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MetZet Classic Papers

On temperature stratification in resting and non-accelerated moving air[E1]

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Translated from German and edited by Esther Volken[1], Alena Giesche[2,3*] and Stefan Brönnimann[2,3*]

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Abstract
This is the edited and translated version of the article by Max Margules "Über Temperaturschichtung in stationär bewegter und in ruhender Luft" (On temperature stratification in resting and non-accelerated moving air), which was published in 1906 in the "Hann-Volume" of the Meteorologische Zeitschrift (p. 243–254), a volume dedicated to the 40 year anniversary of Hann's editorship of the journal.

Keywords: Classic paper, Margules, thermal wind, fronts, cyclones

Superscript numbers indicate original footnotes (translated at the bottom of the page), E… numbers indicate editorial endnotes (at the end of the article), square brackets [] indicate editorial comments in the text. There were some square brackets in the original text that were replaced with {}.

The large horizontal temperature differences on Earth are maintained in a state of dynamic balance, in which heat supply is the most significant factor. In the ideal case of frictionless air movement, such differences could continue to exist without any warming process, once established. There are temperature distributions that remain stationary for days, induced by movement of air masses with inhomogeneous temperatures. During this time, supply and removal of heat do not seem to have a significant effect. Examples of this can be found in Hann's studies on barometer maxima[E2]. In stationary high pressure areas, enormous warm air masses accumulate, and at some distance further away, there are cold air masses; the difference can amount to 10°[E3] within 500 km and persist in vertical layers of several kilometres. Nevertheless, the cold air does not displace the warm air. Other smaller-scale examples include temperature differences between 5 and 10°, which can exist for many hours or even days in nearby locations.

Helmholtz showed, in his first paper[E4] on atmospheric movement, that two rotating air rings that are axial relative to the Earth can remain stationary, even if their temperature is erratically [sprunghaft] different on the same level. The significance of this study goes far beyond this particular case. I try to show a derivation of the Helmholtz equation that may be applied more easily to meteorological problems. Furthermore, I will add some aspects of constant temperature gradients. From the known relations between pressure distribution and wind for stationary frictionless movements, other relations that show the relation between horizontal temperature gradients and vertical change in wind velocity can be derived quite easily.[E5]

Some reflections on the damping of movements by mixing will follow. If the vertical stratification is stable, then the mixing of the air from different heights will occur against gravity, using kinetic energy.

The threshold for the vertical temperature gradient in dry air is generally stated to be 1° per 100 m; larger gradients are considered to occur in unstable conditions. However, this is only true for air with a uniform composition; if the vapour content is a function of height,

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then the gradient may double. In the case of an unsteady transition, the warmer dry layer may be below without resulting in unstable conditions. The gradients registered by balloons and unmanned ascents have given reason to examine what occurs in the relation between vapour content and density for all dry state cases.

1 The Helmholtz boundary between two air masses of erratically different temperature

We look at the relative movement in the simplest rotating system:

- z axis of rotation, j constant speed of rotation,
- the xy- and parallel planes of rotating standard surfaces.
- g constant gravitational acceleration in direction $-z$.

The diverting force $2jc_h$ acts on a moving mass 1 in the xy-plane perpendicular to the direction of the horizontal velocity c_h in such a way that the mass is pushed away from its present direction of movement, namely it moves against the rotation direction.

A straight path can exist only if the diverting force is compensated by another overriding force, through frictionless air movement by the force of the horizontal pressure decrease.

Two air masses move parallel to the x-axis. Their densities μ and velocities u show erratic differences in the plane, and velocity is constant. We ask under what circumstances such a state can be stationary.

Fig. 1[E6] shows the intersection of the boundary surface with the yz-layer; in all parallel planes the state is the same as in the yz-plane and the direction of movement is perpendicular to the yz- plane.

The conditions for stationary linear movement:

- The force of the horizontal pressure decrease is opposite to the deviating force.
- The force of the vertical pressure decrease is opposite to the gravitational force.

Two pairs of equations[E8] result from this if, in addition, p refers to the pressure and the masses have the indices 1 and 2:

$$\frac{1}{\mu_1}\frac{\delta p_1}{\delta y} = 2ju_1 \qquad \frac{1}{\mu_2}\frac{\delta p_2}{\delta y} = 2ju_2$$
$$\frac{1}{\mu_1}\frac{\delta p_1}{\delta z} = -g \qquad \frac{1}{\mu_2}\frac{\delta p_2}{\delta z} = -g \tag{1.1}$$

The equations in the first line apply with the stipulation that, with a positive j, the y-axis is in the former position of the x-axis after a quarter of a rotation.

The pressure may have just one defined value for any point on the boundary surface; the pressure decrease has erratically different values on both sides.

If one assumes that the condition $p_1 = p_2 = p_a$ is true for point a of the boundary surface, then one

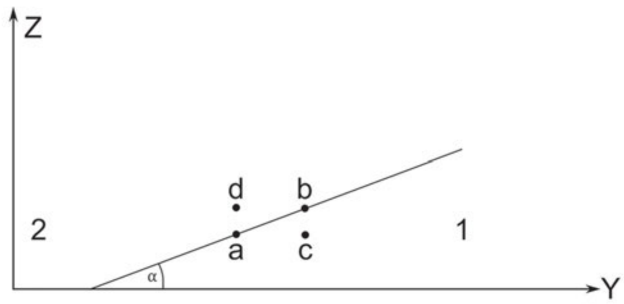

Figure 1: [E7]Schematic figure of the air mass boundary.

should also have the same value p_b around point b, no matter whether one approaches from side 1 or side 2. This requirement defines the incline of the surface.

Two points, c in 1 and d in 2, form a rectangular with a and b, the sides of which we call $d\eta$, $d\xi$. If the distance between a and b is sufficiently small, then the following equations hold true:

In mass 1: In mass 2:

$$p_c - p_a = \frac{\delta p_1}{\delta y} d\eta = 2j\mu_1 u_1 d\eta \qquad p_b - p_d = \frac{\delta p_2}{\delta y} d\eta = 2j\mu_2 u_2 d\eta$$

$$p_b - p_c = \frac{\delta p_1}{\delta z} d\xi = -g\mu_1 d\xi \qquad p_d - p_a = \frac{\delta p_2}{\delta z} d\xi = -g\mu_2 d\xi$$

The two resulting terms for $p_b - p_a$ are set equal to each other. The inclination of the boundary surface (the angle α between the tangent at the boundary line and the y-axis) is determined by:

$$\tan\alpha = \frac{d\xi}{d\eta} = \frac{2j}{g}\frac{\mu_1 u_1 - \mu_2 u_2}{\mu_1 - \mu_2} \tag{1.2}$$

The values of both pairs μ, u should be set in point a. If density and velocity around each mass are known as analytical functions of y, z, then replacing the coordinates by η, ξ leads to equation (1.2) as the differential equation for the boundary line. It is analogue to the equation that Helmholtz derived for rotating axial rings, and holds true for air as well as for incompressible liquids because the assumed movement meets the continuity equation. Its integral is $p_1 = p_2$.

For air with a uniform composition, the equation can be expressed as follows, if T refers to the absolute temperature:

$$\tan\alpha = \frac{2j}{g}\frac{T_2 u_1 - T_1 u_2}{T_2 - T_1} \tag{1.2a}$$

In order to get sense of the size of the inclination angle, we assume:

- $g = 9.8\,\mathrm{m\,sec^{-2}}$, $j = 0.00007292\,\mathrm{sec^{-1}}$ (like at the Earth's pole),
  ($\frac{g}{2j}$ corresponds to a velocity of $67000\,\mathrm{m\,sec^{-1}}$)

- $T_1 = 273\,°$, $T_2 = 283°$, $u_1 = 10\,\mathrm{m\,sec^{-1}}$, $u_2 = 0$

It follows that $\alpha = \mathrm{arc}\,0°\,14'\,29''$.

With a constant α, the boundary surface would rise by 1000 m at a distance of 237 km. (For comparison, the inclination of a surface of constant pressure with a horizontal gradient of 1 mm Hg[E9] is $0°\,0'\,19''$ on the ground, and this surface rises by 10.5 m at a distance of 111 km.)

The boundary surface becomes steeper if the difference between the velocities increases or if the difference between the temperatures decreases.

We allow ourselves to apply the relations derived from a simple rotating system to the atmosphere at all latitudes provisionally, without considering the influence of the vertical components of the deviating force. The angle of inclination changes with the latitude in the same way as its sine, all other things being equal.

The state is stable if the colder mass lies within the acute angle of the wedge between the boundary surface and horizon. If the interface between the boundary surface and the standard surface falls within a parallel of latitude or a meridian of longitude, then the following scheme results:

Assuming that the warmer side is windless:

The cold mass [lies]	poleward	towards the equator	westwards	eastwards
	with easterly wind	with westerly wind	with wind from the pole	with wind from the equator

According to equation (1.2), the only relevant factor is the difference of the horizontal transport of air parallel to the discontinuity surface $(\mu_1 u_1 - \mu_2 u_2)$ on both sides of the surface. It is therefore possible that there is easterly wind with lesser transport on the warm side, or westerly wind on both sides, though in the latter case it would be weaker on the cold side.

Besides the wind that is required to maintain the inclined boundary surface, there may be other components of air movement; only those that are perpendicular to the boundary surface are excluded.

Calculations regarding the ideal case of frictionless stationary movement are of relevance to us only if they lead to specific ideas about the relationships between observed phenomena. The Helmholtz boundary example demonstrates that air masses with different temperatures may coexist on the same level in a rotating system, that is, the potentially colder mass below and the warmer mass above do not expand.

Where there are horizontal temperature differences in extensive air masses, there also exist horizontal pressure differences. These set the air into motion, and kinetic energy increases until a dynamic balance is achieved. Now the state may remain almost stationary if friction loss is replaced, or, alternatively, temperature and pressure differences on the same level may slowly decrease as the velocity and therefore the inclination of the boundary surface become smaller. In this case, kinetic energy is fuelled at the cost of potential energy. However, the kinetic energy is consumed to a greater degree by friction, as well as by another process: the mixing of air from different altitudes dampens the movement – we will come back to this.

Figure 2: [E6]Temperature in Vienna and Bratislava in °C on Dec 3 1898.

Even if the inclination of the boundary surface is a mere fraction of a degree, the potential energy of the system may be great, with temperature differences of 10°, reaching altitudes of some kilometres. This may suffice to generate a storm. However, there is no reason to expect an increase in kinetic energy if a nearly stationary state has been reached.

Similar considerations apply if there is a continuous transition of the temperature instead of a sharp boundary. If the mass is far from a dynamic balance, then the wind will increase until the damping effects come into force. If the air mass enters into a state that is comparable to a stationary state with frictionless movement, then it will become calm if there is no supply of external energy.[E10]

There are phenomena in relatively small areas that may be easier to understand by using the Helmholtz boundary. The following example may serve as an illustration:

Fig. 2[E6] stems from a former publication (temperature levels in Lower Austria [Temperaturstufen in Niederösterreich], Jahrbuch der C.A. für Meteorologie [ZAMG Yearbook] 1899, Vienna 1900).[E11]

On December 3rd, 1898, the temperature in Vienna is continuously about 7 to 8°, warmer than in Bratislava [Pressburg] for 12 hours; the distance between the two locations is 60 km, the temperature is nearly stationary between 2 p.m. and 10 p.m. The boundary surface, if we assume that there is one, approaches Vienna towards the evening; only 30 km eastwards in Orth and Siebenbrunn, the lower temperature may be measured. At 10 a.m., the rapid warming reaches Vienna with a westerly wind; the westerly wind continues with a velocity of 30 km per hour. The air that passes Vienna would have to reach Bratislava within two hours. However, the continuous temperature difference shows that it does not arrive there. The warm current rises near Vienna and blows over a colder area. According to the observations, the boundary surface may incidentally cut the surface meridionally. The shift takes place so slowly that the state can be assumed to be nearly stationary. In order to maintain the surface with calm conditions on the western side there would have to be southerly wind in the cold area. The lively westerly wind in Vienna does not contribute to this if it remains constant in the vicinity of the surface. One expects wind with a southerly component in Bratislava.

Here, at 2 p.m., there is SW wind of force 4.[E12] At 9 p.m., the SW wind of has a force of 2 since the boundary surface is closer to Vienna. The cold area on the ground extends southwest of Bratislava.

We do not consider the qualitative confirmation to be very important; it is rather about acquiring a certain understanding of how cold and warm air masses are delimited at higher altitudes. This understanding may be checked by using balloon and kite techniques, and, if necessary, be corrected or complemented. Similar cases of continuous temperature differences are not uncommon.

2 Constant horizontal temperature gradient: its relationship with the vertical change of wind velocity

Whereas sudden temperature changes can be found occasionally, steady temperature gradients in a layer can be found anywhere. In such a system, air masses also tend to position themselves in horizontal isothermal layers. An uneven heat supply and already existing movement can hinder them from doing this.

We assume again that the state remains stationary with frictionless motion. Furthermore, we assume, to simplify the calculations, linear motion and u, p, T to be functions of y, z only, in the rotating system introduced previously.

In order to find relationships between these variables, we take a pair of equations (1.1), now without an index; by replacing μ^{-1} by RTp. This takes the following form:

$$RT\frac{\delta lgp}{\delta y} = 2ju \quad RT\frac{\delta lgp}{\delta z} = -g \tag{2.1}$$

They are able to exist only if

$$2j\frac{\delta}{\delta z}\left(\frac{u}{RT}\right) + g\frac{\delta}{\delta y}\left(\frac{1}{RT}\right) = 0$$

For constant R, that is, for dry air or for air with the same vapour content throughout, the last equation becomes

$$\frac{\delta u}{\delta z} = u\frac{\delta lgT}{\delta z} + \frac{g}{2j}\frac{\delta lgT}{\delta y} \tag{2.2}$$

(If vapour content is also a function of y, z, then it is $RT = R_\alpha\theta$, with R_α being the gas constant of dry air and θ being the variable that Guldberg and Mohn[E13] called virtual temperature. In (2.2), θ replaces T.)

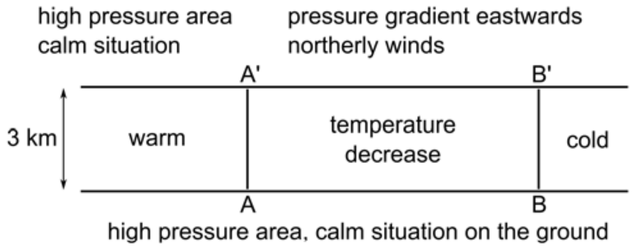

Figure 3: [E7]Idealised cross-section through an air mass with a vertically constant horizontal temperature gradient.

The equation reflects a relation between the vertical change of the velocity and the two temperature gradients perpendicular to the movement.

If the temperature was constant at any level, then the ratio u/T would remain the same at all levels.

The horizontal temperature gradient has a significant effect on the vertical velocity gradient, since $g/2j$ is several thousand times larger than u in the atmosphere.

The relation is easy to recognize if one considers the influence of the temperature on the change in the pressure gradient. If the frictionless stationary movement is eastwards, there is a poleward pressure fall. If the temperature decreases poleward at any level, then the pressure gradient increases with height. Accordingly, in agreement with equation (2.1), the velocity of the westerly wind also increases.

One gets the following scheme:

In an area where the temperature rises	with stationary conditions with increasing height
towards the equator	westerly wind increases, easterly wind decreases
Polewards	westerly wind decreases, easterly wind increases
Westwards	equatorward wind increases, poleward wind decreases
Eastwards	equatorward wind decreases, poleward wind increases

With $T = 273°$ and a temperature gradient of $1°$ per 100 km horizontal distance, the change in velocity is 2.5 m/sec for 1 km of height, if j has the same value as at the pole. The change is 3.5 m/sec with j at 45° latitude.[E14]

At the eastern border of a stationary Hann high pressure area[E15], one usually finds a transition from a warm, almost stationary air mass to a cold air mass that continuously approaches from the north. Let us assume there is steady temperature decrease eastwards above the horizontal line AB (Fig. 3[E6]) at all levels. The area of equal pressure extends on the ground until B. However, at an elevation of 3000 m, it extends only until A', which is located further west. The pressure gradient and wind increase with height. If you assume AB to be 500 km and the temperature drop between A and B and between A' and B' to be 10°, then the increase in wind velocity at 45° latitude at 3 km altitude is $2 \times 3 \times 3.5$ m/sec

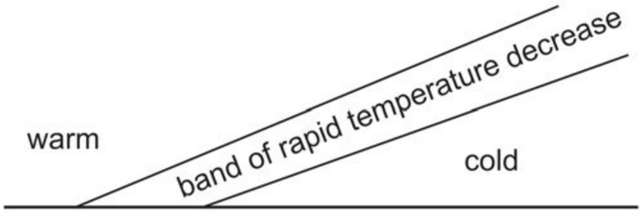

Figure 4: [E7]Idealised cross-section through an air mass with a wedge-shaped horizontal temperature gradient.

with a horizontal temperature gradient of 2° per 100 km. Therefore, with a calm situation on the ground, the northerly wind is 20 m/sec in A' and B'. The pressure difference between A' and B' amounts to between 7 and 8 mm Hg.

If the horizontal temperature gradient is 1° per 10 km, then the change in velocity at the pole and at 45° latitude is already 25 and 35 m/sec, respectively, for every 1 km height. In a vertical band of 4000 m height, the difference in velocity would increase to 100 and 140 m/sec, respectively.

The situation is different if the band with a large temperature fall is inclined; the more it is inclined towards the standard surface, the shorter is the vertical stretch in which velocity changes rapidly (Fig. 4[E6]). Such a band has existence conditions similar to the discontinuity surface of HELMHOLTZ (one easily gets equation (1.2) from equation (2.2) if the temperature difference at both sides of the band is left unchanged and its width is reduced to zero). It may reach high altitudes with rapid temperature decrease everywhere and moderate wind at both sides. In most cases, the horizontal temperature gradient will be unequal at different altitudes and the band will not be sharply delimited.

Analytical remarks regarding 1 and 2[E16]

The equations of motion in the rotating system described above with u, v, w as velocity components, without friction, are:

$$\frac{-1}{\mu}\frac{\delta p}{\delta x} = \frac{du}{dt} + 2jv,$$
$$\frac{-1}{\mu}\frac{\delta p}{\delta y} = \frac{dv}{dt} - 2ju,$$
$$-g - \frac{1}{\mu}\frac{\delta p}{\delta z} = \frac{dw}{dt}$$

We have used the particulate solution $u = f(y,z)$, $v = 0$, $w = 0$ and initially derived equation (1.2) for the discontinuity surface.

With the equation of state for ideal gases, the equations take the form (2.1), resulting in the steady temperature distribution (2.2). Using (2.2) as the differential equation for u, the integral contains a function of y that can be determined if pressure on the ground p_{y0} is given. One obtains:

$$u = \frac{RT}{2j}\frac{\delta lg p_{y_0}}{\delta y} + \frac{g}{2j}T\int_0^z \frac{1}{T^2}\frac{\delta T}{\delta y}dz$$

For stationary movements in coaxial circular paths, similar relationships can be deduced. For the special case that there is only velocity u in direction ρdz and with the cylindrical coordinates $\varepsilon_1 \rho_1 z$ $\{\varepsilon = \arctan(x/y)$ increases with the rotation, $\rho = \sqrt{x^2+y^2}\}$, the equations of motion become:

$$\frac{1}{\mu}\frac{\delta p}{\delta \rho} = 2ju + \frac{u^2}{\rho}, \quad \frac{1}{\mu}\frac{\delta p}{\delta z} = -g$$

The deviating force in the direction of the beam is added to the centrifugal force of the relative movement. If there is a discontinuity surface (ρ, z), the following equation is true, formed from (1.2) by replacing $u_1 u_2$ by:

$$u_1 + \frac{u_1^2}{2j\rho}, \quad u_2 + \frac{u_2^2}{2j\rho}$$

The above equations are true for the vicinity of the pole with the limitation that Earth rotation is not taken into consideration. The same level of accuracy is attained by using a linear coordinate system for another point on the Earth, where the z-axis points towards the zenith, the x-axis eastwards, the y-axis southwards, and the motion in the direction of the x-axis is called westerly wind.

$$\left.\begin{aligned} \frac{-1}{\mu}\frac{\delta p}{\delta x} &= \frac{du}{dt} + 2jv + 2j'w \\ \frac{\delta p}{\delta y} &= \frac{dv}{dt} - 2ju \\ -g - \frac{1}{\mu}\frac{\delta p}{\delta z} &= \frac{dw}{dt} - 2j'u \end{aligned}\right\} \begin{aligned} &v \text{ rotation velocity of the earth} \\ &\vartheta \text{ distance to the pole} \\ &j = v\cos\vartheta, \\ &j' = v\sin\vartheta \end{aligned}$$

In the case of purely meridional movement $\{u = 0, v = f(x,z), w = 0\}$, all remains the same. In the case of west to east movement $\{u = f(y,z), v = 0, w = 0\}$, the vertical component of the deviating force is added. It is a very small effect that HELMHOLTZ describes as follows: the air in westerly wind is lighter, and it is heavier than stationary air with the same density in easterly wind. The equations can be reduced to:

$$\frac{1}{\mu}\frac{\delta p}{\delta y} = 2ju, \quad \frac{1}{\mu}\frac{\delta p}{\delta z} = -g + 2j'u$$

The following inclination of the boundary surface results from this:

$$\tan\alpha = \frac{d\xi}{d\eta} = \frac{2j(\mu_1 u_1 - \mu_2 u_2)}{g(\mu_1 - \mu_2) - 2j(\mu_1 u_1 - \mu_2 u_2)} \qquad (1.2^*)$$

α is the angle between the meridional cut through the surface and the meridian southwards. This equation may be distinguished from (1.2) only by the additional term in the denominator that is not of importance if there is a significant difference in density. If it disappears and only the sudden transition of the velocity remains, then one obtains $\tan\alpha = -\cot\vartheta$ from (1.2*); the dividing line is parallel to the Earth axis. (In this case, one would get a false result from (1.2): vertical boundary surface.) Where there is no significant difference in density, the influence of the vertical component of the deviating force comes into effect.

So far, the Earth curvature has been set to zero. In order to have a complete calculation for at least the case of stationary zonal movement, one starts from the equations of motion for rotating spherical systems. {r = distance from the centre of the Earth, ϑ = distance from the pole southwards, u = positive eastward zonal velocity, j, j' as above and abbreviated

$$U = u + \frac{u^2}{2j'r}\Bigg\}$$

For the special case, the equations may be reduced to

$$\frac{1}{\mu}\frac{\delta p}{r\delta\vartheta} = 2jU, \qquad \frac{1}{\mu}\frac{\delta p}{\delta r} = -g + 2j'U$$

Consequently, one finds an expression for the inclination of the boundary surface which differs from (1.2*) only in U replacing u. The position of the surface determined in this way is identical with that deduced by Helmholtz.

3 Mixing of air masses

We return to Fig. 2 and look only at the thermogram of Vienna. How does it occur that, at 10 a.m., the temperature increases by 8° within a few minutes? From the study of a rather large number of similar cases, the following picture has emerged. A warm air stream flows a few hectometres above the ground; it is separated from the lower lying cold mass by a discontinuity surface or a band of rapid transition. The upper stream, which is warmer in absolute terms at the boundary and potentially warmer within the entire mass, has a greater vertical extension and a higher velocity than the cold layer. It absorbs the cold air slowly, approaches the ground, and finally reaches it. At this moment, the anemograph records the beginning of stronger wind after calm conditions or a change in the wind direction; the thermograph records a strong temperature increase. Therefore, the change does not show the warming of an air mass, but rather the removal of the last bit of a cold layer at that location.

Here, mixing means the transition into a state of weaker vertical stability. One would like to understand the mechanical process. This idea is provided by Helmholtz's studies on wave formation at the boundaries of layers of unequal temperature, and in particular by his remarks about breaking waves[E17].

I only repeat these considerations, which have already been published several years ago[E18], because I would like to add some remarks about the consumption of kinetic energy.

Although the turbulence of air movements increases the effect of internal friction, it can hardly be assumed that, at higher elevations, much of the constantly produced kinetic energy is used up, as has already been mentioned elsewhere. At that point, I believed that damping occurred largely due to the transfer of energy to water in the oceans or to towering objects on land, where the energy quickly disappears due to friction. Stronger damping could take place due to movement countering the horizontal pressure drop or gravity. The former, however, would only give rise to oscillations, and would not result in the permanent loss of active force. Regarding the latter, we have just given an example. The absorption of the cold lower layer by the warm stream represents an air transport against gravity; therefore, kinetic energy must be used to produce the mixture.

A similar situation occurs not only in the case of phenomena that are as striking as sudden warming on the ground. Temperature inversion is not unusual with calm conditions; if the upper layer is windy, then the colder layer becomes more shallow, which goes unnoticed on the ground. Mixing may occur even more often with continuous vertical temperature decrease; it becomes more pronounced and the condition approaches the vertically indifferent equilibrium. Thereby, active force is absorbed.

In the theory named after Espy and Köppen[E19], the difference in the diurnal cycle of wind velocity in air layers close to the ground is presented as a result of nocturnal cooling and daily warming of the bottom air layer. The former makes the vertical stratification more stable. The latter could have the effect that the air rises from the ground in thin threads and that the stratification becomes increasingly indifferent in the lower air mass; in other words, the work required for mixing with the upper stream becomes smaller and the mixing happens more easily. This only depends on the transfer of momentum between the different layers, and not on the loss of kinetic energy.

With high wind velocity at higher elevations, lower layers of cold air several hectometres thick are removed at night (and during days with dense cloud cover); sudden and very quick warming on the ground can then be observed without the influence of insolation.

I believe that stepwise mixing is an important cause for damping stormy air movements, and one that is not inferior to the energy loss on the ground. Movements that do not reach the ground could be eliminated only by mixing.

Work that has to be performed for mixing stratified fluids

In a cylindrical container with a bottom area B there is a fluid with a density = μ_1 and a layer height = h_1, and above it a fluid with a density = μ_2 and a height = h_2; both are at rest. They are non-compressible and mixable.

They would gradually mix anyway due to diffusion, but very slowly. If one wants to achieve complete mixing, one has to expend work. Its amount is:

$$\frac{1}{2}Bh_1h_2g(\mu_1 - \mu_2)$$

If one takes

$$h_1 = h_2 = h \text{ and } \mu_1 = \mu\left(1 + \frac{\sigma}{2}\right),\ \mu_2 = \left(1 - \frac{\sigma}{2}\right),$$

work per unit becomes $\frac{1}{4}gh\sigma$, equivalent to the active force of mass 1. Its velocity is:

$$V = \sqrt{\frac{gh\sigma}{2}}$$

In similar cases, there is an extensive analogy between fluid columns of the same density and gas columns with constant entropy[1]. We calculate the mixing of air masses, each of which is in neutral equilibrium. The upper has higher entropy, and both have the same height h. If we perform this slightly longer calculation, then the temperatures at the boundary are:

$$T_1 = T\left(1 - \frac{\tau}{2}\right), \quad T_2 = T\left(1 + \frac{\tau}{2}\right),$$

(τ is a small fraction)

For a complete mixing of the masses, where stratification is replaced by neutral equilibrium, an amount of work has to be done for each mass unit equivalent to the active force with the velocity

$$V = \sqrt{\frac{gh\tau}{2}}$$

With $h = 500\,\text{m}$, $T_1 = 273°$, and $T_2 = 279°$, V becomes $7.3\,\text{m sec}^{-1}$. If the mixing takes place due to movement alone, any kilogram of the mixture will use $26.6\,\text{kg} \cdot \text{m}^2\,\text{sec}^{-2}$ of the stock of kinetic energy.

We return to incompressible liquids; two masses of the same volume (height = $2h$) are located next to each other in a trough, separated by a vertical wall. The wall is removed and we initially assume that the masses tip over without mixing. The centre of gravity of the system sinks and the active force $\frac{1}{4}gh\sigma$ is released for any unit mass. If the masses had initially been divided into small layers and then been mixed (which does not require any work), then the centre of gravity would have remained unchanged.

If, in the first case, the liquids partly mix while they are tipping over, they reduce the available active force. If they could mix entirely after tipping over (with no significant effect of slow diffusion), then they would have to use all of their remaining kinetic energy.

However, if one had a mixture of stepwise or continuously different density in each of the chambers in stable equilibrium and in a resting state, then the available active force would not suffice to achieve a thorough mixing after removing the wall.

The situation is similar for extended air masses with different entropy at the same height. Wind results if the potentially warmer air masses rise and the potentially colder air masses sink. (For the present, we neglect the less clear processes for air containing water vapour.) If the initial stratification was vertically stable, then the increase in kinetic energy must be smaller than the work required for complete mixing, that is, for achieving a neutral state in the entire mass. Loss due to friction is not considered.

Taken together, streams and mixing lead to a vertically stable stratification. The required active force used for mixing turns into potential and internal energy. Movement could once again result from the increase in these forms of energy; however, on no account could the same amount of kinetic energy, as was absorbed by the mixture, result. (This will be proved at a later point.) A damping effect of the mixing remains.

4 Temperature stratification in stationary air with different vapour content

The conditions for stable equilibrium for air columns with a uniform composition and for columns with vapour saturated air have been given by Lord KELVIN[E21]. They can be derived in the same way for dry air, the vapour content of which is a function of height.

If R_α is the gas constant of dry air, p is air pressure, and p_β is the partial pressure of the vapour, then the gas constant of the mixture is

$$R = \frac{R_a}{1 - 0.3767p_\beta/p} = 287.026 + 109\frac{p_\beta}{p}\left[\frac{m^2}{sec^2\,°\text{C}}\right],$$

therefore, for air with 0, 1, 2 volume percent of vapour

p_β/p	0	0.02	0.02
R	287.026	288.111	289.205

Discontinuity surfaces

If the standard surface forms the border between two layers with a sudden jump in temperature and vapour content, then we assign index 1 to the lower mass and index 2 to the upper mass.

A stable equilibrium exists if the layer with greater density is at the bottom, accordingly, since pressure is the same for both layers at the border, if

$$R_1T_1 < R_2T_2 \tag{4.1}$$

or by introducing the virtual temperature (see above), if $\theta_1 < \theta_2$. If the upper layer contains more vapour, then the layer underneath may be warmer, $R_2 > R_1$, $T_1 > T_2$.[E22]

[1] Energie der Stürme (Energy of storms), Jahrbuch C.A. f. Meteor. (Yearbook of Meteorology) 1903 (Vienna, 1905), Appendix, p. 20.[E20]

Example: Nearly saturated air at 10 °C lies over warmer dry air.

$$T_2 = 283°, \ p_\beta = 9 \,\text{mm Hg}, \ p = 720 \,\text{mm Hg},$$
$$R_2 = R_a \left(1 + 0.3767 \frac{9}{720}\right)$$

With $R_1 = R_\alpha$, $T_1 = 284.3°$ is the maximum permissible value.

If the layer containing more vapour lies below, then a temperature jump at the border is required for stable equilibrium; $R_2 < R_1$, $T_1 < T_2$.

Balloon flights have shown[2] that the upper boundary of a cloud layer is often a discontinuity surface of the temperature. The air above the cloud is warmer. Rising air, where the vapour partly condenses, comes to a halt only below a layer whose density is smaller than the density of the rising mass at the same elevation. If that layer is relatively dry, then there must be a temperature jump between the upper dry layer and the vapour-saturated layer that expands below.

If the density of the two layers only converges at the border, then there is an additional condition. Close to the border of the two layers, the vertical temperature gradient in the upper dry mass may not be larger than that of neutral equilibrium in saturated air of the temperature in the layer below, otherwise a particle rising above the layer would become lighter than the surrounding air and would have to continue to rise due to uplift.[E22]

As an example for the temperature jump at the upper border of vapour rich air, we can use the same numbers that were previously true for the lower border; we only have to exchange the indices.

$$T_1 = 283\,°, \ p_\beta = 9 \,\text{mm Hg}, p = 720 \,\text{mm Hg}.$$

With $R_2 = R_\alpha$, the smallest permissible value for T_2 is 2.3°. If the upper layer has a relative humidity of 50 percent, then the temperature jump is at least 0.6°.

Continuous distribution of vapour content in stationary air

We determine the condition to be indifferent if an air particle is adiabatically moved from elevation z (pressure p) to another layer z' (p') where it arrives with a density identical to that of the air mass in z'. The particle is not affected by any uplift forces.

We assume a dry air column in vertical equilibrium. This condition results in

$$\frac{1}{p}\frac{dp}{dz} = \frac{-g}{RT}$$

If the air particle is adiabatically moved from z to $z' = z + dz$, then its temperature changes from T to T'. Afterwards, if the gas constant R and the specific heat with constant pressure C_p for the composition of air at elevation z hold true, one acquires the following from the condition for adiabatic change of the particle

$$\frac{T'}{T} = \left(1 - \frac{g}{RT}dz\right)^{R/C_p} = 1 - \frac{g}{C_pT}dz$$

The density of the particle at its new position is

$$\frac{p'}{RT'} = \frac{p'}{RT}\left(1 + \frac{g}{C_pT}dz\right)$$

Since RT have been introduced as continuous functions of elevation, the density of its surrounding is

$$\frac{p'}{RT}\left(1 - \frac{1}{R}\frac{dR}{dz}dz - \frac{1}{T}\frac{dT}{dz}dz\right)$$

Both densities are required to be equal, thus

$$\frac{-dT}{dz} = \frac{g}{C_p} + \frac{T}{R}\frac{dR}{dz} \qquad (4.2)$$

A smaller vertical temperature decrease than the one determined here requires a conditionally stable equilibrium.

Equation (4.2) differentiates itself from the equation that is true for neutral equilibrium of an air mass with constant composition through its last element.

By introducing the virtual temperature θ, it becomes

$$\frac{-d\theta}{dz} = \frac{g}{C_p}\frac{\theta}{T}$$

The vertical gradient of θ is almost identical to that of dry air. The vertical gradient of the observed temperature may be much larger or much smaller, depending on whether the vapour content rapidly increases or decreases with height.

The first element on the right side of equation (4.2) has the value 0.01 °C/m. The second element may have an almost identical absolute value if the vapour content changes by 1 percent per volume per 100 metres of height. This is extraordinarily fast, since it may only happen at the boundaries of clouds with temperatures of about 20 °C. Then neutral equilibrium persists with a gradient of 2°/100 m with upwards increasing vapour content, with upward decreasing vapour content and isothermal conditions, or even with a small constant temperature inversion.

Saturated air at an air pressure of 700 mm Hg contains

at	30°	20°	10°	0 °C
	4.5	2.5	1.3	0.65 percent per volume of vapour.

If one assumes that the vapour content decreases downwards by 1 percent per volume per 100 m, then, with a temperature gradient of almost 2°/100 m, the intervals during which a neutral state is still possible, are:

450	250	130	65 m.

[2]Berson, Wissenschaftliche Luftfahrten (Scientific Aviation) 3, 123.[E23]

I do not know whether such a rapid continuous change of the vapour content does actually occur. If vertical temperature gradients of 1.2 or 1.3°/100 m are indicated from balloon flights, then it should be investigated whether this is related to the upward increasing vapour content. The extraordinarily large gradients for intervals of some kilometres, which are occasionally derived from unmanned balloons, appear unlikely.

In the following overview, the cloud represents the vapour rich layer.

Below the cloud:

With a sudden transition of the vapour content, the temperature below the border may be larger by a limited amount than the temperature just above it.

If the vapour content steadily decreases downwards, then the temperature gradient for neutral equilibrium is larger than 1°/100 m.

Above the cloud:

With a discontinuous transition of the vapour content, there must be a temperature jump and the air above the border must be warmer by a limited amount.

If the vapour content decreases steadily upwards, then the gradient for neutral equilibrium is smaller than 1°/100 m.

Vienna, November 1905.

Endnotes

E1 Title Page: Meteorologische Zeitschrift, Hann Volume, to the 40th editing anniversary of J. Hann. Dedicated by friends and colleagues. Revised by Dr. J.M. Pernter, Vienna, and Dr. G. Hellmann, Berlin. With picture and facsimile by J. Hann. 76 text figures and 5 tables, Braunschweig, Publishing and printing: Friedrich Vieweg und Sohn, 1906.

E2 Hann, J., 1890: Das Luftdruckmaximum vom November 1889 in Mitteleuropa, nebst Bemerkungen über die Barometermaxima im Allgemeinen. – Anzeiger der Kaiserlichen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Classe, **9**, 73–77.

E3 Temperature are given here as in the original paper, with a degree symbol ° for temperatures in K and °C for temperatures in degrees Celsius.

E4 Helmholtz‘ work was published in two parts:

Helmholtz. H., 1888a: Über atmosphärische Bewegungen (I). – Sitzber. Preuss. Akad. Wiss. **5**, 647–663

Helmholtz. H., 1888b: Über atmosphärische Bewegungen (II). – Meteorol. Z. **5**, 329–340.

E5 The concepts of geostrophic wind and thermal wind go back to the work of William Ferrel (1856) and Christoph Buys-Ballot (1857).

Ferrel, W., 1856: An essay on the winds and the currents of the ocean. – Nashville H. Med. Surg., **11**, 287–301.

Buys-Ballot, C., 1857: Note sur le rapport de l'intensité et de la direction du vent avec les écarts simultanés du baromètre. – Comptes rendus de l'Académie des Sciences, **45**, 765–768.

E6 Figures are numbered consecutively throughout the Hann-volume (starting with Fig. 57 in Margules' article). In this translation we renumbered the figures. All figures are redrawn.

E7 Original figure has no caption.

E8 On the notation see accompanying paper:

Steinacker, R., S. Brönnimann, 2016: Stationary flow near fronts. – Meteorol. Z. (this issue).

E9 Millimeters of mercury: 1 mm Hg = 1.3332 hPa.

E10 The text from here onward to the end of the section appeared in small print in the original publication.

E11 Margules, M., 1900: Temperaturstufen in Niederösterreich im Winter 1898/99. Anhang zum Jahrbuch der Central-Anstalt für Meteorologie 1899, **26**, Wien.

E12 Wind force was given in Beaufort.

E13 Guldberg, C.M., H. Mohn, 1876: Études sur les mouvements de l'atmosphère. – Christiania (publisher unknown), 99 pp.

Note that Margules uses the symbol θ, which today is mostly used for potential temperature.

E14 In the original text the following paragraph is typed in small print.

E15 Possibly Margules is referring to stationary high-pressure systems with a deep warm layer in their core, such as described in the following publication:

Hann, J., 1890: Das Luftdruckmaximum vom November 1889 in Mitteleuropa, nebst Bemerkungen über die Barometermaxima im Allgemeinen. – Anzeiger der Kaiserlichen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Classe, **9**, 73–77.

E16 In the original text this section is typed in small print.

E17 Helmholtz, H., 1868: Über discontinuierliche Flüssigkeitsbewegungen. – Monatsber. Königl. Preuss. Akad. Wiss. Berlin, **23**, 215–228.

E18 Margules, M., 1901: Über den Arbeitswert einer Luftdruckverteilung und die Erhaltung der Druckunterschiede. – Denkschriften der kaiserlichen Akademie der Wissenschaften, math.-nat. Kl., **73**, 329–345.

E19 ESPY, J.P., 1841: Philosophy of storms. – Charles C. Little and James Brown, Boston, 552 pp.

KÖPPEN, 1879: Die tägliche Periode der Geschwindigkeit und der Richtung des Windes. – Zeitschr. Österr. Ges. Meteorol. **14**, 333–349.

CLEVELAND ABBE wrote in 1902: "It is a significant proof of European ignorance of America, that ESPY's conclusions on the latter subject were independently deduced thirty years later by KÖPPEN, so that Germans are now willing to speak of the ESPY-KÖPPEN explanation, although, according to the established rules of ethics, the credit belongs to ESPY alone."

ABBE, C., 1902: Meteorology and the Position of Science in America. – The North American Review, **174**, No. 547, 833–844.

E20 MARGULES, M., 1905: Über die Energie der Stürme. – Jahrbuch der k. k. Centralanstalt für Meteorol. und Erdmagnetismus in Wien, NF, Bd. **42**.

E21 THOMSON, W. 1848: On an Absolute Thermometric Scale Founded on Carnot's Theory of the Motive Power of Heat. – Phil. Mag. **33**, 100–106.

E22 In the original text the following example is typed in small print.

E23 ASSMANN, R., A. BERSON, 1900: Wissenschaftliche Luftfahrten, ausgeführt vom Deutschen Verein zur Förderung der Luftschiffahrt in Berlin. Vol 3. Zusammenfassungen und Hauptergebnisse. – Braunschweig, Vieweg.