $Z(\mathcal{F}_K)$ for

$$Z(\mathcal{M}) = \sup\{t_i^{-1} : F_i \in \mathcal{M}\}$$

for each measurable $\mathcal{M} \subset \mathcal{F}'$. Since $\{(t_i, F_i)\}$ is a Poisson process, the random sup-measure Z is completely random.

In the other direction, the equality $X(K) = Z(\mathcal{F}_K)$ for all $K \in \mathcal{K}$ yields that

$$\int^e f dX = \int^e f^\vee dZ,$$

by Lemma 2.1, so that the tail dependence functional of X is given by $\ell(f) = \int f^\vee d\nu$, where ν is the control measure of Z . Since $(f+g)^\vee(F) = f^\vee(F) + g^\vee(F)$ for comonotonic functions f and g , the functional ℓ is comonotonic. \square

Remark 6.3. Proposition 6.2 together with Lemma 2.1 can be used to replace the integral $\int^e f dX$ with $\int^e f^\vee dZ$, where the latter integral is taken for a completely random sup-measure and so can be extended for all integrands f , such that f^\vee is integrable with respect to the control measure of Z , see [37].

Theorem 6.4. *For each CRSM X , there is a set-valued function $F : [0, 1] \mapsto \mathcal{F}$ such that $X(K) = Z(F^-(K))$ for a completely random sup-measure Z on $[0, 1]$ and $F^-(K) = \{u \in [0, 1] : F(u) \cap K \neq \emptyset\}$.*

Proof. Corollary 7.3 and the upper semicontinuity of θ yield that X is separable in probability as a process indexed by \mathcal{K} , that is, it satisfies Condition S, see [37] and [32]. Applying [7, Th. 3], we obtain that

$$X(K) \stackrel{d}{\sim} \bigvee_{i \geq 1} \Gamma_i^{-1} f_K(U_i), \quad K \in \mathcal{K},$$

for a Poisson process $\{(\Gamma_i, U_i)\}$ on $\mathbb{R}_+ \times [0, 1]$. By Theorem 4.6, X is a CRSM if and only if $f_K(U_i) = \tau_i \mathbb{1}_{\Xi_i \cap K \neq \emptyset} = \tau_i \mathbb{1}_{U_i \in F^-(K)}$, where $\Xi_i = F(U_i)$ for some set-valued function F . Thus, $X(K) = Z(F^-(K))$, $K \in \mathcal{K}$ for the completely random sup-measure $Z(A) = \bigvee_{i \geq 1} \Gamma_i^{-1} \tau_i \mathbb{1}_{U_i \in A}$. \square

7. MAX-STABLE PROCESSES, SEPARABILITY AND CONTINUITY

Max-stable processes. The sup-derivative $\xi(x) = X(\{x\})$, $x \in \mathbb{E}$, of a max-stable random sup-measure is a max-stable process on \mathbb{E} with upper semicontinuous paths. Conversely, sup-integrals of max-stable process with unit Fréchet marginals and upper semicontinuous paths are max-stable random sup-measures. Sup-derivatives of CRSMs are called *TM processes* in [38]. If \mathbb{E} is finite, then the values of a CRSM on its points build a TM random vector, see Example 10.1.

It should be noted that information on a random sup-measure X can be lost when passing to its sup-derivative ξ . For instance, $\xi(x)$ may almost surely vanish for all $x \in \mathbb{E}$ while X is positive almost surely on all compact balls (with positive radius). This is the case e.g. if X is completely random with a non-atomic control measure that is positive on such balls. Because of this, the max-stable random sup-measures provide a more general setting compared to max-stable processes as studied by their finite-dimensional distributions.

Separability. A random sup-measure X (and the corresponding functionals ℓ and θ) is called *separable* if the distribution of X is uniquely determined by the finite-dimensional distributions of its sup-derivative $\xi(x)$ for x from a countable set $D \subset \mathbb{E}$, that is

$$(21) \quad X(G) = \sup_{x \in D \cap G} X(\{x\}) \quad \text{a.s.,} \quad G \in \mathcal{G}.$$

By expressing the both sides of (21) using the LePage series (13), it is easily seen that a CRSM X is separable if and only if

$$(22) \quad \nu(\mathcal{F}_G) = \nu(\mathcal{F}_{D \cap G}), \quad G \in \mathcal{G},$$

where ν is the measure on \mathcal{F}' associated with the extremal coefficient functional θ of X by (14).

Let \mathcal{I} be the family of finite subsets of \mathbb{E} . A completely alternating functional θ on \mathcal{I} with $\theta(\emptyset) = 0$ can be extended to the capacity on \mathcal{K} by letting

$$(23) \quad \tilde{\theta}(G) = \sup\{\theta(I) : I \subset G, I \in \mathcal{I}\}, \quad G \in \mathcal{G},$$

$$(24) \quad \tilde{\theta}(K) = \inf\{\tilde{\theta}(G) : K \subset G, G \in \mathcal{G}\}, \quad K \in \mathcal{K}.$$

Proposition 7.1. *Let θ be a completely alternating functional on \mathcal{I} with $\theta(\emptyset) = 0$. Then $\tilde{\theta}$ given by (23) and (24) is the smallest extremal coefficient functional that dominates θ . The CRSM with the extremal coefficient functional $\tilde{\theta}$ is separable. Finally, θ is the restriction on \mathcal{I} of a separable extremal coefficient functional if and only if θ and $\tilde{\theta}$ coincide on \mathcal{I} .*

Proof. Let θ' be another extremal coefficient functional that dominates θ on \mathcal{I} . Then θ' dominates $\tilde{\theta}$ on \mathcal{G} and so on \mathcal{K} . Let ν be the measure on \mathcal{F}' determined by $\tilde{\theta}$. Let B be any set from a countable base of the topology on \mathbb{E} , and let $\{I_n\}$ be an increasing sequence of finite sets such that $\theta(I_n) \uparrow \tilde{\theta}(B)$. Since $I_n \uparrow D_B = \cup I_n$, we have $\theta(I_n) \uparrow \nu(\mathcal{F}_{D_B})$. Therefore, $\tilde{\theta}(B) = \nu(\mathcal{F}_{D_B})$. Finally (22) holds for D being the union of D_B over all B from the countable base of the topology and G also belonging to the base of topology. Its validity can be then easily extended for all open G . \square

Continuity. The series representations of max-stable random sup-measures yield the corresponding series representations for max-stable processes. Since these series for TM processes involve indicator functions, it is easy to see that TM processes are never path continuous unless they are a.s. constant.

Proposition 7.2. *If X is a CRSM with the extremal coefficient functional θ , then, for all $K_1, K_2 \in \mathcal{K}$,*

$$\mathbf{P} \{X(K_1) - X(K_2) \leq \varepsilon\} \geq \exp \left\{ -\frac{1}{\varepsilon} (\theta(K_1 \cup K_2) - \theta(K_2)) \right\}.$$

In particular,

$$(25) \quad \mathbf{P} \{|X(K_1) - X(K_2)| > \varepsilon\} \leq \frac{1}{\varepsilon} (2\theta(K_1 \cup K_2) - \theta(K_1) - \theta(K_2)).$$

Proof. Since X is a CRSM,

$$\mathbf{P} \{X(K_1) \leq p\varepsilon, X(K_2) \leq q\varepsilon\} = \exp \left\{ -\frac{\theta_{12} - \theta_2}{p\varepsilon} - \frac{\theta_{12} - \theta_1}{q\varepsilon} - \frac{\theta_1 + \theta_2 - \theta_{12}}{(p \wedge q)\varepsilon} \right\},$$

where $\theta_i = \theta(K_i)$ and $\theta_{ij} = \theta(K_i \cup K_j)$ and $p \wedge q = \min(p, q)$. Hence, for any $n \geq 1$,

$$\begin{aligned} & \mathbf{P}\{X(K_1) - X(K_2) \leq \varepsilon\} \\ & \geq \sum_{k=1}^n \mathbf{P} \{X(K_1) \leq k\varepsilon, X(K_2) \leq k\varepsilon\} - \mathbf{P} \{X(K_1) \leq k\varepsilon, X(K_2) \leq (k-1)\varepsilon\} \\ & = \sum_{k=1}^n \exp \left\{ -\frac{1}{\varepsilon} \left(\frac{\theta_{12} - \theta_2}{k} + \frac{\theta_2}{k} \right) \right\} - \exp \left\{ -\frac{1}{\varepsilon} \left(\frac{\theta_{12} - \theta_2}{k} + \frac{\theta_2}{k-1} \right) \right\} \\ & = \sum_{k=1}^n \exp \left\{ -\frac{\theta_{12} - \theta_2}{\varepsilon k} \right\} \left[\exp \left\{ -\frac{\theta_2}{\varepsilon k} \right\} - \exp \left\{ -\frac{\theta_2}{\varepsilon(k-1)} \right\} \right] \\ & \geq \exp \left\{ -\frac{\theta_{12} - \theta_2}{\varepsilon} \right\} \sum_{k=1}^n \left[\exp \left\{ -\frac{\theta_2}{\varepsilon k} \right\} - \exp \left\{ -\frac{\theta_2}{\varepsilon(k-1)} \right\} \right], \end{aligned}$$

where the last telescoping sum equals $\exp \{-\theta_2/(\varepsilon n)\}$ and converges to 1. \square

Corollary 7.3. *If X is a CRSM, then $X(K_n)$ converges in probability to $X(K)$ for $K \in \mathcal{K}$ and a sequence $K_n \in \mathcal{K}$, $n \geq 1$, if and only if $\theta(K_n) \rightarrow \theta(K)$ and $\theta(K_n \cup K) \rightarrow \theta(K)$.*

Proof. Sufficiency follows from (25). For the necessity, note that the convergence in probability yields the convergence in distribution, and so $\theta(K_n) \rightarrow \theta(K)$. Since $X(K_n \cup K) \rightarrow X(K)$ in probability, $\theta(K_n \cup K) \rightarrow \theta(K)$. \square

Corollary 7.4. *A CRSM is continuous in probability in the Hausdorff metric if and only if its extremal coefficient functional is continuous in the Hausdorff metric. Then X is almost surely continuous at each $K \in \mathcal{K}$ that coincides with the closure of its interior.*

Proof. If K is regular closed and K_n converges to K in the Hausdorff metric, then $K^{-\varepsilon_n} \subset K_n \subset K^{\varepsilon_n}$ for a sequence $\varepsilon_n \downarrow 0$, where $K^r = \{x : B_r(x) \cap K \neq \emptyset\}$ and $K^{-r} = \{x : B_r(x) \subset K\}$ for the closed ball $B_r(x)$ of radius r centred at x . Note that both $X(K^{-\varepsilon_n})$ and $X(K^{\varepsilon_n})$ are monotone sequences that converge in probability to $X(K)$ and so almost surely as well. \square

Corollary 7.5. *The TM process $\xi(x) = X(\{x\})$, $x \in \mathbb{E}$, of a CRSM X is stochastically continuous if and only if $\theta(\{x\})$, $x \in \mathbb{E}$, is continuous.*

8. COUPLING AND CONTINUOUS CHOICE

Ordered coupling. Two random sup-measures X and X' are said to admit the *ordered coupling* (notation $X \preceq X'$) if they can be realised on the same probability space so that with probability one $X(K) \leq X'(K)$ for all $K \in \mathcal{K}$. For this, one needs that

$$\mathbf{P} \{X(K_1) \geq t_1, \dots, X(K_m) \geq t_m\} \geq \mathbf{P} \{X'(K_1) \geq t_1, \dots, X'(K_m) \geq t_m\}$$

for all $m \geq 1$ and $K_1, \dots, K_m \in \mathcal{K}$, see e.g. [18].

Theorem 8.1. *Let X be a max-stable random sup-measure X , such that $X(\mathbb{E})$ is a.s. finite. Then there exist unique CRSMs X_* and X^* , such that $X_* \preceq X \preceq X^*$, and for any other CRSMs X' and X'' such that $X' \preceq X \preceq X''$, we have also $X' \preceq X_*$ and $X^* \preceq X''$.*

Proof. The max-stable random sup-measure X admits the LePage representation (10), where $\{Y_i\}$ are i.i.d. copies of a sup-measure Y . Then $X \preceq X'$ for another max-stable random sup-measure X' if and only if X' has the LePage representation with i.i.d. random sup-measures $\{Y'_i\}$ distributed as Y' such that $Y \preceq Y'$. Since $X(\mathbb{E})$ is finite,

$$\ell(\mathbf{1}_{\mathbb{E}}) = \mathbf{E} \int^e \mathbf{1}_{\mathbb{E}} dY = \mathbf{E} Y(\mathbb{E}) < \infty.$$

Thus, $Y(\mathbb{E})$ is a.s. finite. The minimal CRSM X^* dominating X arises if the corresponding Y^* is chosen to be the smallest random sup-measure Y^* with realisations from $\mathfrak{S}_{\text{ind}}$ that dominates Y , that is $Y^*(K) = Y(\mathbb{E}) \mathbf{1}_{Y(K) > 0}$ for $K \in \mathcal{K}$. Furthermore,

$Y_*(K) = Y(\mathbb{E})$ for $K \subset \{x : Y(\{x\}) = Y(\mathbb{E})\}$ and $Y_*(K) = 0$ otherwise is the largest indicator random sup-measure that is dominated by Y . Finally, construct the CRSMs X^* and X_* by (10) using i.i.d. copies of Y^* and Y_* , respectively. \square

Continuous choice. Upper semicontinuous max-stable processes $\xi(x)$ defined for x from a compact space \mathbb{E} , have been used to model *continuous choice* in [29]. In particular, the random set

$$M = \{x \in \mathbb{E} : \xi(x) = \xi^\vee(\mathbb{E})\} = \{x \in \mathbb{E} : X(\{x\}) = X(\mathbb{E})\}$$

is the set of optimal choices, where $X = \xi^\vee$ is the sup-integral of ξ . The upper semicontinuity assumption on X and ξ yields that M is indeed a random closed set [24, Th. 1.2.27(ii)]. In the following we relax the compactness assumption on \mathbb{E} by only imposing that the max-stable random sup-measure is finite, so that $\xi^\vee(\mathbb{E}) < \infty$ and $M \neq \emptyset$ almost surely. The following theorem immediately recovers and extends a number of results from [29], namely Theorem 4.1 and 4.2 and Corollary 4.1 therein, for not necessarily separable max-stable random sup-measures on not necessarily compact spaces. We do not need to assume that X is separable, since M is defined for each ω from the probability space.

Theorem 8.2. *If X is a finite max-stable random sup-measure with the LePage representation (10), then the set of optimal choices M is distributed as*

$$M_Y = \{x : Y(\{x\}) = Y(\mathbb{E})\}$$

and is independent of $X(\mathbb{E})$. The set M is a singleton if and only if M_Y is a.s. a singleton; in particular, if X is a CRSM, this is possible if and only if X is completely random.

Proof. If X is a finite CRSM, then (15) yields that

$$\xi(x) \stackrel{d}{\sim} \theta(\mathbb{E}) \bigvee_{i \geq 1} \Gamma_i^{-1} \mathbf{1}_{x \in \Xi_i}$$

for an i.i.d. sequence $\{\Xi_i\}$ of a.s. non-empty random closed sets. Then $M = \Xi_1$, and so its distribution can be identified as $\mathbf{P}\{M \cap K \neq \emptyset\} = \theta(K)/\theta(\mathbb{E})$, $K \in \mathcal{K}$. Furthermore, the random sets $M = \Xi_1$ and $\xi^\vee(\mathbb{E}) = \theta(\mathbb{E})\Gamma_1^{-1}$ are independent.

The largest CRSM X_* dominated by X and constructed in Theorem 8.1 shares with X the same values for the maximum and the corresponding arg-max set M . It suffices to note that X_* admits the LePage representation (15) and so its value on \mathbb{E} and the arg-max set are independent, while M has the same distribution as Ξ_1 .

Finally, M is a singleton if and only if M_Y is a singleton. In the CRSM case, this is equivalent to θ being a Radon measure, see Proposition 6.1. \square

9. INVARIANCE AND TRANSFORMATIONS

Bernstein functions. A non-negative function g on $[0, \infty)$ is called a *Bernstein function* if and only if it is continuous on $[0, \infty)$ and its derivatives $g^{(n)}$ on $(0, \infty)$ exist and satisfy $(-1)^{(n+1)}g^{(n)} \geq 0$ for all $n \geq 1$. Each Bernstein function such that $g(0) = 0$ can be represented as

$$(26) \quad g(t) = bt + \int_0^\infty (1 - e^{-st}) \varrho(ds)$$

for $b \geq 0$ and a Radon measure ϱ on $(0, \infty)$ with $\int_0^\infty (s \wedge 1)\varrho(ds) < \infty$, where $s \wedge 1 = \min(s, 1)$, see [33]. Such functions can be also viewed as the continuous non-negative negative definite functions on the semigroup being $[0, \infty)$ with the arithmetic addition, see [4]. A wealth of material on Bernstein functions including many examples can be found in [33].

Proposition 9.1. *Let θ be the extremal coefficient functional of a max-stable random sup-measure. If g be a Bernstein function such that $g(0) = 0$, then the composition $(g \circ \theta)(K) = g(\theta(K))$, $K \in \mathcal{K}$, is an extremal coefficient functional.*

Proof. It is easily seen that $g \circ \theta(\emptyset) = 0$ and $g \circ \theta$ is upper semicontinuous by the continuity of g . The functional $g \circ \theta$ is also completely alternating, since the complete alternation and negative definiteness are equivalent on the idempotent (and in particular 2-divisible) semigroup $(\mathcal{K}, \cup, \emptyset)$ [4, Cor. 4.6.8 and p. 120] and Bernstein functions preserve this property [4, Prop. 3.2.9 and p. 114]. By Lemma 3.5 the functional $g \circ \theta$ is an extremal coefficient functional. \square

Example 9.2. The Bernstein functions $g_\alpha(t) = t^\alpha$, $\alpha \in (0, 1)$ can be represented in the form (26) with $\varrho(ds) = \frac{\alpha}{\Gamma(1-\alpha)}s^{-(\alpha+1)}ds$, see [4, p. 78]. Hence, if θ is an extremal coefficient functional, so is θ^α for $\alpha \in (0, 1)$.

Rearrangement invariance. Assume that X is a max-stable random sup-measure and let μ be a Radon measure on the Borel σ -algebra in \mathbb{E} . The tail dependence functional $\ell(f)$ (and the max-stable random sup-measure X) is said to be *rearrangement invariant* (or symmetric) with respect to μ if $\ell(f_1) = \ell(f_2)$ if $\mu(\{f_1 \geq t\}) = \mu(\{f_2 \geq t\})$ for all $t > 0$.

If X is a CRSM, then $\ell(f)$ is rearrangement invariant if and only if $\theta(K)$ is *symmetric*, meaning that $\theta(K_1) = \theta(K_2)$ whenever $\mu(K_1) = \mu(K_2)$. For general stable sup-measures the symmetry of θ does not imply the rearrangement invariance of ℓ , see [40, Ex. 4].

Example 9.3. Let μ be the counting measure on $\mathbb{E} = \{1, \dots, d\}$, so that a CRSM X is determined by $X(\{i\}) = \xi_i$, $i = 1, \dots, d$. The rearrangement invariance of X means that $\theta(K)$ depends only on $\mu(K)$, i.e. the cardinality of K . The normalised functional $\theta(K)$ is the capacity functional of a random set $\Xi \subset \{1, \dots, d\}$. By the rearrangement invariance, conditionally on $\mu(\Xi) = k$, the random set Ξ can equally likely be any

k -tuple of points from $\{1, \dots, d\}$. Thus,

$$\theta(K) = c \left(\sum_{k=1}^{d-m} \frac{\binom{d}{k} - \binom{d-m}{k}}{\binom{d}{k}} p_k + \sum_{k=d-m+1}^d p_k \right) = c \left(1 - p_0 - \sum_{k=1}^{d-m} \frac{\binom{d-m}{k}}{\binom{d}{k}} p_k \right),$$

where m is the cardinality of K , p_0, \dots, p_d is a probability distribution on $\{0, \dots, d\}$, and $c = \theta(\mathbb{E})$.

Example 9.4. Let \mathbb{E} be a countable set with the discrete topology and the counting measure μ . Assume that $\theta(\mathbb{E})$ is finite, so that the normalised θ defines a random closed set Ξ . The capacity functional of Ξ is rearrangement invariant if and only if the sequence $\{\mathbb{1}_{i \in \Xi}, i \geq 1\}$ is exchangeable. By the de Finetti theorem such a sequence is conditionally i.i.d. Thus, given a random variable $\zeta \in [0, 1]$, Ξ consists of all points in \mathbb{E} , independently chosen with probability ζ and

$$\theta(K) = c(1 - \mathbf{E}[(1 - \zeta)^{\mu(K)}])$$

for $c > 0$ yields all rearrangement invariant finite extremal coefficient functionals on a countable space. The random set Ξ with the capacity functional given by the normalised θ is the support of the Cox (doubly stochastic Poisson) process whose intensity measure is given by $(-\log(1 - \zeta))\mu$. If $\zeta = 1$, then $\Xi = \mathbb{E}$.

Any rearrangement invariant extremal coefficient functional can be represented as $\theta(K) = g(\mu(K))$ for a monotone function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $g(0) = 0$ and θ is completely alternating. This is the case, for instance, if g is a Bernstein function.

Example 9.5. The rearrangement invariant tail dependence functional ℓ of a CRSM can be extended to L^∞ and then, applied to $-f$, becomes a coherent risk measure of f , see [8]. One of the most important coherent risk measures is the average value at risk that appears if $\theta(K) = g_\alpha(\mu(K))$ for $g_\alpha(t) = \frac{1}{\alpha}(t \wedge \alpha)$ with a fixed $\alpha \in (0, 1]$ and a probability measure μ , see [12, Ex. 4.65]. However, $g_\alpha(\mu(K))$ is not alternating of order 3 and so is not completely alternating and consequently is not an extremal coefficient functional. Indeed, assume that μ is non-atomic and fix disjoint sets K_1, K_2, K_3, K_4 with equal measures $p/4$ for some $p \in (0, 1]$ such that $\alpha \in [(3/4)p, p)$. Then

$$\begin{aligned} \Delta_{K_1} \Delta_{K_2} \Delta_{K_3} (g_\alpha \circ \mu)(K_4) &= \frac{1}{\alpha} \left(\sum_{k=0}^3 (-1)^k \binom{3}{k} \left(\frac{(k+1)p}{4} \wedge \alpha \right) \right) \\ &= \frac{1}{\alpha} \left(\frac{p}{4} \wedge \alpha - 3 \frac{2p}{4} \wedge \alpha + 3 \frac{3p}{4} \wedge \alpha - \frac{4p}{4} \wedge \alpha \right) = \frac{p}{\alpha} - 1 > 0. \end{aligned}$$

In particular, this example shows that the convex set \mathbf{M} of probability measures having density with respect to μ bounded by a constant $c > 1$ does not yield a tail dependence functional by (18).

Under the assumption that the reference measure μ is a probability measure and $\theta(\mathbb{E}) = 1$, each rearrangement invariant extremal coefficient functional can be expressed

as

$$\theta(K) = \int_{(0,1]} s^{-1}(\mu(K) \wedge s)\kappa(ds) + \mathbf{1}_{K \neq \emptyset}\kappa(\{0\})$$

for a probability measure κ on $[0, 1]$, see [12, Th. 4.87] and [19]. Example 9.5 shows that κ concentrated at a single point does not yield a valid extremal coefficient functional.

Example 9.6. If $g(t) = t^\alpha$ for $\alpha \in (0, 1)$, then $\kappa(dt) = \alpha(1 - \alpha)t^{\alpha-1}dt$. The corresponding functional $\theta(K) = \mu(K)^\alpha$ is an extremal coefficient functional and $\ell(-f)$ is the proportional hazard risk measure.

Stationarity and self-similarity. A random sup-measure on $\mathbb{E} = \mathbb{R}^d$ is called *stationary* if $X(\cdot + x) \stackrel{d}{\sim} X(\cdot)$ for all $x \in \mathbb{E}$. It is called *self-similar* with exponent H if $X(c \cdot) \stackrel{d}{\sim} c^H X$ for all $c > 0$. Stationary and self-similar random sup-measures are the only possible scaling limits of extremal processes on $[0, \infty)$, cf. [28].

It is immediate that a CRSM is stationary (resp. self-similar) if and only if its extremal coefficient functional satisfies $\theta(K + x) = \theta(K)$ (resp. $\theta(cK) = c^H \theta(K)$) for $K \in \mathcal{K}$. However, the non-uniqueness of the spectral measure π in the LePage representation of max-stable random sup-measures (10) implies that non-stationary (or non-selfsimilar) Y may result in stationary (or self-similar) random sup-measures. In particular, the CRSM given by (10) is stationary if and only if $\mathbf{E}Y(K) = \mathbf{E}Y(K + x)$ for all $x \in \mathbb{R}^d$. In other words, the first order stationarity of Y implies the stationarity of X .

Example 9.7. Let $\mathbb{E} = \mathbb{R}$, and let ζ be a positive random variable with density $(rg(r))^{-1}$, $r > 0$, for an appropriately chosen function g . Set $\Xi = \{\log \zeta\}$ and $\tau = g(\zeta)$ in Remark 4.7, so that $Y(K) = g(\zeta)\mathbf{1}_{\log \zeta \in K}$, $K \in \mathcal{K}$. While Ξ is not stationary, $\mathbf{E}Y(K) = \int_0^\infty \mathbf{1}_{\log r \in K} r^{-1} dr$ equals the Lebesgue measure of K and so is translation invariant.

Remark 9.8. A generic construction of stationary CRSM works as follows. Considering the Poisson process $\{(t_i, v_i, F_i)\}$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{F}'$ with the intensity measure $\lambda \otimes \lambda_d \otimes \nu$ (where λ_d is the Lebesgue measure in \mathbb{R}^d) and let

$$(27) \quad X(K) = \bigvee_{i \geq 1} t_i^{-1} \mathbf{1}_{(F_i + v_i) \cap K \neq \emptyset}, \quad K \in \mathcal{K}.$$

The extremal coefficient functional of X is

$$\theta(K) = \int_{\mathcal{F}'} \lambda_d(K + \check{F})\nu(dF),$$

where $\check{F} = \{-x : x \in F\}$. If θ is normalised and corresponds to the random closed set Ξ , then the latter simplifies to $\theta(K) = \mathbf{E}\lambda_d(K + \Xi)$.

A similar construction with $F_i + v_i$ replaced by $s_i F_i$ on $\mathbb{E} = \mathbb{R}^d \setminus \{0\}$ for the Poisson process $\{s_i, i \geq 1\}$ of intensity $\alpha s^{\alpha-1} ds$, $s > 0$, yields self-similar CRSMs. These constructions can be also applied to obtain stationary and self-similar versions of the tail dependence functional of general max-stable random sup-measures.

10. EXAMPLES OF CRSM SUP-MEASURES

Example 10.1 (TM random vectors). If $\mathbb{E} = \{1, \dots, d\}$ is a finite set, then the CRSM corresponds to a semi-simple max-stable random vector, whose distribution is uniquely determined by its extremal coefficients $\theta(K) = \ell(\mathbf{1}_K)$, $K \subset \{1, \dots, d\}$. The comonotonicity property of ℓ is equivalent to

$$\ell(u) = (u_d - u_{d-1})\theta(\{d\}) + (u_{d-1} - u_{d-2})\theta(\{d-1, d\}) + \dots + u_1\theta(\{1, \dots, d\})$$

for $u_1 \leq \dots \leq u_d$. Thus, the CRSMs on a finite carrier space become TM random vectors studied in [38]. In this case, each CRSM is necessarily finite and the series representation (15) yields the series representation of TM random vectors from [38]. Proposition 6.2 becomes [38, Eq. (10)]. It is easy to see that a stable sup-measure X is a CRSM if and only if its finite dimensional distributions are TM random vectors.

Example 10.2. If $\Xi = F$ is a deterministic closed set, then $\theta(K) = \mathbf{1}_{K \cap F \neq \emptyset}$, and the corresponding CRSM constructed by (15) is the indicator sup-measure $X(K) = \zeta \mathbf{1}_{K \cap F \neq \emptyset}$, where ζ is the unit Fréchet random variable.

Example 10.3. Let $\Xi = [\zeta, \infty)$ on $\mathbb{E} = \mathbb{R}_+$, where ζ is a non-negative random variable, so that $\theta(K) = \mathbf{P}\{\zeta \leq \sup K\}$ is the capacity functional of Ξ . The corresponding CRSM is given by $X(K) = \eta(\sup K)$ for the increasing max-stable process

$$\eta(t) = \bigvee_{i \geq 1} \Gamma_i^{-1} \mathbf{1}_{\zeta_i \leq t}, \quad t \geq 0,$$

where $\{\zeta_i\}$ are i.i.d. copies of ζ .

Example 10.4. Assume that $\mathbb{E} = \mathbb{R}^d$ and let $\Xi = \xi + M$, where ξ is a random vector and M is a deterministic compact set. The CRSM constructed by (15) using i.i.d. copies of Ξ can be obtained as $X(K) = Z(K + \check{M})$, where Z is a completely random sup-measure with the control measure being the distribution of ξ and $K + \check{M} = \{x - y : x \in K, y \in M\}$. The extremal coefficient functional of X is $\theta(K) = \mathbf{P}\{\xi \in K + \check{M}\}$.

Example 10.5. Let $\theta(K)$ be the perimeter of a convex set K in $\mathbb{E} = \mathbb{R}^2$. The corresponding measure ν on \mathcal{F}' such that $\nu(\mathcal{F}_K) = \theta(K)$ is the Haar measure on the affine Grassmannian $A(1, 2)$ that consists of all lines in \mathbb{R}^2 , see e.g. [36, p. 582]. The measure $\lambda \otimes \nu$ defines a stationary and isotropic marked line process $\{(t_i, L_i)\}$ on $\mathbb{R}_+ \times A(1, 2)$, see [36, p. 124]. The LePage series (13) defines a CRSM X such that $X(K)$ equals the maximum of t_i^{-1} for all lines L_i that hit K .

Example 10.6. Let Ξ be a random closed subset of $\mathbb{E} = \mathbb{R}_+$ with distribution μ . For $\beta \in (0, 1)$, define

$$\theta(K) = \int_0^\infty \mathbf{P}\{\Xi + v \cap K \neq \emptyset\} \beta v^{\beta-1} dv.$$

The LePage representation of the corresponding CRSM X turns into

$$X(K) \stackrel{d}{\sim} \bigvee_{i \geq 1} t_i^{-1} \mathbf{1}_{\Xi_i + v_i \cap K \neq \emptyset}, \quad K \in \mathcal{K},$$

where $\{(t_i, v_i, \Xi_i)\}$ is the Poisson process on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{F}'$ with the intensity $dt\beta v^{\beta-1}dv d\mu$. This sup-measure X is the central object in [20] for Ξ being the range of a stable subordinator of order $(1 - \beta)$. Then Ξ coincides in distribution with $s\Xi$ for all $s > 0$, so that $\theta(sK) = s^\beta\theta(K)$ for all $K \in \mathcal{K}$ and $s > 0$. Furthermore, $\theta(K + s) = \theta(K)$ meaning that X is stationary. This CRSM X is the sup-vague limit of appropriately rescaled random sup-measures arising from a stationary symmetric α -stable sequence (here $\alpha = 1$) whose dynamics is driven by a Markov chain with regularly varying first entrance time.

Example 10.7. Let θ be the capacity functional of the random set Ξ being the path of the standard Brownian motion W_t , $t \geq 0$, in \mathbb{R}^d for $d \geq 3$ that starts at zero. The corresponding CRSM is constructed by (15) and has the tail dependence functional $\ell(f) = \mathbf{E} \sup_{t \geq 0} f(W_t)$ for $f \in \text{USC}_0$.

Example 10.8. Assume that $\mathbb{E} = \mathbb{R}^d$. The measure ν on \mathcal{F}' related to the extremal coefficient functional by (14) is supported by convex sets if and only if θ is additive on convex sets meaning that

$$\theta(K_1 \cup K_2) + \theta(K_1 \cap K_2) = \theta(K_1) + \theta(K_2)$$

for all convex K_1, K_2 such that $K_1 \cup K_2$ is also convex, see [23, Th. 5.1.1]. This property is also known under the name of *C-additivity*, such functional θ is also called a *valuation*, see [35, Ch. 6]. Assuming that θ is monotone and invariant for rigid motions (equivalently X is stationary and isotropic), the Hadwiger theorem [35, Th. 4.2.7] establishes that

$$(28) \quad \theta(K) = \sum_{i=0}^d a_i V_i(K)$$

for all convex compact K , where $a_1, \dots, a_d \geq 0$ and $V_0(K), \dots, V_d(K)$ are intrinsic volumes of K .

The functional $a_i V_i(K)$ defines a Poisson process of intensity $a_i H_{d-i}$, where H_{d-i} is the normalised Haar measure on the affine Grassmannian $A(d-i, d)$ that consists of all $(d-i)$ -dimensional affine subspaces of \mathbb{R}^d . Thus, θ yields a measure ν on \mathcal{F}' that corresponds to a superposition of such processes on Grassmannians of varying dimension, see also [23, Th. 5.4.2]. The sets F_i in (13) are affine subspaces of \mathbb{R}^d .

Example 10.9. Let $\theta(K) = c\mathbf{E}\lambda_d(\Xi + \check{K})$ for a constant $c > 0$ and a random compact set Ξ with distribution ν . The corresponding measure ν on \mathcal{F}' is translation invariant and so can be disintegrated into the product $\lambda_d \otimes c\nu$. Then the LePage series from (13) can be obtained from the Poisson process $\{(t_i, x_i, F_i)\}$ in $\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{K}$ with the intensity measure $\lambda \otimes \lambda_d \otimes \nu$, so that (27) yields a stationary CRSM with the extremal coefficient functional θ . In order to ensure that X is finite on compact sets, it is required that the Lebesgue measure of the sum of Ξ and the unit ball is integrable. The sup-derivative ξ of X is the so-called *storm process* generated by indicator functions, see [21]. The random set Ξ determines the shape of the random indicator function called a storm, while the points x_i control the locations of storms whose strengths are then given by t_i^{-1} . In [21] the random set Ξ is chosen to be the Poisson polygon.

The functional θ is additive on convex sets, and so admits the representation (28). For example, if $\Xi = B_\xi$ is the ball of random radius ξ centred at the origin, then the Steiner formula from convex geometry [35, p. 208] yields that

$$\theta(K) = \sum_{i=0}^d V_i(K) \mathbf{E} \xi^{d-i}$$

for each convex compact set K . In particular, X shares the same distribution with the CRSM from Example 10.8 (with $a_i = \mathbf{E} \xi^{d-i}$) on any chain $K_1 \subset K_2 \subset \dots \subset K_m$ of convex sets.

Example 10.10. Let W be a centred Gaussian process on \mathbb{R}^d with stationary increments, that is, the law of $\{W(x+y) - W(y)\}_{x \in \mathbb{R}^d}$ does not depend on $y \in \mathbb{R}^d$. Specifying $W(0) = 0$, the law of W is uniquely determined by its variogram $\gamma(x, y) = \mathbf{E}(W(x) - W(y))^2$. The Brown–Resnick process associated to γ [17] is defined by the LePage series representation

$$\xi(x) = \bigvee_{i \geq 1} \Gamma_i^{-1} \exp \left(W_i(x) - \frac{\gamma(x, 0)}{2} \right), \quad x \in \mathbb{R}^d,$$

where $\{W_i, i \geq 1\}$ are i.i.d. copies of W independent of $\{\Gamma_i, i \geq 1\}$. Its sup-integral $X(K) = \sup\{\xi(x) : x \in K\}$, $K \in \mathcal{K}$, is a stationary max-stable random sup-measure. However, it is not a CRSM, since its sup-derivative ξ has continuous paths. This also follows from the fact that $\xi(x_1), \dots, \xi(x_m)$ follows multivariate Hüsler–Reiss distributions [16] that are not spectrally discrete and so do not correspond to TM random vectors, cf. Example 10.1. This has been illustrated in [38, Fig. 3] by plotting the dependency sets (which are a finite-dimensional analogue of the sets of measures \mathbf{M} in (19)) and which are not polytopes.

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