

Normal approximation on Poisson spaces: Mehler’s formula, second order Poincaré inequalities and stabilization

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Abstract We prove a new class of inequalities, yielding bounds for the normal approximation in the Wasserstein and the Kolmogorov distance of functionals of a general Poisson process (Poisson random measure). Our approach is based on an iteration of the classical Poincaré inequality, as well as on the use of Malliavin operators, of Stein’s method, and of an (integrated) Mehler’s formula, providing a representation of the Ornstein-Uhlenbeck semigroup in terms of thinned Poisson processes. Our estimates only involve first and second order difference operators, and have consequently a clear geometric interpretation. In particular we will show that our results are perfectly tailored to deal with the normal approximation of geometric functionals displaying a weak form of stabilization, and with non-linear functionals of Poisson shot-noise processes. We discuss two examples of stabilizing functionals in great detail: (i) the edge length of the k -nearest neighbour graph, (ii) intrinsic volumes of k -faces of Voronoi tessellations. In all these examples we obtain rates of convergence (in the Kolmogorov and the Wasserstein distance) that one can reasonably conjecture to be

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optimal, thus significantly improving previous findings in the literature. As a necessary step in our analysis, we also derive new lower bounds for variances of Poisson functionals.

Keywords Central limit theorem · Chaos expansion · Kolmogorov distance · Malliavin calculus · Mehler's formula · Nearest neighbour graph · Poincaré inequality · Poisson process · Spatial Ornstein-Uhlenbeck process · Stabilization · Stein's method · Stochastic geometry · Voronoi tessellation · Wasserstein distance

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1 Introduction

1.1 Overview and motivation

We consider a Poisson process (Poisson random measure) η on a measurable space $(\mathbb{X}, \mathcal{X})$, with σ -finite intensity measure λ , see [21]. Let $F = f(\eta)$ be a measurable function of η . The Poincaré inequality (see [27, 57] and the references therein) states that the variance of a square-integrable Poisson functional F can be bounded as

$$\mathrm{Var} F \leq \mathbb{E} \int (D_x F)^2 \lambda(dx), \quad (1.1)$$

where the difference operator $D_x F$ is defined as $D_x F := f(\eta + \delta_x) - f(\eta)$. Here, $\eta + \delta_x$ is the configuration arising by adding to η a point at $x \in \mathbb{X}$. Consequently, a small expectation of $\|DF\|^2$ leads to small fluctuations of F , where DF is a shorthand notation for the mapping (discrete gradient) $x \mapsto D_x F$ (also depending on η)

and where $\|v\| := (\int v^2 d\lambda)^{1/2}$ denotes the $L^2(\lambda)$ -norm of a measurable function $v : \mathbb{X} \rightarrow \mathbb{R}$. The principal aim of this paper is to combine estimates of the type (1.1) with a suitable version of Stein’s method (see e.g. [33]), in order to establish explicit bounds on the normal approximation of a general functional of the type $F = f(\eta)$. To do so, we will partially follow the route pioneered in references [10,34], in the framework of the normal approximation of functionals of Gaussian fields.

Indeed, the estimate (1.1) can be regarded as the Poisson space counterpart of the famous Chernoff-Nash-Poincaré inequality of Gaussian analysis (see [12,32]), stating that, if $X = (X_1, \dots, X_d)$ is an i.i.d. standard Gaussian vector and f is a smooth mapping on \mathbb{R}^d , then

$$\text{Var } f(X) \leq \mathbb{E}\|\nabla f(X)\|^2. \tag{1.2}$$

Motivated by problems in random matrix theory, Chatterjee [10] has extended (1.2) to a second order Poincaré inequality, by proving the following bound: if f is twice differentiable, then (for a suitable constant C uniquely depending on the variance of $f(X)$)

$$d_{TV}(f(X), N) \leq C \mathbb{E} \left[\|\text{Hess } f(X)\|_{op}^4 \right]^{1/4} \times \mathbb{E} \left[\|\nabla f(X)\|^4 \right]^{1/4}, \tag{1.3}$$

where d_{TV} is the total variation distance between the laws of two random variables, $\|\cdot\|_{op}$ stands for the usual operator norm, and N is a Gaussian random variable with the same mean and variance as $f(X)$. The main intuition behind relation (1.3) is the following: if the L^4 norm of $\|\text{Hess } f(X)\|_{op}$ is negligible with respect to that of $\|\nabla f(X)\|$, then f is close to an affine transformation, and therefore the distribution of $f(X)$ must be close to Gaussian. This class of second order results has been further generalized in [34] to the framework of functionals F of an infinite-dimensional isonormal Gaussian process X over a Hilbert space \mathfrak{H} , in which case the estimate (1.3) becomes (with obvious notation)

$$d_{TV}(F, N) \leq C \mathbb{E} \left[\|D^2 F\|_{op}^4 \right]^{1/4} \times \mathbb{E} \left[\|DF\|_{\mathfrak{H}}^4 \right]^{1/4}, \tag{1.4}$$

where D and D^2 stand, respectively, for the first and second Malliavin derivatives associated with X (see also [33, Chapter 5]). We notice immediately that, in general, the estimates (1.3)–(1.4) yield presumably suboptimal rates of convergence, that is: if F_n is a sequence of centred smooth functionals of X such that $\text{Var } F_n =: v_n^2 \rightarrow \infty$ and $\tilde{F}_n = F_n/v_n$ converges to N in distribution, then (1.3)–(1.4) yield an upper bound on the quantity $d_{TV}(\tilde{F}_n, N)$ of the order of $v_n^{-1/2}$, even when the expected rate is of the order of v_n^{-1} ; see for instance the examples discussed in [34, Sect. 6].

In this paper we shall establish and apply a new class of second order Poincaré inequalities, involving general square-integrable functionals of the Poisson process η . The counterparts of the operators ∇ and Hess appearing in (1.3) will be, respectively, the difference operator D , and the second order difference operator D^2 , which acts on a random variable F by generating the symmetric random mapping $D^2 F : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}, (x, y) \mapsto D_{x,y}^2 F := D_y(D_x F)$.

In view of the discrete nature of D and D^2 , our bounds will have a significantly more complex structure than those appearing in (1.3)–(1.4). We will see that our estimates yield the presumably optimal rates of convergence in the normal approximation of non-linear functionals of Poisson shot-noise processes, as well as in two geometric applications displaying a stabilizing nature (see e.g. [45,49]). All these rates have previously been outside the scope of existing techniques. We observe that the problem of formally establishing the optimality of Berry-Esseen bounds in the context of geometric functionals of a Poisson measure is largely open-see [4] for some partial results in this direction.

Our approach relies heavily on the normal approximation results proved in [17,37,50], which are in turn derived from a combination of Stein’s method and Malliavin calculus. However, in order to apply these results as efficiently as possible, we need to establish a general Mehler’s formula for Poisson processes (see [46] for a special case), providing a representation of the inverse Ornstein-Uhlenbeck generator in terms of the thinned Poisson process. The development and application of this formula is arguably one of the crucial contributions of our work.

1.2 Main results

Our main findings will provide upper bounds on the Wasserstein distance and the Kolmogorov distance between the law of a standardized Poisson functional and a that of a standard normal random variable. Here, the Wasserstein distance between the laws of two random variables Y_1, Y_2 is defined as

$$d_W(Y_1, Y_2) = \sup_{h \in \text{Lip}(1)} |\mathbb{E}h(Y_1) - \mathbb{E}h(Y_2)|,$$

where $\text{Lip}(1)$ is the set of all functions $h : \mathbb{R} \rightarrow \mathbb{R}$ with a Lipschitz-constant less than or equal to one. The Kolmogorov distance between the laws of Y_1, Y_2 is given by

$$d_K(Y_1, Y_2) = \sup_{x \in \mathbb{R}} |\mathbb{P}(Y_1 \leq x) - \mathbb{P}(Y_2 \leq x)|.$$

This is the supremum distance between the distribution functions of Y_1 and Y_2 .

Our bound on the Wasserstein distance $d_W(F, N)$ (where F is a Poisson functional with zero mean and unit variance, and N is a standard Gaussian random variable) is stated in the forthcoming Theorem 1.1, and is expressed in terms of the following three parameters, whose definition involves exclusively the random functions DF and D^2F :

$$\begin{aligned} \gamma_1 &:= 2 \left[\int \left[\mathbb{E}(D_{x_1} F)^2 (D_{x_2} F)^2 \right]^{1/2} \left[\mathbb{E}(D_{x_1, x_3}^2 F)^2 (D_{x_2, x_3}^2 F)^2 \right]^{1/2} \lambda^3(d(x_1, x_2, x_3)) \right]^{1/2}, \\ \gamma_2 &:= \left[\int \mathbb{E}(D_{x_1, x_3}^2 F)^2 (D_{x_2, x_3}^2 F)^2 \lambda^3(d(x_1, x_2, x_3)) \right]^{1/2}, \\ \gamma_3 &:= \int \mathbb{E}|D_x F|^3 \lambda(dx). \end{aligned}$$

As is customary, throughout the paper we shall use the following notation: L^2_η indicates the class of all square-integrable functionals of the Poisson measure η ; by $\text{dom } D$ we denote the collection of those $F \in L^2_\eta$ such that

$$\mathbb{E} \int (D_x F)^2 \lambda(dx) < \infty. \tag{1.5}$$

Theorem 1.1 *Let $F \in \text{dom } D$ be such that $\mathbb{E}F = 0$ and $\text{Var } F = 1$, and let N be a standard Gaussian random variable. Then,*

$$d_W(F, N) \leq \gamma_1 + \gamma_2 + \gamma_3.$$

The numbers γ_1 and γ_2 control the size of the fluctuations of the second order difference operator D^2F in a relative and an absolute way. Therefore, a small value of $\gamma_1 + \gamma_2$ indicates that F is close to an element of the first Wiener chaos of η , that is, of the L^2 space generated by the linear functionals of $\hat{\eta} := \eta - \lambda$ (see e.g. [27, 38]). Moreover, a small value of γ_3 heuristically indicates that the projection of F on the first Wiener chaos of η is close in distribution to a Gaussian random variable (see e.g. [37, Corollary 3.4]).

In order to state our bound on the Kolmogorov distance $d_K(F, N)$, we will need the following additional terms (carrying heuristic interpretations similar to those of $\gamma_1, \gamma_2, \gamma_3$):

$$\begin{aligned} \gamma_4 &:= \frac{1}{2} \left[\mathbb{E}F^4 \right]^{1/4} \int \left[\mathbb{E}(D_x F)^4 \right]^{3/4} \lambda(dx), \\ \gamma_5 &:= \left[\int \mathbb{E}(D_x F)^4 \lambda(dx) \right]^{1/2}, \\ \gamma_6 &:= \left[\int 6 \left[\mathbb{E}(D_{x_1} F)^4 \right]^{1/2} \left[\mathbb{E}(D^2_{x_1, x_2} F)^4 \right]^{1/2} + 3 \mathbb{E}(D^2_{x_1, x_2} F)^4 \lambda^2(d(x_1, x_2)) \right]^{1/2}. \end{aligned}$$

Theorem 1.2 *Let $F \in \text{dom } D$ be such that $\mathbb{E}F = 0$ and $\text{Var } F = 1$, and let N be a standard Gaussian random variable. Then,*

$$d_K(F, N) \leq \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6.$$

In view of [40], our approach can be extended so as to yield bounds for the normal approximation of multivariate Poisson functionals, while [35] allows in principle to deal with Poisson approximations in the total variation distance. Details and applications of such extensions will be reported elsewhere.

The following example illustrates how the general bounds in Theorems 1.1 and 1.2 can be evaluated for linear functionals of the underlying Poisson process.

Example 1.3 Consider the simple case where $F = I_1(f) = \int f(x) \hat{\eta}(dx)$ is an element of the first Wiener chaos of η , where the deterministic kernel $f \in L^2(\lambda)$ is such

that $\|f\|_{L^2(\lambda)} = 1$, where we adopt the notation $\|f\|_{L^p(\lambda)} := (\int |f|^p d\lambda)^{1/p}$, $p > 0$. Then, one has that $DF = f$, $D^2F = 0$,

$$\mathbb{E}F^2 = \|f\|_{L^2(\lambda)}^2 = 1 \quad \text{and} \quad \mathbb{E}F^4 = 3 + \|f\|_{L^4(\lambda)}^4,$$

where we have implicitly applied a multiplication formula for Wiener-Itô integrals such as the one stated in [38, Theorem 6.5.1]. In this framework, Theorems 1.1 and 1.2 imply, respectively, that

$$d_W(F, N) \leq \|f\|_{L^3(\lambda)}^3$$

and

$$d_K(F, N) \leq \|f\|_{L^3(\lambda)}^3 \times \left(1 + \frac{3^{1/4}}{2} + \frac{\|f\|_{L^4(\lambda)}}{2}\right) + \|f\|_{L^4(\lambda)}^2.$$

Analogous bounds for the Wasserstein distance can also be inferred from [37, Corollary 3.4], whereas the bounds for the Kolmogorov distance can alternatively be deduced from the main results of [50]. The most straightforward application of these bounds for the normal approximation of first order Wiener-Itô integrals corresponds to the case where $\mathbb{X} = \mathbb{R}_+$, λ equals the Lebesgue measure, and $F = F_t = I_1(f_t)$, with $t > 0$ and $f_t(x) = t^{-1/2}\mathbf{1}\{x \leq t\}$. In this case, one has that F_t is a rescaled centred Poisson random variable with parameter t , and the previous estimates become

$$d_W(F_t, N) \leq \frac{1}{\sqrt{t}} \quad \text{and} \quad d_K(F_t, N) \leq \frac{1}{\sqrt{t}} \times \left(2 + \frac{3^{1/4}}{2} + \frac{1}{2t^{1/4}}\right),$$

yielding rates of convergence (as $t \rightarrow \infty$) that are consistent with the usual Berry-Esseen estimates.

An important and highly non-trivial application of our general bounds concerns the asymptotic analysis of Poisson functionals enjoying some weak form of stabilization (see [43, 49], as well as Sect. 1.3c below). Our main result in this respect is Theorem 6.1, showing how the ‘second order interactions’ associated with a given Poisson functional can be quantified in order to yield explicit bounds in normal approximations, without making any further assumptions on the state space. In order to motivate the reader, we shall now present an important consequence of our Theorem 6.1. For $t \geq 1$, we let η_t be a Poisson process with intensity measure $\lambda_t = t\lambda$, with λ a fixed finite measure on \mathbb{X} .

Proposition 1.4 *Let $F_t \in L^2_{\eta_t}$, $t \geq 1$, and assume there are finite constants $p_1, p_2, c > 0$ such that*

$$\mathbb{E}|D_x F_t|^{4+p_1} \leq c, \quad \lambda\text{-a.e. } x \in \mathbb{X}, \quad t \geq 1, \tag{1.6}$$

and

$$\mathbb{E}|D^2_{x_1, x_2} F_t|^{4+p_2} \leq c, \quad \lambda^2\text{-a.e. } (x_1, x_2) \in \mathbb{X}^2, \quad t \geq 1. \tag{1.7}$$

Moreover, assume that $\text{Var } F_t/t > v, t \geq 1$, with $v > 0$ and that

$$m := \sup_{x \in \mathbb{X}, t \geq 1} \int \mathbb{P} \left(D_{x,y}^2 F_t \neq 0 \right)^{p_2/(16+4p_2)} \lambda_t(dy) < \infty. \tag{1.8}$$

Let N be a standard Gaussian random variable. Then, there exists a finite constant C , depending uniquely on c, p_1, p_2, v, m and $\lambda(\mathbb{X})$, such that

$$\max \left\{ d_W \left(\frac{F_t - \mathbb{E}F_t}{\sqrt{\text{Var } F_t}}, N \right), d_K \left(\frac{F_t - \mathbb{E}F_t}{\sqrt{\text{Var } F_t}}, N \right) \right\} \leq C t^{-1/2}, \quad t \geq 1.$$

The crucial assumption (1.8) is intimately connected with the theory of stabilization, developed in [6,41,43–45] and many other references; see [49] for a survey. Indeed, if $\mathbb{X} \subset \mathbb{R}^d$ is compact and λ equals the restriction of the Lebesgue measure to \mathbb{X} , then (1.8) requires to bound

$$\int_{\mathbb{X}} t \mathbb{P}(D_{x,y}^2 F_t \neq 0)^\alpha dy,$$

uniformly in $x \in \mathbb{X}$ and $t \geq 1$ for suitable $\alpha > 0$, a task which is often simplified by some sort of translation invariance of $D^2 F_t$. Assume, for instance, that there exist finite random variables $R_t(x, \eta_t)$ (radii of stabilization), such that $D_x F_t$ does only depend on the restriction of η_t to the ball $B^d(x, R_t(x, \eta_t))$, where $B^d(x, r)$ is our generic notation for a ball with radius r centred at x : formally, this means that, for every $t \geq 1$ and every $x \in \mathbb{X}$,

$$D_x f_t(\eta_t) = D_x f_t(\eta_t \cap B^d(x, R_t(x, \eta_t))),$$

where $F_t \equiv f_t(\eta_t)$. Then, we need to show that

$$\sup_{x \in \mathbb{X}, t \geq 1} \int_{\mathbb{X}} t \mathbb{P}(y \in B^d(x, R_t(x, \eta_t)) \text{ or } R_t(x, \eta_t + \delta_y) \neq R_t(x, \eta_t))^\alpha dy < \infty.$$

This is a close relative of the concept of strong stabilization (or add-one cost stabilization) introduced in [43].

In Sect. 7 we shall illustrate the power of Proposition 1.4 and its more general version, Theorem 6.1, by fully developing two geometric applications, namely optimal Berry-Esseen bounds in the normal approximation of:

- (i) statistics (including the total edge count and the total length) based on a k -nearest neighbour graph with a Poisson input (generalising and improving previous estimates from [3,6,7,41,43,45]);
- (ii) the intrinsic volumes of k -faces associated with a Poisson-Voronoi tessellation (improving the results in [3,6,18,41,43,45]).

In our opinion, the new connection between the Stein-Malliavin approach and the theory of stabilization has a great potential for further generalisations and applications.

For example, the results of this paper are used in [53] to derive central limit theorems for edge-length functionals of the radial spanning tree.

A second application (of a completely different nature) will be developed in Sect. 8, where we apply our main theorems to study integrals of non-linear functionals $\varphi(X_t)$ of general Poisson shot-noise processes $(X_t)_{t \in \mathbb{Y}}$. Here, \mathbb{Y} is a measurable space and X_t is a first order Wiener-Itô integral w.r.t. $\hat{\eta}$, whose integrand deterministically depends on the parameter $t \in \mathbb{Y}$. Such Wiener-Itô integrals provide fundamental examples of random fields, see [54] for a recent survey. The numbers γ_i can be controlled by the first and second derivative of φ and our bounds yield optimal rates of convergence in central limit theorems. We will now illustrate our results with the (very) special example of a Ornstein-Uhlenbeck Lévy process (see e.g. [5,37]). To do so, we let η be a Poisson process on $\mathbb{R} \times \mathbb{R}$ with intensity measure $\lambda(du, dx) = \nu(du) dx$, where the (Lévy) measure ν on \mathbb{R} satisfies

$$\int |u|^\alpha \nu(du) < \infty, \quad \alpha \in \{1, 4 + 4p\},$$

for some $p > 0$. Define

$$X_t := \int \mathbf{1}\{x \leq t\} u e^{-(t-x)} \hat{\eta}(d(u, x)), \quad t \in \mathbb{R},$$

and

$$F_T := \int_0^T \varphi(X_t) dt, \quad T > 0,$$

where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function satisfying, for some $C' > 0$,

$$|\varphi(r)| + |\varphi'(r)| + |\varphi''(r)| \leq C'(1 + |r|^p), \quad r \in \mathbb{R}.$$

Proposition 1.5 *Let $(X_t)_{t \in \mathbb{R}}$ and F_T be as above and let N be a standard Gaussian random variable. Assume that*

$$\text{Var } F_T \geq \sigma T, \quad T \geq t_0, \tag{1.9}$$

for some $\sigma, t_0 > 0$. Then there is a constant $C > 0$ depending on σ, p, C' and t_0 such that

$$d_K \left(\frac{F_T - \mathbb{E}F_T}{\sqrt{\text{Var } F_T}}, N \right) \leq \frac{C}{\sqrt{T}}, \quad T \geq t_0.$$

A sufficient condition for (1.9) to hold is that $\tilde{\varphi}(r) := \mathbb{E}[\varphi(X_0 + r) - \varphi(X_0)]$ satisfies $\int_0^u \tilde{\varphi}(r) dr \neq 0$ for some $u > 0$ in the support of ν , see Lemma 8.4 and the subsequent discussion. Proposition 1.5 largely extends the CLTs proved in [17,37,40], which only addressed linear and quadratic functionals.

For the sake of conciseness, in Sects. 7 and 8 we will only present estimates involving the Kolmogorov distance d_K ; since the bound in Theorem 1.1 is actually simpler than that in Theorem 1.2, analogous estimates hold also for the Wasserstein distance.

1.3 Further connections with the existing literature

(a) *Malliavin-Stein techniques.* The present paper represents the latest instalment in a very active direction of research, based on the combined use of Stein's method and Malliavin calculus in order to deduce explicit bounds for probabilistic approximations on the Poisson space. We refer the reader to [37, 40] for the first bounds of this type in the context of CLTs in the Wasserstein and smoother distances, and to [47] for the first panoply of results indicating that Malliavin-Stein techniques can be of great value for deriving quantitative CLTs in a geometric setting. Other relevant contributions in this area are: [17, 50] for explicit bounds in the Kolmogorov distance; [19] for applications to additive functionals of Boolean models; [8, 15, 24, 25, 29] for applications to geometric U -statistics and other non-linear functionals; [51] for CLTs in the framework of Poisson-Voronoi approximations; [35, 52] for Poisson approximations and related geometric applications. It is important to notice that all these references are based on bounds involving not only the discrete gradient D , but also the so-called pseudo-inverse of the Ornstein-Uhlenbeck generator, denoted in what follows by L^{-1} (see (2.9) for a definition). Dealing with L^{-1} is usually a very delicate issue, and typically requires excellent estimates on the kernels appearing in a given Wiener-Itô chaotic decomposition (Wiener chaoses are indeed the eigenspaces of L^{-1} ; see again (2.9)). While quantitative CLTs with optimal rates can still be obtained even for random variables with an infinite chaotic expansion (see e.g. [19]), the involved computation can be of a daunting technicality. The estimates appearing in Theorem 1.1 and Theorem 1.2 above are the first bounds obtained by Malliavin-Stein techniques that have a purely geometric nature, and therefore do not require detailed knowledge of the chaotic decomposition of the underlying random variable F .

(b) *Other second order inequalities.* An early attempt at developing second-order Poincaré inequalities on the Poisson space can be found in Vázquez [56]. One should note that the results developed in this work are quite limited in scope, as the author does not make use of Mehler's formula, in such a way that the only treatable examples are those referring to random variables living in a fixed Wiener chaos.

(c) *Stabilization.* As made clear by the title and by the previous discussion, we regard the hypothesis (1.8) of Proposition 1.4 as a weak form of stabilization. The powerful and far-reaching concept of stabilization in the context of central limit theorems was introduced in its actual form by Penrose and Yukich in [43, 44] and Baryshnikov and Yukich in [6], building on the set of techniques introduced by Kesten and Lee [23] and Lee [30]. This notion typically applies to a collection of geometric functionals $\{F_t : t \geq 1\}$ of the type

$$F_t = \int_H h_t(x, \eta_t \cap H) \eta_t(dx), \quad t \geq 1, \quad (1.10)$$

where $H \subset \mathbb{R}^d$ has finite Lebesgue measure, η_t is a Poisson process on \mathbb{R}^d with intensity $t\ell_d$ (where ℓ_d denotes Lebesgue measure), $h_1 : H \times \mathbf{N} \rightarrow \mathbb{R}$ is some measurable translation-invariant mapping (with \mathbf{N} indicating the class of σ -finite configurations on \mathbb{R}^d ; see Sect. 2), and $h_t(x, \mu) := h_1(t^{1/d}x, t^{1/d}\mu)$. There are many versions of stabilization. Add-one cost stabilization (see [43, 44] and the discussion after Proposition 1.4) requires the differences $D_x F_t = F_t(\eta_t + \delta_x) - F_t$, to stabilize around any $x \in H$ in a similar sense as described after Proposition 1.4. Exponential stabilization requires stabilization of the functions $h_t(x, \eta_t \cap H)$ as well as an exponential tail-behaviour of the associated radii of stabilization; see [45] for more details.

One important result in the area that is relevant for our paper (see e.g. [42, Sect. 2], [45, Theorem 2.1] or [49, Theorem 4.26]) is that, if $\text{Var } F_t \sim t$, some moment conditions are satisfied, and F_t are exponentially stabilizing, then the random variables $\tilde{F}_t := (F - \mathbb{E}F_t)/\sqrt{\text{Var } F_t}$ verify a CLT, when $t \rightarrow \infty$, with a rate of convergence in the Kolmogorov distance of the order of $t^{-1/2}A(t)$, where $A(t)$ is some positive function slowly diverging to infinity (e.g., $A(t) = (\log t)^\alpha$, for $\alpha > 0$); to the best of our knowledge, the question of whether the factor $A(t)$ could be removed at all has remained open until now. If the functionals F_t appearing in Proposition 1.4 are given in the form (1.10), then it is not difficult (albeit quite technical) to translate (1.6)–(1.8) in terms of assumptions on the kernels h_t , see also Remark 6.2. We do not give the details here.

However, as demonstrated by Proposition 1.4 and its generalizations, our approach enjoys at least two fundamental advantages with respect to classical stabilization techniques: (i) the assumptions in our quantitative results do not require that the functionals F_t are represented in the special form (1.10), and (ii) the rates of convergence given by our estimates seem to be systematically better (when both approaches apply). As already discussed, in Sect. 7.1 we will provide optimal Berry-Esseen bounds for the total edge length of the k -nearest neighbour graph over Poisson inputs; this is a typical stabilizing functional for which our rates improve those in the existing literature. The advantage of our method seems to be that we exploit Stein's method using operators from stochastic analysis that are intrinsically associated with the model at hand, and therefore we do not need to rely on a discretised version of the problem.

(d) *Iteration of Efron-Stein.* The idea of proving normal approximation results by controlling second order interactions between random points is also successfully applied in [9], where the author combines Stein's method with an iteration of the Efron-Stein inequality, in order to deduce explicit rates of convergence in a number of CLTs involving geometric functionals over binomial inputs. At this stage, it is difficult to compare our techniques with those of [9] for mainly two reasons: (i) while it is possible to apply the results from [9] to compute Berry-Esseen bounds for Poisson functionals (as demonstrated in the recent reference [11], dealing with the total length of the minimal spanning tree over a Poisson input), this operation can only be accomplished via a discretisation argument that is completely absent from our approach, and (ii) the techniques of [9] are specifically devised for deducing rates of convergence in the Wasserstein distance, whereas bounds in the Kolmogorov distance are only obtained

by using the standard relation between d_K and d_W (see e.g. [33, formula (C.2.6)]), a strategy which cannot be expected, in general, to yield the optimal rates.

1.4 Plan

This paper is organized as follows. After some preliminaries in Sect. 2, Mehler’s formula is established in Sect. 3. It is the crucial argument in the proofs of Theorems 1.1 and 1.2, which are given in Sect. 4. Section 5 contains the proof of several lower bounds for variances, which are helpful in applications, in order to ensure that variances do not degenerate. In Sect. 6 we consider the normal approximation of stabilizing Poisson functionals. Sections 7 and 8 contain the applications of our results to problems from stochastic geometry and to non-linear functionals of shot-noise processes.

2 Preliminaries

The reader is referred to the monograph [46] as well as to the paper [27] for any unexplained definition or result.

Let $(\mathbb{X}, \mathcal{X})$ be an arbitrary measurable space and let λ be a σ -finite measure on \mathbb{X} such that $\lambda(\mathbb{X}) > 0$. For $p > 0$ and $n \in \mathbb{N}$ we denote by $L^p(\lambda^n)$ the set of all measurable functions $f : \mathbb{X}^n \rightarrow \mathbb{R}$ such that $\int |f|^p d\lambda^n < \infty$. We call a function $f : \mathbb{X}^n \rightarrow \mathbb{R}$ symmetric if it is invariant under permutations of its arguments, and denote by $L_s^p(\lambda^n)$ the set of all $f \in L^p(\lambda^n)$ such that there exists a symmetric function \hat{f} verifying $f = \hat{f}$, λ^n -a.e. In particular, the class $L_s^2(\lambda^n)$ is a Hilbert subspace of $L^2(\lambda^n)$. For $f, g \in L^2(\lambda^n)$ we define the inner product $\langle f, g \rangle_n := \int fg d\lambda^n$ and the norm $\|f\|_n := (\int f^2 d\lambda^n)^{1/2}$.

For the rest of this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space. For $p > 0$, let $L^p(\mathbb{P})$ denote the space of all random variables $Y : \Omega \rightarrow \mathbb{R}$ such that $\mathbb{E}|Y|^p < \infty$. Let η be a Poisson process in \mathbb{X} with intensity measure λ defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As usual we interpret η as a random element in the space \mathbf{N} of integer-valued σ -finite measures μ on \mathbb{X} equipped with the smallest σ -field \mathcal{N} making the mappings $\mu \mapsto \mu(B)$ measurable for all $B \in \mathcal{X}$; see [21] or [27]. We write $\hat{\eta}$ for the compensated random (signed) measure $\eta - \lambda$. By L_p^η , $p > 0$, we denote the space of all random variables $F \in L^p(\mathbb{P})$ such that $F = f(\eta)$, \mathbb{P} -a.s., for some measurable function $f : \mathbf{N} \rightarrow \mathbb{R}$; such a function f (which is uniquely determined by F up to sets of \mathbb{P} -measure zero) is customarily called a representative of F .

Let $f : \mathbf{N} \rightarrow \mathbb{R}$ be a measurable function. For $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{X}$ we define

$$D_{x_1, \dots, x_n}^n f(\mu) := \sum_{J \subset \{1, 2, \dots, n\}} (-1)^{n-|J|} f\left(\mu + \sum_{j \in J} \delta_{x_j}\right), \quad \mu \in \mathbf{N}, \tag{2.1}$$

where $|J|$ denotes the number of elements of J . This shows that the n -th order difference operator D_{x_1, \dots, x_n}^n is symmetric in x_1, \dots, x_n , and that $(\mu, x_1, \dots, x_n) \mapsto D_{x_1, \dots, x_n}^n f(\mu)$ is measurable. For fixed $\mu \in \mathbf{N}$ the latter mapping is abbreviated as

$D^n f(\mu)$. For $F \in L^2_\eta$ with representative f we define $D^n F = D^n f(\eta)$. By the multivariate Mecke equation (see e.g. [27, formula (2.10)]) this definition does λ^n -a.e. and \mathbb{P} -a.s. not depend on the choice of the representative f .

For $F, G \in L^2_\eta$ the Fock space representation derived in [27] says that

$$\mathbb{E}[FG] = \mathbb{E}[F]\mathbb{E}[G] + \sum_{n=1}^\infty n! \langle f_n, g_n \rangle_n, \tag{2.2}$$

where $f_n := \frac{1}{n!} \mathbb{E} D^n F$, $g_n := \frac{1}{n!} \mathbb{E} D^n G$ and $f_n, g_n \in L^2_s(\lambda^n)$. If $F = G$, (2.2) yields a formula for the variance of F .

Let us denote by $I_n(g)$ the n th order Wiener-Itô integral of $g \in L^2_s(\lambda^n)$ with respect to $\hat{\eta}$. The Definition (3.1) in [27] requires a representation of the form $\eta = \sum_{n=1}^{\eta(\mathbb{X})} \delta_{X_n}$, where (X_n) is a sequence of random variables in \mathbb{X} . If $(\mathbb{X}, \mathcal{X})$ is a Borel space (see [21]), then such a sequence always exists. In the general case there is a Poisson process $\eta' \stackrel{d}{=} \eta$, such that the above representation holds for η' . In view of our results (Theorems 1.1 and 1.2) it is no loss of generality to assume throughout the paper that $\eta = \eta'$.

The Wiener-Itô integrals satisfy the isometry and orthogonality relation

$$\mathbb{E} I_n(g) I_m(h) = n! \mathbf{1}\{m = n\} \langle g, h \rangle_n, \quad g \in L^2_s(\lambda^n), h \in L^2_s(\lambda^m), n, m \in \mathbb{N}. \tag{2.3}$$

It is a well-known fact that every $F \in L^2_\eta$ admits a representation of the type

$$F = \mathbb{E} F + \sum_{n=1}^\infty I_n(f_n), \tag{2.4}$$

where $f_n = \frac{1}{n!} \mathbb{E} D^n F$ and the right-hand side converges in $L^2(\mathbb{P})$. Such a representation is known as Wiener-Itô chaos expansion of F ; in this general and explicit form the result was proved in [27].

Given F as in (2.4), we write $F \in \text{dom } D$ if

$$\sum_{n=1}^\infty n n! \|f_n\|_n^2 < \infty. \tag{2.5}$$

In this case, we have that, \mathbb{P} -a.s. and for λ -a.e. $x \in \mathbb{X}$,

$$D_x F = \sum_{n=1}^\infty n I_{n-1}(f_n(x, \cdot)), \tag{2.6}$$

cf. [27] and the references given there. Applying the Fock space relation (2.2) to $D_x F$, it is easy to see that (2.5) is actually equivalent to the integrability condition (1.5); see [39, Lemma 3.1]. The restriction of D to the space $\text{dom } D$ is usually called the Malliavin derivative operator associated with the Poisson measure η .

One can think of the difference operator D as an operator mapping a random variable to a random function. The Skorohod integral (or Kabanov-Skorohod integral-see [22]) δ maps a random function g from the space $L^2_\eta(\mathbb{P} \otimes \lambda)$ of all elements of $L^2(\mathbb{P} \otimes \lambda)$ that are $\mathbb{P} \otimes \lambda$ -a.e. of the form $g(\omega, x) = \tilde{g}(\eta(\omega), x)$ for some measurable \tilde{g} to a random

variable. Its domain $\text{dom } \delta$ is the set of all $g \in L^2_\eta(\mathbb{P} \otimes \lambda)$ having Wiener-Itô chaos expansions

$$g(x) = \sum_{n=0}^\infty I_n(g_n(x, \cdot)), \quad x \in \mathbb{X}, \tag{2.7}$$

with $I_0(c) = c$ for $c \in \mathbb{R}$ and measurable functions $g_n : \mathbb{X}^{n+1} \rightarrow \mathbb{R}, n \in \mathbb{N} \cup \{0\}$, that are symmetric in the last n variables and satisfy

$$\sum_{n=0}^\infty (n+1)! \|\tilde{g}_n\|_{n+1}^2 < \infty. \tag{2.8}$$

Here $\tilde{h} : \mathbb{X}^m \rightarrow \mathbb{R}$ stands for the symmetrization

$$\tilde{h}(x_1, \dots, x_m) = \frac{1}{m!} \sum_{\sigma \in \text{Per}(m)} h(x_{\sigma(1)}, \dots, x_{\sigma(m)})$$

of $h : \mathbb{X}^m \rightarrow \mathbb{R}$, where $\text{Per}(m)$ denotes the group of all permutations of $\{1, \dots, m\}$. Now the Skorohod integral of $g \in \text{dom } \delta$ is defined as

$$\delta(g) = \sum_{n=0}^\infty I_{n+1}(\tilde{g}_n).$$

The third operator we will use in the following is the so-called Ornstein-Uhlenbeck generator. Its domain is given by all $F \in L^2_\eta$ satisfying

$$\sum_{n=1}^\infty n^2 n! \|f_n\|_n^2 < \infty.$$

In this case one defines

$$LF = - \sum_{n=1}^\infty n I_n(f_n).$$

The (pseudo) inverse L^{-1} of L is given by

$$L^{-1}F := - \sum_{n=1}^\infty \frac{1}{n} I_n(f_n). \tag{2.9}$$

Note that the random variable $L^{-1}F$ is a well-defined element of L^2_η for every $F \in L^2_\eta$.

The following integration by parts formula [27, 46] shows that the difference operator and the Skorohod integral can be seen as dual operators.

Lemma 2.1 *Let $F \in \text{dom } D$ and $g \in \text{dom } \delta$. Then*

$$\mathbb{E} \int D_x F g(x) \lambda(dx) = \mathbb{E}[F \delta(g)].$$

In the previous formula, one needs that $F \in \text{dom } D$ (although $D_x F$ is still defined otherwise). Obviously, $\mathbf{1}\{F > t\}$ is in L^2_η for any $F \in L^2_\eta$ and $t \in \mathbb{R}$, but it is unclear whether $\mathbf{1}\{F > t\} \in \text{dom } D$ whenever the underlying intensity λ is such that $\lambda(\mathbb{X}) = \infty$. To overcome this difficulty, we shall need the following special integration by parts formula, for which we assume slightly more than $g \in \text{dom } D$ (because of the missing symmetrization in (2.10)).

Lemma 2.2 *Let $F \in L^2_\eta, s \in \mathbb{R}$ and $g \in L^2_\eta(\mathbb{P} \otimes \lambda)$ such that*

$$\sum_{n=0}^\infty (n + 1)! \|g_n\|_{n+1}^2 < \infty \tag{2.10}$$

and assume that $D_x \mathbf{1}\{F > s\} g(x) \geq 0, \mathbb{P}$ -a.s., λ -a.e. $x \in \mathbb{X}$. Then $g \in \text{dom } \delta$ and

$$\mathbb{E} \int D_x \mathbf{1}\{F > s\} g(x) \lambda(dx) = \mathbb{E}[\mathbf{1}\{F > s\} \delta(g)].$$

Proof By the Cauchy-Schwarz inequality, (2.10) implies (2.8) so that $g \in \text{dom } \delta$. In Lemma 2.3 in [50] the assertion is stated for $g \in \text{dom } D$ with finite chaos expansions. But the proof still holds for $g \in L^2(\mathbb{P} \otimes \lambda)$ satisfying (2.10) since an application of the Cauchy-Schwarz inequality shows that one can interchange summation and integration in Eq. (9) in [50]. □

We next state a basic isometry property of the Skorohod integral. Although special cases of this result are well-known (see e.g. [46]) we give the proof for the sake of completeness.

Proposition 2.3 *Let $g \in L^2_\eta(\mathbb{P} \otimes \lambda)$ be such that*

$$\mathbb{E} \iint (D_y g(x))^2 \lambda(dx) \lambda(dy) < \infty. \tag{2.11}$$

Then, g satisfies (2.10) and $g \in \text{dom } \delta$, and

$$\mathbb{E} \delta(g)^2 = \mathbb{E} \int g(x)^2 \lambda(dx) + \mathbb{E} \iint D_y g(x) D_x g(y) \lambda(dx) \lambda(dy). \tag{2.12}$$

Proof Suppose that g is given by (2.7). Assumption (2.11) implies that $g(x) \in \text{dom } D$ for λ -a.e. $x \in \mathbb{X}$. We therefore deduce from (2.6) that

$$h(x, y) := D_y g(x) = \sum_{n=1}^\infty n I_{n-1}(g_n(x, y, \cdot))$$

\mathbb{P} -a.s. and for λ^2 -a.e. $(x, y) \in \mathbb{X}^2$. Using assumption (2.11) together with (2.3), one infers that

$$\sum_{n=1}^{\infty} nn! \|\tilde{g}_n\|_{n+1}^2 \leq \sum_{n=1}^{\infty} nn! \|g_n\|_{n+1}^2 = \mathbb{E} \iint (D_y g(x))^2 \lambda(dx) \lambda(dy) < \infty, \tag{2.13}$$

yielding that g satisfies (2.10) and that $g \in \text{dom } \delta$. Now define

$$g^{(m)}(x) := \sum_{n=0}^m I_n(g_n(x, \cdot)), \quad x \in \mathbb{X},$$

for $m \in \mathbb{N}$. Then

$$\mathbb{E} \delta(g^{(m)})^2 = \sum_{n=0}^m \mathbb{E} I_{n+1}(\tilde{g}_n)^2 = \sum_{n=0}^m (n+1)! \|\tilde{g}_n\|_{n+1}^2.$$

Using the symmetry of the functions g_n it is easy to see that the latter sum equals

$$\sum_{n=0}^m n! \int g_n^2 d\lambda^{n+1} + \sum_{n=1}^m nn! \iint g_n(x, y, z) g_n(y, x, z) \lambda^2(d(x, y)) \lambda^{n-1}(dz), \tag{2.14}$$

with the obvious interpretation of the integration with respect to λ^0 . On the other hand, we have from (2.6) that

$$D_y g^{(m)}(x) = \sum_{n=1}^m n I_{n-1}(g_n(x, y, \cdot))$$

so that

$$\mathbb{E} \int g^{(m)}(x)^2 \lambda(dx) + \mathbb{E} \iint D_y g^{(m)}(x) D_x g^{(m)}(y) \lambda(dx) \lambda(dy)$$

coincides with (2.14). Hence

$$\mathbb{E} \delta(g^{(m)})^2 = \mathbb{E} \int g^{(m)}(x)^2 \lambda(dx) + \mathbb{E} \iint D_y g^{(m)}(x) D_x g^{(m)}(y) \lambda(dx) \lambda(dy). \tag{2.15}$$

From the equality part in (2.13) it follows that $h_m(x, y) := D_y g^{(m)}(x)$ converges to h in $L^2(\mathbb{P} \otimes \lambda^2)$. Similarly, $h'_m(x, y) := D_x g^{(m)}(y)$ converges towards $h'(x, y) := D_x g(y)$. Together with

$$\mathbb{E} \int g^{(m)}(x)^2 \lambda(dx) = \sum_{n=1}^m n! \|g_n\|_{n+1}^2 \rightarrow \sum_{n=1}^{\infty} n! \|g_n\|_{n+1}^2 = \mathbb{E} \int g(x)^2 \lambda(dx)$$

as $m \rightarrow \infty$ we can now conclude that the right-hand side of (2.15) tends to the right-hand side of the asserted identity (2.12). On the other hand,

$$\begin{aligned} \mathbb{E}\delta \left(g - g^{(m)} \right)^2 &= \sum_{n=m+1}^{\infty} \mathbb{E}I_{n+1}(\tilde{g}_n)^2 = \sum_{n=m+1}^{\infty} (n+1)! \|\tilde{g}_n\|_{n+1}^2 \\ &\leq \sum_{n=m+1}^{\infty} (n+1)! \|g_n\|_{n+1}^2 \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$. This concludes the proof. □

We will also exploit the following consequence of Proposition 2.3.

Corollary 2.4 *Let $g \in \text{dom } \delta$. Then*

$$\mathbb{E}\delta(g)^2 \leq \mathbb{E} \int g(x)^2 \lambda(dx) + \mathbb{E} \iint (D_y g(x))^2 \lambda(dx) \lambda(dy). \tag{2.16}$$

Proof We can assume that the right-hand side of (2.16) is finite. Now the Cauchy-Schwarz inequality implies that

$$\mathbb{E} \iint D_y g(x) D_x g(y) \lambda(dx) \lambda(dy) \leq \mathbb{E} \iint (D_y g(x))^2 \lambda(dx) \lambda(dy),$$

and the result follows from Proposition 2.3. □

In the sequel, we use the following Poincaré inequality to establish that a Poisson functional is in L^2_η and to bound its variance.

Proposition 2.5 *For $F \in L^1_\eta$,*

$$\mathbb{E}F^2 \leq (\mathbb{E}F)^2 + \mathbb{E} \int (D_x F)^2 \lambda(dx). \tag{2.17}$$

In particular, $F \in L^2_\eta$ if the right-hand side is finite.

Proof For $F \in L^2_\eta$, (2.17) is a special case of Theorem 1.2 in [27]. We extend it to $F \in L^1_\eta$ by the following truncation argument. For $s > 0$ we define

$$F_s = \mathbf{1}\{F > s\}s + \mathbf{1}\{-s \leq F \leq s\}F - \mathbf{1}\{F < -s\}s$$

By definition of F_s we have $F_s \in L^2_\eta$ and $|D_x F_s| \leq |D_x F|$ for λ -a.e. $x \in \mathbb{X}$. Together with the Poincaré inequality for L^2 -functionals we obtain that

$$\mathbb{E}F_s^2 \leq (\mathbb{E}F_s)^2 + \mathbb{E} \int (D_x F_s)^2 \lambda(dx) \leq (\mathbb{E}F_s)^2 + \mathbb{E} \int (D_x F)^2 \lambda(dx).$$

By the monotone convergence theorem and the dominated convergence theorem, respectively, we have that $\mathbb{E}F_s^2 \rightarrow \mathbb{E}F^2$ and $\mathbb{E}F_s \rightarrow \mathbb{E}F$ as $s \rightarrow \infty$. Hence letting $s \rightarrow \infty$ in the previous inequality yields the assertion. \square

3 Mehler’s formula

For $F \in L^1_\eta$ with representative f we define

$$P_s F := \int \mathbb{E}[f(\eta^{(s)} + \mu) \mid \eta] \Pi_{(1-s)\lambda}(d\mu), \quad s \in [0, 1], \tag{3.1}$$

where $\eta^{(s)}$ is a s -thinning of η and where $\Pi_{\lambda'}$ denotes the distribution of a Poisson process with intensity measure λ' . The thinning $\eta^{(s)}$ can be defined by first representing η as a finite or countable sum of Dirac measures at random points (as explained in Sect. 2, this is no restriction of generality) and then removing these points independently of each other with probability $1 - s$, see [21, p.226]. This does also show that the mapping $(\omega, s) \mapsto P_s F(\omega)$ can be assumed to be measurable. Since

$$\Pi_\lambda = \mathbb{E} \int \mathbf{1}\{\eta^{(s)} + \mu \in \cdot\} \Pi_{(1-s)\lambda}(d\mu), \tag{3.2}$$

the definition of $P_s F$ does almost surely not depend on the choice of the representative of F . Equation (3.2) implies in particular that

$$\mathbb{E}P_s F = \mathbb{E}F, \quad F \in L^1_\eta, \tag{3.3}$$

while Jensen’s inequality and (3.2) yield the contractivity property

$$\mathbb{E}[|P_s F|^p] \leq \mathbb{E}[|F|^p], \quad s \in [0, 1], F \in L^1_\eta, \tag{3.4}$$

for every $p \geq 1$.

Lemma 3.1 *Let $F \in L^2_\eta$. Then, for all $n \in \mathbb{N}$ and $s \in [0, 1]$,*

$$D^n_{x_1, \dots, x_n}(P_s F) = s^n P_s D^n_{x_1, \dots, x_n} F, \quad \lambda^n\text{-a.e. } (x_1, \dots, x_n) \in \mathbb{X}^n, \mathbb{P}\text{-a.s.} \tag{3.5}$$

In particular

$$\mathbb{E}D^n_{x_1, \dots, x_n} P_s F = s^n \mathbb{E}D^n_{x_1, \dots, x_n} F, \quad \lambda^n\text{-a.e. } (x_1, \dots, x_n) \in \mathbb{X}^n. \tag{3.6}$$

Proof Let $s \in [0, 1]$. To begin with, we assume that the representative of F is given by $f(\mu) = e^{-\int v d\mu}$ for some $v : \mathbb{X} \rightarrow [0, \infty)$ such that $\lambda(\{v > 0\}) < \infty$. By the definition of a thinning,

$$\mathbb{E} \left[e^{-\int v d\eta^{(s)}} \mid \eta \right] = \exp \left[\int \log \left((1-s) + se^{-v(y)} \right) \eta(dy) \right],$$

and it follows from Lemma 12.2 in [21] that

$$\int \exp \left(- \int v \, d\mu \right) \Pi_{(1-s)\lambda}(d\mu) = \exp \left[- (1-s) \int (1 - e^{-v}) \, d\lambda \right].$$

Hence, the definition (3.1) of the operator P_s implies that the following function $f_s : \mathbf{N} \rightarrow \mathbb{R}$ is a representative of $P_s F$:

$$f_s(\mu) := \exp \left[- (1-s) \int (1 - e^{-v}) \, d\lambda \right] \exp \left[\int \log \left((1-s) + s e^{-v(y)} \right) \mu(dy) \right].$$

Therefore we obtain for any $x \in \mathbb{X}$ that

$$D_x P_s F = f_s(\eta + \delta_x) - f_s(\eta) = s(e^{-v(x)} - 1) f_s(\eta) = s(e^{-v(x)} - 1) P_s F.$$

This identity can be iterated to yield for all $n \in \mathbb{N}$ and all $(x_1, \dots, x_n) \in \mathbb{X}^n$ that

$$D_{x_1, \dots, x_n}^n P_s F = s^n \prod_{i=1}^n (e^{-v(x_i)} - 1) P_s F.$$

On the other hand we have \mathbb{P} -a.s. that

$$P_s D_{x_1, \dots, x_n}^n F = P_s \prod_{i=1}^n (e^{-v(x_i)} - 1) F = \prod_{i=1}^n (e^{-v(x_i)} - 1) P_s F,$$

so that (3.5) holds for Poisson functionals of the given form.

By linearity, (3.5) extends to all F with a representative in the set \mathbf{G} of all linear combinations of functions f as above. In view of [27, Lemma 2.1], for any $F \in L^2_\eta$ with representative f there are $f^k \in \mathbf{G}, k \in \mathbb{N}$, satisfying $F^k := f^k(\eta) \rightarrow F = f(\eta)$ in $L^2(\mathbb{P})$ as $k \rightarrow \infty$. The contractivity property (3.4) implies that

$$\mathbb{E}(P_s F^k - P_s F)^2 = \mathbb{E}(P_s (F^k - F))^2 \leq \mathbb{E}(F^k - F)^2 \rightarrow 0,$$

as $k \rightarrow \infty$. Taking $B \in \mathcal{X}$ with $\lambda(B) < \infty$, it therefore follows from [27, Lemma 2.3] that

$$\mathbb{E} \int_{B^n} |D_{x_1, \dots, x_n}^n P_s F^k - D_{x_1, \dots, x_n}^n P_s F| \lambda^n(d(x_1, \dots, x_n)) \rightarrow 0,$$

as $k \rightarrow \infty$. On the other hand we obtain from the Fock space representation (2.2) that $\mathbb{E}|D_{x_1, \dots, x_n}^n F| < \infty$ for λ^n -a.e. $(x_1, \dots, x_n) \in \mathbb{X}^n$ so that linearity of P_s and (3.4) imply

$$\begin{aligned} & \mathbb{E} \int_{B^n} |P_s D_{x_1, \dots, x_n}^n F^k - P_s D_{x_1, \dots, x_n}^n F| \lambda^n(d(x_1, \dots, x_n)) \\ & \leq \int_{B^n} \mathbb{E} |D_{x_1, \dots, x_n}^n (F^k - F)| \lambda^n(d(x_1, \dots, x_n)). \end{aligned}$$

Again, this latter integral tends to 0 as $k \rightarrow \infty$. Since (3.5) holds for any F^k , we obtain together with the triangle inequality that

$$\mathbb{E} \int_{B^n} |D_{x_1, \dots, x_n}^n P_s F - s^n P_s D_{x_1, \dots, x_n}^n F| \lambda^n(d(x_1, \dots, x_n)) = 0.$$

Therefore (3.5) holds $\mathbb{P} \otimes (\lambda_B)^n$ -a.e., and hence also $\mathbb{P} \otimes \lambda^n$ -a.e.

Taking the expectation in (3.5) and using (3.3) proves (3.6). □

The following result yields a pathwise representation of the inverse Ornstein-Uhlenbeck operator using the operators $\{P_s : s \in [0, 1]\}$.

Theorem 3.2 *Let $F \in L^2_\eta$. If $\mathbb{E}F = 0$, then we have \mathbb{P} -a.s. that*

$$L^{-1}F = - \int_0^1 s^{-1} P_s F ds. \tag{3.7}$$

Proof Assume that F is given as in (2.4). Because of (3.4), we have that $P_s F \in L^2_\eta$. Applying (2.4) to $P_s F$ and using (3.6), we therefore infer that

$$P_s F = \mathbb{E}F + \sum_{n=1}^\infty s^n I_n(f_n) = \sum_{n=1}^\infty s^n I_n(f_n), \quad \mathbb{P}\text{-a.s.}, s \in [0, 1], \tag{3.8}$$

where we have used the fact that F is centred. Relation (3.8) can be used to show that the integral on the right-hand side of (3.7) is \mathbb{P} -a.s. finite and defines a square-integrable random variable. To see this, just apply (in order) Jensen’s inequality, (3.8) and (2.3) to deduce that

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^1 s^{-1} |P_s F| ds \right)^2 \right] & \leq \mathbb{E} \left[\int_0^1 s^{-2} \mathbb{E} |P_s F|^2 ds \right] \\ & = \sum_{n=1}^\infty n! \|f_n\|_n^2 \int_0^1 s^{2n-2} ds < \infty. \end{aligned}$$

Now,

$$L^{-1} \left(\sum_{n=1}^m I_n(f_n) \right) = - \sum_{n=1}^m \frac{1}{n} I_n(f_n) = - \int_0^1 s^{-1} \sum_{n=1}^m s^n I_n(f_n) ds, \quad m \geq 1.$$

Since L^{-1} is a continuous operator from L^2_η into itself, and in view of (2.9), we need to show that the right-hand side of the above expression converges in L^2_η , as $m \rightarrow \infty$,

to the right-hand of side of (3.7). Taking into account (3.8) we hence have to show that

$$R_m := \int_0^1 s^{-1} \left(P_s F - \sum_{n=1}^m s^n I_n(f_n) \right) ds = \int_0^1 s^{-1} \left(\sum_{n=m+1}^\infty s^n I_n(f_n) \right) ds$$

converges in L^2_η to zero. We obtain

$$\mathbb{E}R_m^2 \leq \int_0^1 s^{-2} \mathbb{E} \left(\sum_{n=m+1}^\infty s^n I_n(f_n) \right)^2 ds = \sum_{n=m+1}^\infty n! \|f_n\|_n^2 \int_0^1 s^{2n-2} ds,$$

which tends to zero as $m \rightarrow \infty$. □

We also record the following important consequence of (3.7).

Corollary 3.3 *For every $F \in L^2_\eta$ such that $\mathbb{E}F = 0$,*

$$-DL^{-1}F = \int_0^1 P_s DF ds, \quad \mathbb{P} \otimes \lambda\text{-a.e.}, \tag{3.9}$$

and

$$-D^2L^{-1}F = \int_0^1 s P_s D^2F ds, \quad \mathbb{P} \otimes \lambda^2\text{-a.e.} \tag{3.10}$$

Proof In the proof of Theorem 3.2 we have seen that

$$\mathbb{E} \int_0^1 s^{-2} (P_s F)^2 ds \leq \sum_{n=1}^\infty n! \|f_n\|_n^2 < \infty.$$

In particular,

$$\int_0^1 s^{-1} |P_s F| ds < \infty, \quad \mathbb{P}\text{-a.s.} \tag{3.11}$$

Furthermore, Lemma 3.1 implies that, for λ -a.e. $x \in \mathbb{X}$,

$$\mathbb{E} \int_0^1 s^{-1} |D_x P_s F| ds = \int_0^1 \mathbb{E} |P_s D_x F| ds \leq \mathbb{E} |D_x F|,$$

where we have used $\mathbb{E} |D_x F| < \infty$ for λ -a.e. $x \in \mathbb{X}$ (which follows from (2.2)) and the contractivity property (3.4) to get the inequality. We obtain that

$$\int_0^1 s^{-1} |D_x P_s F| ds < \infty, \quad \lambda\text{-a.e. } x \in \mathbb{X}, \quad \mathbb{P}\text{-a.s.} \tag{3.12}$$

Denoting by f_s a representative of $P_s F$, we derive from (3.11) and (3.12) that

$$\int_0^1 s^{-1} |f_s(\eta + \delta_x)| ds < \infty \quad \text{and} \quad \int_0^1 s^{-1} |f_s(\eta)| ds < \infty,$$

for λ -a.e. $x \in \mathbb{X}$ and \mathbb{P} -a.s. Hence, the difference operator of the right-hand side of (3.7) is the integrated difference operator and (3.9) follows from Lemma 3.1.

The proof of (3.10) is similar, using that $\mathbb{E}|D_{x_1, x_2}^2 F| < \infty$ for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$. □

In the following, we shall refer to the identity (3.7) as Mehler’s formula. Note that this formula can be written as

$$L^{-1} F = - \int_0^\infty T_s F ds,$$

where $T_s F := P_{e^{-s}} F$ for $s \geq 0$. The family $\{T_s : s \geq 0\}$ of operators describes a special example of Glauber dynamics. From (3.8), it follows in particular that

$$T_s F = \mathbb{E} F + \sum_{n=1}^\infty e^{-ns} I_n(f_n), \quad \mathbb{P}\text{-a.s.}, \quad s \geq 0, \tag{3.13}$$

which was proven for the special case of a finite Poisson process with a diffuse intensity measure in [46]. One should note that, for the sake of brevity, in this paper we slightly deviate from the standard terminology adopted in a Gaussian framework, where the equivalent of (3.13) and (3.7) are called, respectively, ‘Mehler’s formula’ and ‘integrated Mehler’s formula’ (see e.g. [33]).

We conclude this section with two useful inequalities.

Lemma 3.4 *For $F \in L^2_\eta$ and $p \geq 1$ we have*

$$\mathbb{E}|D_x L^{-1} F|^p \leq \mathbb{E}|D_x F|^p, \quad \lambda\text{-a.e. } x \in \mathbb{X},$$

and

$$\mathbb{E}|D_{x,y}^2 L^{-1} F|^p \leq \mathbb{E}|D_{x,y}^2 F|^p, \quad \lambda^2\text{-a.e. } (x, y) \in \mathbb{X}^2.$$

Proof Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a representative of F . Combining (3.9) and the definition of P_s leads to

$$\mathbb{E}|D_x L^{-1} F|^p = \mathbb{E} \left| \int_0^1 \int \mathbb{E}[D_x f(\eta^{(s)} + \mu) \mid \eta] \Pi_{(1-s)\lambda}(d\mu) ds \right|^p, \quad \lambda\text{-a.e. } x \in \mathbb{X}.$$

An application of Jensen’s inequality with respect to the integrals and the conditional expectation yields

$$\mathbb{E}|D_x L^{-1} F|^p \leq \mathbb{E} \int_0^1 \int \mathbb{E}[|D_x f(\eta^{(s)} + \mu)|^p \mid \eta] \Pi_{(1-s)\lambda}(d\mu) ds, \quad \lambda\text{-a.e. } x \in \mathbb{X}.$$

Because of (3.2) the right-hand side can be simplified to $\mathbb{E}|D_x f(\eta)|^p$, which concludes the proof of the first inequality. By (3.10) and analogous arguments as above, we obtain that, for λ^2 -a.e. $(x, y) \in \mathbb{X}^2$,

$$\begin{aligned} \mathbb{E}|D_{x,y}^2 L^{-1} F|^p &= \mathbb{E} \left| \int_0^1 \int s \mathbb{E}[D_{x,y}^2 f(\eta^{(s)} + \mu) \mid \eta] \Pi_{(1-s)\lambda}(d\mu) ds \right|^p \\ &\leq \mathbb{E} \int_0^1 \int \mathbb{E}[|D_{x,y}^2 f(\eta^{(s)} + \mu)|^p \mid \eta] \Pi_{(1-s)\lambda}(d\mu) ds = \mathbb{E}|D_{x,y}^2 F|^p, \end{aligned}$$

which is the second inequality. □

4 Proofs of Theorems 1.1 and 1.2

4.1 Ancillary computations

Our proofs of Theorems 1.1 and 1.2 are based on the following bounds, taken respectively from [37, Theorem 3.1] and [17, Theorem 3.1], concerning the Wasserstein and the Kolmogorov distance between the law of a given $F \in L^2_\eta$ satisfying $\mathbb{E}F = 0$ and $F \in \text{dom } D$, and the law of a standard Gaussian random variable N :

$$d_W(F, N) \leq \mathbb{E} \left| 1 - \int (D_x F)(-D_x L^{-1} F) \lambda(dx) \right| + \mathbb{E} \int (D_x F)^2 |D_x L^{-1} F| \lambda(dx), \tag{4.1}$$

$$\begin{aligned} d_K(F, N) &\leq \mathbb{E} \left| 1 - \int (D_x F)(-D_x L^{-1} F) \lambda(dx) \right| \\ &\quad + \frac{\sqrt{2\pi}}{8} \mathbb{E} \int (D_x F)^2 |D_x L^{-1} F| \lambda(dx) \\ &\quad + \frac{1}{2} \mathbb{E} \int (D_x F)^2 |F| |D_x L^{-1} F| \lambda(dx) \\ &\quad + \sup_{t \in \mathbb{R}} \mathbb{E} \int (D_x \mathbf{1}\{F > t\})(D_x F) |D_x L^{-1} F| \lambda(dx). \end{aligned} \tag{4.2}$$

One should note that the estimate (4.2) improves a previous result in [50]. The above bounds (4.1)–(4.2) are rather general. However, both can be quite difficult to evaluate if one uses the representation (2.9) of the inverse Ornstein-Uhlenbeck generator since this requires the explicit knowledge of the kernels $f_n, n \in \mathbb{N}$, of the Fock space representation, which is usually not the case for a given Poisson functional. Our main tool for overcoming this problem is Mehler’s formula. In the following result it is combined with the Poincaré inequality, in order to control the first summand on the right-hand sides of (4.1) and (4.2).

Proposition 4.1 For $F, G \in \text{dom } D$ with $\mathbb{E}F = \mathbb{E}G = 0$, we have

$$\begin{aligned} & \mathbb{E} \left(\mathbb{C}\text{ov}(F, G) - \int (D_x F)(-D_x L^{-1} G) \lambda(dx) \right)^2 \\ & \leq 3 \int [\mathbb{E}(D_{x_1, x_3}^2 F)^2 (D_{x_2, x_3}^2 F)^2]^{1/2} [\mathbb{E}(D_{x_1} G)^2 (D_{x_2} G)^2]^{1/2} \lambda^3(d(x_1, x_2, x_3)) \\ & \quad + \int [\mathbb{E}(D_{x_1} F)^2 (D_{x_2} F)^2]^{1/2} [\mathbb{E}(D_{x_1, x_3}^2 G)^2 (D_{x_2, x_3}^2 G)^2]^{1/2} \lambda^3(d(x_1, x_2, x_3)) \\ & \quad + \int [\mathbb{E}(D_{x_1, x_3}^2 F)^2 (D_{x_2, x_3}^2 F)^2]^{1/2} [\mathbb{E}(D_{x_1, x_3}^2 G)^2 (D_{x_2, x_3}^2 G)^2]^{1/2} \lambda^3(d(x_1, x_2, x_3)). \end{aligned}$$

Proof We can of course assume that the three integrals on the right-hand side of the inequality are finite—otherwise there is nothing to prove. Let $f, g : \mathbb{N} \rightarrow \mathbb{R}$ be representatives of F and G . Combining (2.2) with (2.6) and (2.9), we have

$$\mathbb{E} \int (D_x F)(-D_x L^{-1} G) \lambda(dx) = \mathbb{C}\text{ov}(F, G).$$

Start by assuming that

$$\int |D_y((D_x F)(-D_x L^{-1} G))| \lambda(dx) < \infty, \quad \lambda\text{-a.e. } y \in \mathbb{X}, \quad \mathbb{P}\text{-a.s.} \quad (4.3)$$

Then, the integral

$$\int (D_x f(\eta + \delta_y))(-D_x \tilde{g}(\eta + \delta_y)) \lambda(dx),$$

where \tilde{g} is a representative of $L^{-1}G$, exists and is finite \mathbb{P} -a.s. for λ -a.e. $y \in \mathbb{X}$. Consequently, \mathbb{P} -a.s. for λ -a.e. $y \in \mathbb{X}$,

$$D_y \int (D_x F)(-D_x L^{-1} G) \lambda(dx) = \int D_y((D_x F)(-D_x L^{-1} G)) \lambda(dx),$$

and

$$\left| D_y \int (D_x F)(-D_x L^{-1} G) \lambda(dx) \right| \leq \int |D_y((D_x F)(-D_x L^{-1} G))| \lambda(dx).$$

Together with the Poincaré inequality (see Proposition 2.5) this yields

$$\begin{aligned} A & := \mathbb{E} \left(\mathbb{C}\text{ov}(F, G) - \int (D_x F)(-D_x L^{-1} G) \lambda(dx) \right)^2 \\ & \leq \mathbb{E} \int \left(D_y \int (D_x F)(-D_x L^{-1} G) \lambda(dx) \right)^2 \lambda(dy) \\ & \leq \mathbb{E} \int \left(\int |D_y((D_x F)(-D_x L^{-1} G))| \lambda(dx) \right)^2 \lambda(dy) := B. \end{aligned}$$

Of course, if assumption (4.3) is not satisfied, then the estimate $A \leq B$ (as defined above) continues (trivially) to hold. Since, for any $x, y \in \mathbb{X}$,

$$D_y((D_x F)(-D_x L^{-1}G)) = (D_{x,y}^2 F)(-D_x L^{-1}G) + (D_x F)(-D_{x,y}^2 L^{-1}G) + (D_{x,y}^2 F)(-D_{x,y}^2 L^{-1}G),$$

we obtain that

$$\mathbb{E} \left(\text{Cov}(F, G) - \int (D_x F)(-D_x L^{-1}G) \lambda(dx) \right)^2 \leq 3(I_1 + I_2 + I_3) \tag{4.4}$$

with

$$I_1 := \mathbb{E} \int \left(\int |(D_{x,y}^2 F)(-D_x L^{-1}G)| \lambda(dx) \right)^2 \lambda(dy),$$

$$I_2 := \mathbb{E} \int \left(\int |(D_x F)(-D_{x,y}^2 L^{-1}G)| \lambda(dx) \right)^2 \lambda(dy),$$

$$I_3 := \mathbb{E} \int \left(\int |(D_{x,y}^2 F)(-D_{x,y}^2 L^{-1}G)| \lambda(dx) \right)^2 \lambda(dy).$$

We will now use Mehler’s formula to derive upper bounds for I_1, I_2 and I_3 . By combining (3.9) with the definition of P_s and Fubini’s Theorem, we see that

$$\begin{aligned} & \int |(D_{x,y}^2 F)(-D_x L^{-1}G)| \lambda(dx) \\ &= \int \left| D_{x,y}^2 F \int_0^1 P_s D_x G ds \right| \lambda(dx) \\ &= \int \left| D_{x,y}^2 f(\eta) \int_0^1 \int \mathbb{E} \left[D_x g(\eta^{(s)} + \mu) \mid \eta \right] \Pi_{(1-s)\lambda}(d\mu) ds \right| \lambda(dx) \\ &\leq \int_0^1 \int \mathbb{E} \left[\int |D_{x,y}^2 f(\eta) D_x g(\eta^{(s)} + \mu)| \lambda(dx) \mid \eta \right] \Pi_{(1-s)\lambda}(d\mu) ds. \end{aligned}$$

Now an application of Jensen’s inequality with respect to the outer integrals and the conditional expectation as well as the Cauchy-Schwarz inequality lead to

$$\begin{aligned} & \left(\int |(D_{x,y}^2 F)(-D_x L^{-1}G)| \lambda(dx) \right)^2 \\ &\leq \int_0^1 \int \mathbb{E} \left[\int |D_{x_1,y}^2 f(\eta) D_{x_2,y}^2 f(\eta) D_{x_1} g(\eta^{(s)} + \mu) D_{x_2} g(\eta^{(s)} + \mu)| \lambda^2(d(x_1, x_2)) \mid \eta \right] \\ &\quad \times \Pi_{(1-s)\lambda}(d\mu) ds \\ &= \int |D_{x_1,y}^2 f(\eta) D_{x_2,y}^2 f(\eta)| \\ &\quad \times \int_0^1 \int \mathbb{E} [|D_{x_1} g(\eta^{(s)} + \mu) D_{x_2} g(\eta^{(s)} + \mu)| \mid \eta] \Pi_{(1-s)\lambda}(d\mu) ds \lambda^2(d(x_1, x_2)) \end{aligned}$$

$$\begin{aligned} &\leq \int \left| D_{x_1,y}^2 f(\eta) D_{x_2,y}^2 f(\eta) \right| \\ &\quad \times \left[\int_0^1 \int \mathbb{E} \left[(D_{x_1} g(\eta^{(s)} + \mu))^2 (D_{x_2} g(\eta^{(s)} + \mu))^2 \mid \eta \right] \Pi_{(1-s)\lambda}(\mathrm{d}\mu) \mathrm{d}s \right]^{1/2} \lambda^2(\mathrm{d}(x_1, x_2)). \end{aligned}$$

Using the Cauchy-Schwarz inequality again, we obtain that

$$\begin{aligned} I_1 &\leq \int \left[\mathbb{E} \int_0^1 \int \mathbb{E} [(D_{x_1} g(\eta^{(s)} + \mu))^2 (D_{x_2} g(\eta^{(s)} + \mu))^2 \mid \eta] \Pi_{(1-s)\lambda}(\mathrm{d}\mu) \mathrm{d}s \right]^{1/2} \\ &\quad \times \left[\mathbb{E} (D_{x_1,y}^2 f(\eta))^2 (D_{x_2,y}^2 f(\eta))^2 \right]^{1/2} \lambda^3(\mathrm{d}(x_1, x_2, y)). \end{aligned}$$

By (3.2), the first part of the integrand simplifies to $[\mathbb{E}(D_{x_1} G)^2 (D_{x_2} G)^2]^{1/2}$ so that

$$I_1 \leq \int \left[\mathbb{E} (D_{x_1,y}^2 F)^2 (D_{x_2,y}^2 F)^2 \right]^{1/2} \left[\mathbb{E} (D_{x_1} G)^2 (D_{x_2} G)^2 \right]^{1/2} \lambda^3(\mathrm{d}(x_1, x_2, y)).$$

For I_2 and I_3 we obtain in a similar way by using (3.10) that

$$\begin{aligned} &\left(\int |(D_x F)(-D_{x,y}^2 L^{-1} G)| \lambda(\mathrm{d}x) \right)^2 \\ &\leq \left(\int |D_x f(\eta)| \int_0^1 s \int \mathbb{E} [|D_{x,y}^2 g(\eta^{(s)} + \mu)| \mid \eta] \Pi_{(1-s)\lambda}(\mathrm{d}\mu) \mathrm{d}s \lambda(\mathrm{d}x) \right)^2 \\ &= \left(\int_0^1 \int \mathbb{E} \left[s \int |D_x f(\eta) D_{x,y}^2 g(\eta^{(s)} + \mu)| \lambda(\mathrm{d}x) \mid \eta \right] \Pi_{(1-s)\lambda}(\mathrm{d}\mu) \mathrm{d}s \right)^2 \\ &\leq \int_0^1 u^2 \mathrm{d}u \times \int_0^1 \int \mathbb{E} \left[\int |D_{x_1} f(\eta) D_{x_2} f(\eta) D_{x_1,y}^2 g(\eta^{(s)} + \mu) \right. \\ &\quad \left. \times D_{x_2,y}^2 g(\eta^{(s)} + \mu)| \lambda^2(\mathrm{d}(x_1, x_2)) \mid \eta \right] \Pi_{(1-s)\lambda}(\mathrm{d}\mu) \mathrm{d}s \\ &\leq \frac{1}{3} \int \left[\int_0^1 \int \mathbb{E} [(D_{x_1,y}^2 g(\eta^{(s)} + \mu))^2 (D_{x_2,y}^2 g(\eta^{(s)} + \mu))^2 \mid \eta] \Pi_{(1-s)\lambda}(\mathrm{d}\mu) \mathrm{d}s \right]^{1/2} \\ &\quad \times |D_{x_1} f(\eta) D_{x_2} f(\eta)| \lambda^2(\mathrm{d}(x_1, x_2)) \end{aligned}$$

and

$$\begin{aligned} &\left(\int |(D_{x,y}^2 F)(-D_{x,y}^2 L^{-1} G)| \lambda(\mathrm{d}x) \right)^2 \\ &\leq \left(\int |D_{x,y}^2 f(\eta)| \int_0^1 \int \mathbb{E} [s |D_{x,y}^2 g(\eta^{(s)} + \mu)| \mid \eta] \Pi_{(1-s)\lambda}(\mathrm{d}\mu) \mathrm{d}s \lambda(\mathrm{d}x) \right)^2 \end{aligned}$$

$$\begin{aligned}
 &= \left(\int_0^1 \int \mathbb{E} \left[s \int |D_{x,y}^2 f(\eta) D_{x,y}^2 g(\eta^{(s)} + \mu)| \lambda(dx) \mid \eta \right] \Pi_{(1-s)\lambda}(d\mu) ds \right)^2 \\
 &\leq \int_0^1 u^2 du \times \int_0^1 \int \mathbb{E} \left[\int |D_{x_1,y}^2 f(\eta) D_{x_2,y}^2 f(\eta) D_{x_1,y}^2 g(\eta^{(s)} + \mu) \right. \\
 &\quad \times \left. D_{x_2,y}^2 g(\eta^{(s)} + \mu)| \lambda^2(d(x_1, x_2)) \mid \eta \right] \Pi_{(1-s)\lambda}(d\mu) ds \\
 &= \frac{1}{3} \int |D_{x_1,y}^2 f(\eta) D_{x_2,y}^2 f(\eta)| \\
 &\quad \times \int_0^1 \int \mathbb{E}[|D_{x_1,y}^2 g(\eta^{(s)} + \mu) D_{x_2,y}^2 g(\eta^{(s)} + \mu)| \mid \eta] \Pi_{(1-s)\lambda}(d\mu) ds \lambda^2(d(x_1, x_2)) \\
 &\leq \frac{1}{3} \int |D_{x_1,y}^2 f(\eta) D_{x_2,y}^2 f(\eta)| \\
 &\quad \times \left[\int_0^1 \int \mathbb{E}[(D_{x_1,y}^2 g(\eta^{(s)} + \mu))^2 (D_{x_2,y}^2 g(\eta^{(s)} + \mu))^2 \mid \eta] \Pi_{(1-s)\lambda}(d\mu) ds \right]^{1/2} \\
 &\quad \times \lambda^2(d(x_1, x_2)).
 \end{aligned}$$

As before a combination of the Cauchy-Schwarz inequality and (3.2) leads to

$$I_2 \leq \frac{1}{3} \int \left[\mathbb{E}(D_{x_1} F)^2 (D_{x_2} F)^2 \right]^{1/2} \left[\mathbb{E}(D_{x_1,y}^2 G)^2 (D_{x_2,y}^2 G)^2 \right]^{1/2} \lambda^3(d(x_1, x_2, y))$$

and

$$I_3 \leq \frac{1}{3} \int \left[\mathbb{E}(D_{x_1,y}^2 F)^2 (D_{x_2,y}^2 F)^2 \right]^{1/2} \left[\mathbb{E}(D_{x_1,y}^2 G)^2 (D_{x_2,y}^2 G)^2 \right]^{1/2} \lambda^3(d(x_1, x_2, y)).$$

Combining the inequalities for I_1 , I_2 and I_3 with (4.4) yields the assertion. □

4.2 Proofs

We can now proceed to the proof of our main results.

Proof of Theorem 1.1 For the first summand on the right-hand side of (4.1) we obtain, by the Cauchy-Schwarz inequality and Proposition 4.1 in the case $G = F$, that

$$\mathbb{E} \left| 1 - \int (D_x F)(-D_x L^{-1} F) \lambda(dx) \right| \leq \sqrt{\mathbb{E} \left(1 - \int (D_x F)(-D_x L^{-1} F) \lambda(dx) \right)^2} \leq \gamma_1 + \gamma_2.$$

For the second part of the bound in (4.1), Hölder’s inequality and Lemma 3.4 yield

$$\begin{aligned} \mathbb{E} \int (D_x F)^2 |D_x L^{-1} F| \lambda(\mathrm{d}x) &\leq \int [\mathbb{E}|D_x F|^3]^{2/3} [\mathbb{E}|D_x L^{-1} F|^3]^{1/3} \lambda(\mathrm{d}x) \\ &\leq \int \mathbb{E}|D_x F|^3 \lambda(\mathrm{d}x), \end{aligned}$$

which concludes the proof. □

Proof of Theorem 1.2 Observe that the first and the second summand in (4.2) can be treated exactly as in the proof of Theorem 1.1. Hölder’s inequality and Lemma 3.4 yield that

$$\begin{aligned} \mathbb{E} \int (D_x F)^2 |F| |D_x L^{-1} F| \lambda(\mathrm{d}x) &\leq \int [\mathbb{E}(D_x F)^4]^{1/2} [\mathbb{E}F^4]^{1/4} [\mathbb{E}|D_x L^{-1} F|^4]^{1/4} \lambda(\mathrm{d}x) \\ &\leq [\mathbb{E}F^4]^{1/4} \int [\mathbb{E}(D_x F)^4]^{3/4} \lambda(\mathrm{d}x) = 2\gamma_4. \end{aligned}$$

To conclude the proof, assume that $\gamma_5 + \gamma_6 < \infty$ (otherwise, there is nothing to prove). We shall first show that the random function $g(x) := D_x F |D_x L^{-1} F|$ verifies the integrability condition

$$A := \mathbb{E} \int g(x)^2 \lambda(\mathrm{d}x) + \mathbb{E} \iint (D_y g(x))^2 \lambda(\mathrm{d}x) \lambda(\mathrm{d}y) < \infty.$$

By the trivial inequality $||a| - |b|| \leq |a - b|$, which is true for all $a, b \in \mathbb{R}$, we have that

$$|D_y |D_x L^{-1} F|| \leq |D_{x,y}^2 L^{-1} F|.$$

Thus, we infer that

$$\begin{aligned} |D_y((D_x F)|D_x L^{-1} F)| &\leq |D_{x,y}^2 F| |D_x L^{-1} F| + |D_x F| |D_{x,y}^2 L^{-1} F| \\ &\quad + |D_{x,y}^2 F| |D_{x,y}^2 L^{-1} F|, \end{aligned}$$

whence

$$\begin{aligned} A &\leq \mathbb{E} \int (D_x F)^2 (D_x L^{-1} F)^2 \lambda(\mathrm{d}x) + 3\mathbb{E} \iint (D_{x,y}^2 F)^2 (D_x L^{-1} F)^2 \\ &\quad + (D_x F)^2 (D_{x,y}^2 L^{-1} F)^2 + (D_{x,y}^2 F)^2 (D_{x,y}^2 L^{-1} F)^2 \lambda(\mathrm{d}x) \lambda(\mathrm{d}y). \end{aligned}$$

Now the Cauchy-Schwarz inequality and Lemma 3.4 yield that

$$\begin{aligned}
 A &\leq \int \mathbb{E}(D_x F)^4 \lambda(dx) + 3 \iint [\mathbb{E}(D_{x,y}^2 F)^4]^{1/2} [\mathbb{E}(D_x F)^4]^{1/2} \\
 &\quad + [\mathbb{E}(D_x F)^4]^{1/2} [\mathbb{E}(D_{x,y}^2 F)^4]^{1/2} + \mathbb{E}(D_{x,y}^2 F)^4 \lambda(dx) \lambda(dy) \\
 &\leq \gamma_5^2 + \gamma_6^2 < \infty.
 \end{aligned}$$

By virtue of Proposition 2.3, this yields that g satisfies (2.10). The integration by parts formula in Lemma 2.2 together with the Jensen inequality (and the fact that indicators are bounded by 1) now imply that

$$\begin{aligned}
 \mathbb{E} \int D_x \mathbf{1}\{F > t\} (D_x F) |D_x L^{-1} F| \lambda(dx) &= \mathbb{E} \mathbf{1}\{F > t\} \delta((DF) |DL^{-1} F|) \\
 &\leq [\mathbb{E} \delta((DF) |DL^{-1} F|)^2]^{1/2}.
 \end{aligned}$$

Finally, Corollary 2.4 and the upper bound for A above imply that

$$\mathbb{E} \delta((DF) |DL^{-1} F|)^2 \leq A \leq \gamma_5^2 + \gamma_6^2.$$

Combining all these bounds with (4.2) concludes the proof. □

We conclude the section by recording a useful inequality, which we shall apply throughout the paper in order to bound the fourth moment of a given random variable F in terms of the difference operator DF . In particular, such an estimate is crucial for dealing with the quantity γ_4 appearing in the statement of Theorem 1.2.

Lemma 4.2 *Let $F \in L^2_\eta$ be such that $\mathbb{E}F = 0$ and $\text{Var } F = 1$. Then,*

$$\mathbb{E}F^4 \leq \max \left\{ 256 \left[\int [\mathbb{E}(D_z F)^4]^{1/2} \lambda(dz) \right]^2, 4 \int \mathbb{E}(D_z F)^4 \lambda(dz) + 2 \right\}.$$

Moreover, if the right-hand side is finite, $F \in \text{dom } D$.

Proof By virtue of the Poincaré inequality (see Proposition 2.5) (as applied to the random variable F^2) and of a straightforward computation, we obtain

$$\begin{aligned}
 \mathbb{E}F^4 &= \text{Var } F^2 + (\mathbb{E}F^2)^2 = \text{Var } F^2 + 1 \\
 &\leq \int \mathbb{E}(D_z(F^2))^2 \lambda(dz) + 1 = \int \mathbb{E}(2F(D_z F) + (D_z F)^2)^2 \lambda(dz) + 1 \\
 &\leq \int 8\mathbb{E}F^2(D_z F)^2 + 2\mathbb{E}(D_z F)^4 \lambda(dz) + 1 \\
 &\leq 8[\mathbb{E}F^4]^{1/2} \int [\mathbb{E}(D_z F)^4]^{1/2} \lambda(dz) + 2 \int \mathbb{E}(D_z F)^4 \lambda(dz) + 1 \\
 &\leq \max \left\{ 16[\mathbb{E}F^4]^{1/2} \int [\mathbb{E}(D_z F)^4]^{1/2} \lambda(dz), 4 \int \mathbb{E}(D_z F)^4 \lambda(dz) + 2 \right\},
 \end{aligned}$$

which implies the inequality. The conclusion $F \in \text{dom } D$ follows from (1.5) and the Cauchy-Schwarz inequality. \square

5 Lower bounds for variances

5.1 General bounds

In what follows, we will apply our main bounds to sequences of standardized random variables of the form $(F - \mathbb{E}F)/\sqrt{\text{Var}(F)}$, where F is e.g. some relevant geometric quantity. Our aim in this section is to prove several new analytical criteria, allowing one to deduce explicit lower bounds for variances. Our approach, which we believe is of independent interest, is based on the use of difference operators, and is perfectly tailored to deal with geometric applications.

We start by proving two criteria, ensuring that the variance of a given random variable is non-zero or greater than a constant, respectively.

Lemma 5.1 *Let $F \in L^2_\eta$ and let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a representative of F . Then $\text{Var } F = 0$ if and only if*

$$\mathbb{E} \left[f \left(\eta + \sum_{i \in I_1} \delta_{x_i} \right) - f \left(\eta + \sum_{i \in I_2} \delta_{x_i} \right) \right] = 0, \quad \lambda^k\text{-a.e. } (x_1, \dots, x_k) \in \mathbb{X}^k, \quad (5.1)$$

for all $k \in \mathbb{N}$ and $I_1, I_2 \subset \{1, \dots, k\}$ such that $I_1 \cup I_2 = \{1, \dots, k\}$.

Proof Let us assume that $\text{Var } F = 0$. Then it follows from (2.2) that

$$\mathbb{E} D^n_{x_1, \dots, x_n} F = 0, \quad \lambda^n\text{-a.e. } (x_1, \dots, x_n) \in \mathbb{X}^n, \quad (5.2)$$

for all $n \in \mathbb{N}$. Now it can be shown by induction that

$$\mathbb{E} \left[f \left(\eta + \sum_{i=1}^n \delta_{x_i} \right) - f(\eta) \right] = 0, \quad \lambda^n\text{-a.e. } (x_1, \dots, x_n) \in \mathbb{X}^n. \quad (5.3)$$

The case $n = 1$ coincides with (5.2). For $n \geq 2$ we have (using (2.1))

$$\begin{aligned} \mathbb{E} D^n_{x_1, \dots, x_n} F &= \mathbb{E} \sum_{I \subset \{1, \dots, n\}} (-1)^{n-|I|} f \left(\eta + \sum_{i \in I} \delta_{x_i} \right) \\ &= \sum_{I \subset \{1, \dots, n\}} (-1)^{n-|I|} \mathbb{E} [f \left(\eta + \sum_{i \in I} \delta_{x_i} \right) - f(\eta)] + \sum_{I \subset \{1, \dots, n\}} (-1)^{n-|I|} \mathbb{E} f(\eta). \end{aligned}$$

Here, the second summand is zero due to the alternating sign. The induction hypothesis yields that in the first sum only the summand for $I = \{1, \dots, n\}$ remains for λ^n -a.e. $(x_1, \dots, x_n) \in \mathbb{X}^n$, which proves (5.3).

By (5.3) we obtain that

$$\begin{aligned} & \mathbb{E} \left[f \left(\eta + \sum_{i \in I_1} \delta_{x_i} \right) - f \left(\eta + \sum_{i \in I_2} \delta_{x_i} \right) \right] \\ &= \mathbb{E} \left[f \left(\eta + \sum_{i \in I_1} \delta_{x_i} \right) - f(\eta) \right] - \mathbb{E} \left[f \left(\eta + \sum_{i \in I_2} \delta_{x_i} \right) - f(\eta) \right] = 0 \end{aligned}$$

for λ^n -a.e. $(x_1, \dots, x_n) \in \mathbb{X}^n$.

The other direction holds since (5.1) for all $k \in \mathbb{N}$ and all subsets I_1, I_2 implies (5.2) for all $n \in \mathbb{N}$, which is equivalent to $\text{Var } F = 0$. □

The next theorem provides a quantitative bound for the case that the variance is not zero.

Theorem 5.2 *Let $F \in L^2_\eta$ with a representative $f : \mathbb{N} \rightarrow \mathbb{R}$ and assume that there are $k \in \mathbb{N}, I_1, I_2 \subset \{1, \dots, k\}$ with $I_1 \cup I_2 = \{1, \dots, k\}, U \subset \mathbb{X}^k$ measurable and $c > 0$ such that*

$$\left| \mathbb{E} \left[f \left(\eta + \sum_{i \in I_1} \delta_{x_i} \right) - f \left(\eta + \sum_{i \in I_2} \delta_{x_i} \right) \right] \right| \geq c, \quad \lambda^k\text{-a.e. } (x_1, \dots, x_k) \in U. \tag{5.4}$$

Then,

$$\text{Var } F \geq \frac{c^2}{4^{k+1}k!} \min_{\emptyset \neq J \subset \{1, \dots, k\}} \inf_{\substack{V \subset U \\ \lambda^k(V) \geq \lambda^k(U)/2^{k+1}}} \lambda^{|J|}(\Pi_J(V)),$$

where Π_J stands for the projection onto the components whose indices belong to J .

Proof We begin with the special case $I_1 = \{1, \dots, k\}$ and $I_2 = \emptyset$. For $x_1, \dots, x_k \in \mathbb{X}$ and an index set $J = \{j_1, \dots, j_{|J|}\} \subset \{1, \dots, k\}$ we put $D_{x_J}^{|J|} F = D_{x_{j_1}, \dots, x_{j_{|J|}}}^{|J|} F$. Now it follows from (2.1) that

$$\begin{aligned} \sum_{\emptyset \neq J \subset \{1, \dots, k\}} D_{x_J}^{|J|} F &= \sum_{\emptyset \neq J \subset \{1, \dots, k\}} \sum_{I \subset J} (-1)^{|J|-|I|} f \left(\eta + \sum_{i \in I} \delta_{x_i} \right) \\ &= \sum_{I \subset \{1, \dots, k\}} (-1)^{|I|} f \left(\eta + \sum_{i \in I} \delta_{x_i} \right) \sum_{I \subset J \subset \{1, \dots, k\}, J \neq \emptyset} (-1)^{|J|} \\ &= f \left(\eta + \sum_{i \in \{1, \dots, k\}} \delta_{x_i} \right) - f(\eta), \end{aligned}$$

where we have used that the interior alternating sum is zero except for $I = \{1, \dots, k\}$ and $I = \emptyset$. Combining this with (5.4), we obtain that, for $(x_1, \dots, x_k) \in U$,

$$c \leq \left| \mathbb{E} \left[f \left(\eta + \sum_{i \in \{1, \dots, k\}} \delta_{x_i} \right) - f(\eta) \right] \right| = \left| \mathbb{E} \sum_{\emptyset \neq J \subset \{1, \dots, k\}} D_{x_J}^{|J|} F \right| \leq \sum_{\emptyset \neq J \subset \{1, \dots, k\}} \left| \mathbb{E} D_{x_J}^{|J|} F \right|.$$

Hence, there must be a non-empty set $I_0 \subset \{1, \dots, k\}$ and a set $V \subset U$ such that

$$|\mathbb{E} D_{x_{I_0}}^{|I_0|} F| \geq \frac{c}{2^k}, \quad \lambda^k\text{-a.e. } (x_1, \dots, x_k) \in V, \quad \text{and} \quad \lambda^k(V) \geq \frac{1}{2^k} \lambda^k(U).$$

Thus, it follows from (2.2) that

$$\text{Var } F \geq \frac{c^2}{4^k k!} \min_{\emptyset \neq J \subset \{1, \dots, k\}} \inf_{\substack{V \subset U \\ \lambda^k(V) \geq \lambda^k(U)/2^k}} \lambda^{|J|}(\Pi_J(V)). \tag{5.5}$$

For arbitrary $I_1, I_2 \subset \{1, \dots, k\}$ with $I_1 \cup I_2 = \{1, \dots, k\}$ we can deduce from (5.4) that there is a set $\tilde{U} \subset U$ such that

$$\left| \mathbb{E} \left[f \left(\eta + \sum_{i \in I_1} \delta_{x_i} \right) - f(\eta) \right] \right| \geq \frac{c}{2}, \quad \lambda^k\text{-a.e. } (x_1, \dots, x_k) \in \tilde{U},$$

and

$$\left| \mathbb{E} \left[f \left(\eta + \sum_{i \in I_2} \delta_{x_i} \right) - f(\eta) \right] \right| \geq \frac{c}{2}, \quad \lambda^k\text{-a.e. } (x_1, \dots, x_k) \in U \setminus \tilde{U}.$$

Without loss of generality, we can assume that $\lambda^k(\tilde{U}) \geq \lambda^k(U)/2$. Hence, it follows from (5.5) that

$$\begin{aligned} \text{Var } F &\geq \frac{(c/2)^2}{4^k k!} \min_{\emptyset \neq J \subset \{1, \dots, k\}} \inf_{\substack{V \subset \tilde{U} \\ \lambda^k(V) \geq \lambda^k(\tilde{U})/2^k}} \lambda^{|J|}(\Pi_J(V)) \\ &\geq \frac{c^2}{4^{k+1} k!} \min_{\emptyset \neq J \subset \{1, \dots, k\}} \inf_{\substack{V \subset U \\ \lambda^k(V) \geq \lambda^k(U)/2^{k+1}}} \lambda^{|J|}(\Pi_J(V)), \end{aligned}$$

which concludes the proof. □

5.2 The case of Poisson processes in Euclidean space

Some of our results can be further simplified if we assume that \mathbb{X} is a subset of \mathbb{R}^d . In this case we use the following notation. Recall that $B^d(x, r)$ is a closed ball in \mathbb{R}^d with centre x and radius r , $B_r^d = B^d(0, r)$, and $B^d = B_1^d$, and that ℓ_d is the Lebesgue measure in \mathbb{R}^d . For a compact set A let ∂A be its boundary. We also let $r(A)$ stand for the inradius of a compact convex set A , and use the symbol κ_d to denote the volume of B_1^d .

Throughout this subsection, we assume that $\eta_t, t > 0$, is the restriction of a stationary Poisson process in \mathbb{R}^d to a measurable set $H \subset \mathbb{R}^d$ whose intensity measure λ_t is t times the restriction of ℓ_d to H . By \mathbf{N}_H we denote the set of all locally finite point configurations in H .

Theorem 5.3 *Let $F \in L^2_{\eta_t}$ and let $f : \mathbf{N}_H \rightarrow \mathbb{R}$ be a representative of F . Let $k \in \mathbb{N}$ and $I_1, I_2 \subset \{1, \dots, k\}$ with $I_1 \cup I_2 = \{1, \dots, k\}$ and define*

$$g(x_1, \dots, x_k) := \left| \mathbb{E} \left[f \left(\eta_t + \sum_{i \in I_1} \delta_{x_i} \right) - f \left(\eta_t + \sum_{i \in I_2} \delta_{x_i} \right) \right] \right|, \quad (x_1, \dots, x_k) \in \mathbb{R}^{dk}.$$

Assume that there are $\hat{x}_1, \dots, \hat{x}_k \in \mathbb{R}^d$ such that g is continuous in $(\hat{x}_1, \dots, \hat{x}_k)$ and that there is a constant $c > 0$ such that $g(\hat{x}_1, \dots, \hat{x}_k) \geq c$. Moreover, let $A \subset \mathbb{R}^d$ and $\varepsilon > 0$ be such that

$$g(\hat{x}_1 + z, \hat{x}_2 + y_2 + z, \dots, \hat{x}_k + y_k + z) = g(\hat{x}_1, \hat{x}_2 + y_2, \dots, \hat{x}_k + y_k)$$

for all $z \in A$ and $y_2, \dots, y_k \in B_\varepsilon^d$. Then

$$\tau := \sup\{r \in (0, \varepsilon) : g(\hat{x}_1, \hat{x}_2 + y_2, \dots, \hat{x}_k + y_k) > c/2 \text{ for all } y_2, \dots, y_k \in B_r^d\} > 0$$

and

$$\text{Var } F \geq \frac{c^2}{4 \cdot 8^{k+1} k!} \min_{j=1, \dots, k} 2^{-d(k-j)} (t \kappa_d \tau^d)^{j-1} t \ell_d(A).$$

Proof The continuity of g in $(\hat{x}_1, \dots, \hat{x}_k)$ and the assumption $g(\hat{x}_1, \dots, \hat{x}_k) \geq c$ ensure that $\tau > 0$. Now we define

$$U := \{(\hat{x}_1 + z, \hat{x}_2 + y_2 + z, \dots, \hat{x}_k + y_k + z) : z \in A, y_2, \dots, y_k \in B_\tau^d\}.$$

Note that $g > c/2$ holds ℓ_d^k -a.e. on U . A straightforward computation shows that

$$\lambda_t^k(U) = (t \kappa_d \tau^d)^{k-1} t \ell_d(A). \tag{5.6}$$

In order to apply Theorem 5.2, we have to compute

$$\min_{\emptyset \neq J \subset \{1, \dots, k\}} \inf_{\substack{V \subset U \\ \lambda_t^k(V) \geq \lambda_t^k(U)/2^{k+1}}} \lambda_t^{|J|}(\Pi_J(V)).$$

Let $\emptyset \neq J \subset \{1, \dots, k\}$ and let x_J be the components of $x = (x_1, \dots, x_k) \in \mathbb{R}^{dk}$ whose indices belong to J . By definition of U , we have for all $i, j \in \{1, \dots, k\}$ and for all $(y_1, \dots, y_k) \in U$ that $y_i - y_j \in B^d(\hat{x}_i - \hat{x}_j, 2\tau)$. This means that, for any given $y_J \in \mathbb{R}^{d|J|}$,

$$\lambda_t^{k-|J|} \left(\{y_{J^c} \in \mathbb{R}^{d(k-|J|)} : (y_J, y_{J^c}) \in U\} \right) \leq (2^d t \kappa_d \tau^d)^{k-|J|}.$$

For any $V \subset U$, this provides the second inequality in

$$\begin{aligned} \lambda_t^k(V) &\leq t^k \int_{(\mathbb{R}^d)^k} \mathbf{1}\{x_J \in \Pi_J(V)\} \mathbf{1}\{(x_1, \dots, x_k) \in U\} dx_1 \dots dx_k \\ &\leq t^{|J|} \int_{(\mathbb{R}^d)^{|J|} } \mathbf{1}\{x_J \in \Pi_J(V)\} (2^d t \kappa_d \tau^d)^{k-|J|} dx_J \\ &= \lambda_t^{|J|}(\Pi_J(V)) (2^d t \kappa_d \tau^d)^{k-|J|}. \end{aligned}$$

Consequently, for any $V \subset U$ and $\emptyset \neq J \subset \{1, \dots, k\}$ we have

$$\lambda_t^{|J|}(\Pi_J(V)) \geq (2^d t \kappa_d \tau^d)^{-(k-|J|)} \lambda_t^k(V).$$

Together with (5.6), we obtain that

$$\begin{aligned} &\min_{\emptyset \neq J \subset \{1, \dots, k\}} \inf_{\substack{V \subset U \\ \lambda_t^k(V) \geq \lambda_t^k(U)/2^{k+1}}} \lambda_t^{|J|}(\Pi_J(V)) \\ &\geq \min_{j=1, \dots, k} \left(2^d t \kappa_d \tau^d \right)^{-(k-j)} 2^{-k-1} \left(t \kappa_d \tau^d \right)^{k-1} t \ell_d(A) \\ &\geq 2^{-k-1} \min_{j=1, \dots, k} 2^{-d(k-j)} (t \kappa_d \tau^d)^{j-1} t \ell_d(A), \end{aligned}$$

which concludes the proof. □

For a family of Poisson functionals $(F_t)_{t \geq 1}$ depending on the intensity of the underlying Poisson process the following corollary ensures that the asymptotic variance does not degenerate.

Corollary 5.4 *Let $F_t \in L^2_{\eta_t}$ with a representative $f_t : \mathbf{N}_H \rightarrow \mathbb{R}$ for $t \geq 1$. Let $k \in \mathbf{N}$ and $I_1, I_2 \subset \{1, \dots, k\}$ with $I_1 \cup I_2 = \{1, \dots, k\}$ and define*

$$g_t(x_1, \dots, x_k) := \left| \mathbb{E} \left[f_t \left(\eta_t + \sum_{i \in I_1} \delta_{x_i} \right) - f_t \left(\eta_t + \sum_{i \in I_2} \delta_{x_i} \right) \right] \right|, \quad (x_1, \dots, x_k) \in \mathbb{R}^{dk},$$

for $t \geq 1$. Assume that there are $\hat{x}_1, \dots, \hat{x}_k \in \mathbb{R}^d$ such that g_1 is continuous in $(\hat{x}_1, \dots, \hat{x}_k)$ and $g_1(\hat{x}_1, \dots, \hat{x}_k) > 0$. Moreover, let $A \subset \mathbb{R}^d$ with $\ell_d(A) > 0$ and $\varepsilon > 0$ be such that

$$g_t(\hat{x}_1 + z, \hat{x}_1 + z + t^{-1/d}(\hat{x}_2 - \hat{x}_1 + y_2), \dots, \hat{x}_1 + z + t^{-1/d}(\hat{x}_k - \hat{x}_1 + y_k)) = g_1(\hat{x}_1, \hat{x}_2 + y_2, \dots, \hat{x}_k + y_k)$$

for all $z \in A \cup \{0\}$, $y_2, \dots, y_k \in B_\varepsilon^d$ and $t \geq 1$. Then there is a constant $\sigma > 0$ such that $\text{Var } F_t \geq \sigma t$ for $t \geq 1$.

Proof Define $c := g_1(\hat{x}_1, \dots, \hat{x}_k)$. For any $t \geq 1$ let $\varepsilon_t = t^{-1/d}\varepsilon$ and

$$\begin{aligned} \tau_t &:= \sup\{r \in (0, \varepsilon_t) : g_t(\hat{x}_1, \hat{x}_1 + t^{-1/d}(\hat{x}_2 - \hat{x}_1) \\ &\quad + y_2, \dots, \hat{x}_1 + t^{-1/d}(\hat{x}_k - \hat{x}_1) + y_k) > c/2 \text{ for all } y_2, \dots, y_k \in B_r^d\} \\ &= t^{-1/d} \sup\{r \in (0, \varepsilon) : g_1(\hat{x}_1, \hat{x}_2 + y_2, \dots, \hat{x}_k + y_k) > c/2 \text{ for all } y_2, \dots, y_k \in B_r^d\} \\ &= t^{-1/d} \tau_1. \end{aligned}$$

Choosing $\hat{x}_i^{(t)} = \hat{x}_1 + t^{-1/d}(\hat{x}_i - \hat{x}_1)$, $i \in \{1, \dots, k\}$, we can apply Theorem 5.3 for every $t \geq 1$, which yields the assertion. □

6 Stabilizing Poisson functionals

The following result is the main bound used in the geometric applications discussed in Sects. 7.1 and 7.2 and the underlying result of Proposition 1.4.

Theorem 6.1 *Let $F \in \text{dom } D$ with $\text{Var } F > 0$ and denote by N a standard Gaussian random variable. Assume that there are constants $c_1, c_2, p_1, p_2 > 0$ such that*

$$\mathbb{E}|D_x F|^{4+p_1} \leq c_1, \quad \lambda\text{-a.e. } x \in \mathbb{X}, \tag{6.1}$$

and

$$\mathbb{E}|D_{x_1, x_2}^2 F|^{4+p_2} \leq c_2, \quad \lambda^2\text{-a.e. } (x_1, x_2) \in \mathbb{X}^2, \tag{6.2}$$

and let $\bar{c} = \max\{1, c_1, c_2\}$. Then

$$\begin{aligned} d_W\left(\frac{F - \mathbb{E}F}{\sqrt{\text{Var } F}}, N\right) &\leq \frac{3\bar{c}}{\text{Var } F} \left[\int \left(\int \mathbb{P}(D_{x_1, x_2}^2 F \neq 0)^{p_2/(16+4p_2)} \lambda(dx_2) \right)^2 \lambda(dx_1) \right]^{1/2} \\ &\quad + \frac{\bar{c}}{(\text{Var } F)^{3/2}} \int \mathbb{P}(D_x F \neq 0)^{(1+p_1)/(4+p_1)} \lambda(dx) \end{aligned}$$

and

$$d_K\left(\frac{F - \mathbb{E}F}{\sqrt{\text{Var } F}}, N\right) \leq \frac{3\bar{c}}{\text{Var } F} \left[\int \left(\int \mathbb{P}(D_{x_1, x_2}^2 F \neq 0)^{p_2/(16+4p_2)} \lambda(dx_2) \right)^2 \lambda(dx_1) \right]^{1/2} \\ + \frac{\bar{c} \Gamma_F^{1/2}}{\text{Var } F} + \frac{2\bar{c} \Gamma_F}{(\text{Var } F)^{3/2}} + \frac{\bar{c} \Gamma_F^{5/4} + 2\bar{c} \Gamma_F^{3/2}}{(\text{Var } F)^2} \\ + \frac{\sqrt{6\bar{c}} + \sqrt{3\bar{c}}}{\text{Var } F} \left[\int \mathbb{P}(D_{x_1, x_2}^2 F \neq 0)^{p_2/(8+2p_2)} \lambda^2(d(x_1, x_2)) \right]^{1/2}.$$

with

$$\Gamma_F := \int \mathbb{P}(D_x F \neq 0)^{p_1/(8+2p_1)} \lambda(dx).$$

Proof For the proof we estimate the right-hand sides of the bounds in Theorems 1.1 and 1.2. It follows from Hölder’s inequality and the assumptions (6.1) and (6.2) that

$$\mathbb{E}(D_x F)^4 \leq \mathbb{P}(D_x F \neq 0)^{p_1/(4+p_1)} [\mathbb{E}|D_x F|^{4+p_1}]^{4/(4+p_1)} \\ \leq c_1^{4/(4+p_1)} \mathbb{P}(D_x F \neq 0)^{p_1/(4+p_1)}, \\ \mathbb{E}|D_x F|^3 \leq c_1^{3/(4+p_1)} \mathbb{P}(D_x F \neq 0)^{(1+p_1)/(4+p_1)}$$

for λ -a.e. $x \in \mathbb{X}$ and

$$\mathbb{E}(D_{x_1, x_2}^2 F)^4 \leq \mathbb{P}(D_{x_1, x_2}^2 F \neq 0)^{p_2/(4+p_2)} [\mathbb{E}|D_{x_1, x_2}^2 F|^{4+p_2}]^{4/(4+p_2)} \\ \leq c_2^{4/(4+p_2)} \mathbb{P}(D_{x_1, x_2}^2 F \neq 0)^{p_2/(4+p_2)}$$

for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$. Together with further applications of Hölder’s inequality, we obtain that

$$\gamma_1 \leq \frac{2c_1^{1/(4+p_1)} c_2^{1/(4+p_2)}}{\text{Var } F} \\ \times \left[\int (\mathbb{P}(D_{x_1, x_3}^2 F \neq 0) \mathbb{P}(D_{x_2, x_3}^2 F \neq 0))^{p_2/(16+4p_2)} \lambda^3(d(x_1, x_2, x_3)) \right]^{1/2}, \\ \gamma_2 \leq \frac{c_2^{2/(4+p_2)}}{\text{Var } F} \left[\int (\mathbb{P}(D_{x_1, x_3}^2 F \neq 0) \mathbb{P}(D_{x_2, x_3}^2 F \neq 0))^{p_2/(8+2p_2)} \lambda^3(d(x_1, x_2, x_3)) \right]^{1/2}, \\ \gamma_3 \leq \frac{c_1^{3/(4+p_1)}}{(\text{Var } F)^{3/2}} \int \mathbb{P}(D_x F \neq 0)^{(1+p_1)/(4+p_1)} \lambda(dx), \\ \gamma_4 \leq \frac{c_1^{3/(4+p_1)}}{2(\text{Var } F)^2} [\mathbb{E}(F - \mathbb{E}F)^4]^{1/4} \int \mathbb{P}(D_x F \neq 0)^{p_1/(8+2p_1)} \lambda(dx), \\ \gamma_5 \leq \frac{c_1^{2/(4+p_1)}}{\text{Var } F} \left[\int \mathbb{P}(D_x F \neq 0)^{p_1/(4+p_1)} \lambda(dx) \right]^{1/2},$$

$$\begin{aligned} \gamma_6 \leq & \frac{\sqrt{6}c_1^{1/(4+p_1)}c_2^{1/(4+p_2)}}{\text{Var } F} \left[\int \mathbb{P}(D_{x_1,x_2}^2 F \neq 0)^{p_2/(8+2p_2)} \lambda^2(d(x_1, x_2)) \right]^{1/2} \\ & + \frac{\sqrt{3}c_2^{2/(4+p_2)}}{\text{Var } F} \left[\int \mathbb{P}(D_{x_1,x_2}^2 F \neq 0)^{p_2/(4+p_2)} \lambda^2(d(x_1, x_2)) \right]^{1/2}. \end{aligned}$$

By Lemma 4.2, we have

$$\begin{aligned} \frac{\mathbb{E}(F - \mathbb{E}F)^4}{(\text{Var } F)^2} & \leq \max \left\{ 256c_1^{4/(4+p_1)} \left[\int \mathbb{P}(D_x F \neq 0)^{p_1/(8+2p_1)} \lambda(dx) \right]^2 / (\text{Var } F)^2, \right. \\ & \quad \left. 4c_1^{4/(4+p_1)} \int \mathbb{P}(D_x F \neq 0)^{p_1/(4+p_1)} \lambda(dx) / (\text{Var } F)^2 + 2 \right\} \\ & \leq \max \left\{ 256c_1^{4/(4+p_1)} \Gamma_F^2 / (\text{Var } F)^2, 4c_1^{4/(4+p_1)} \Gamma_F / (\text{Var } F)^2 + 2 \right\} \end{aligned}$$

so that

$$\gamma_4 \leq \frac{c_1^{3/(4+p_1)}}{(\text{Var } F)^{3/2}} \Gamma_F + \frac{c_1^{4/(4+p_1)}}{(\text{Var } F)^2} \Gamma_F^{5/4} + \frac{2c_1^{4/(4+p_1)}}{(\text{Var } F)^2} \Gamma_F^{3/2}.$$

Combining all these estimates concludes the proof. □

Remark 6.2 As discussed in the introduction, Poisson functionals occurring in stochastic geometry are often given in the representation

$$F = \int h(y, \eta) \eta(dy)$$

with a measurable function $h : \mathbb{X} \times \mathbf{N} \rightarrow \mathbb{R}$. Such a Poisson functional has the first and second order difference operators

$$D_x F = \int D_x h(y, \eta) \eta(dy) + h(x, \eta + \delta_x), \quad x \in \mathbb{X},$$

and

$$\begin{aligned} D_{x_1,x_2}^2 F & = \int D_{x_1,x_2}^2 h(y, \eta) \eta(dy) + D_{x_1} h(x_2, \eta + \delta_{x_2}) \\ & \quad + D_{x_2} h(x_1, \eta + \delta_{x_1}), \quad x_1, x_2 \in \mathbb{X}. \end{aligned}$$

Proof of Proposition 1.4 The assertion follows by deducing the order in t for all summands in both bounds appearing in the statement of Theorem 6.1. □

7 Applications to stochastic geometry

7.1 k -nearest neighbour graph

Let η_t be a homogeneous Poisson process of intensity $t > 0$ in a compact convex observation window $H \subset \mathbb{R}^d$ with interior points. For $k \in \mathbb{N}$ the k -nearest neighbour graph is constructed by connecting two distinct points $x, y \in \eta_t$ whenever x is one of the k -nearest neighbours of y or y is one of the k -nearest neighbours of x . In the following, we investigate for $\alpha \geq 0$ the sum $L_t^{(\alpha)}$ of the α -th powers of the edge lengths of the k -nearest neighbour graph, that is

$$L_t^{(\alpha)} = \frac{1}{2} \sum_{(x,y) \in \eta_{t,\neq}^2} \mathbf{1}\{x \text{ } k\text{-nearest neighbour of } y \text{ or vice versa}\} \|x - y\|^\alpha.$$

Here and in the following we identify simple point processes with their support and denote by $\eta_{t,\neq}^2$ the set of all pairs of distinct points of η_t . For $\alpha = 0$, $L_t^{(\alpha)}$ is the number of edges and for $\alpha = 1$ the total edge length. We are in particular interested in the asymptotic behaviour of $L_t^{(\alpha)}$ for $t \rightarrow \infty$.

Central limit theorems for the total edge length of the k -nearest neighbour graph were studied in [3,6,7,41,43,45]. The first quantitative bound for the Kolmogorov distance of order of $(\log t)^{1+3/4}t^{-1/4}$ was deduced by Avram and Bertsimas in [3]. This bound was improved to the order of $(\log t)^{3d}t^{-1/2}$ by Penrose and Yukich in [45], and the problem has remained open until now of whether the logarithmic factor in the rate of convergence could be removed at all. As shown in the following statement, the answer is indeed positive.

Theorem 7.1 *Let N be a standard Gaussian random variable. Then there are constants $C_\alpha, \alpha \geq 0$, only depending on k, H and α such that*

$$d_K \left(\frac{L_t^{(\alpha)} - \mathbb{E}L_t^{(\alpha)}}{\sqrt{\text{Var } L_t^{(\alpha)}}}, N \right) \leq C_\alpha t^{-1/2}, \quad t \geq 1.$$

We prepare the proof of Theorem 7.1 by the following asymptotic result for the variance of $L_t^{(\alpha)}$. Although it follows from the exact variance asymptotics available in the literature (see [43, Theorem 6.1], for example), we provide an independent proof, both for the sake of completeness, and in order to illustrate the application of the lower bounds for variances established in Sect. 5.

Lemma 7.2 *For any $\alpha \geq 0$ there is a constant $\sigma_\alpha > 0$ depending on k, H and α such that*

$$\text{Var } L_t^{(\alpha)} \geq \sigma_\alpha t^{1-2\alpha/d}, \quad t \geq 1.$$

Throughout the proofs of the previous results we consider the Poisson functionals $F_t = t^{\alpha/d} L_t^{(\alpha)}$. By $l_t^{(\alpha)} : \mathbf{N}_H \rightarrow \mathbb{R}$ we denote a representative of F_t . For $x \in \mathbb{R}^d$ and

$\mu \in \mathbf{N}_H$ we denote by $N(x, \mu)$ the k -nearest neighbours of x with respect to the points of μ that are distinct from x . A crucial fact for controlling the difference operators of F_t is given in the following Lemma:

Lemma 7.3 *For $x \in H$ and $\mu \in \mathbf{N}_H$ let*

$$R(x, \mu) = \max \left\{ \sup\{\|z_1 - z_2\| : z_1 \in \mu, x \in N(z_1, \mu + \delta_x), z_2 \in N(z_1, \mu)\}, \sup_{z \in N(x, \mu)} \|z - x\| \right\}.$$

Then,

$$D_x l_t^{(\alpha)}(\mu) = D_x l_t^{(\alpha)}(\mu \cap B^d(x, 3R(x, \mu))).$$

Proof Inserting the point x can generate new edges and delete existing edges. The new edges are all emanating from x and are, by definition of $R(x, \mu)$, within $B^d(x, R(x, \mu))$. An edge between two points $z_1, z_2 \in \mu$ is deleted if the following situations (i) and (ii) are simultaneously verified: (i) z_1 has x as a k -nearest neighbour and z_2 was a k -nearest neighbour of z_1 before x was added, or z_2 has x as a k -nearest neighbour and z_1 was a k -nearest neighbour of z_2 before x was added, and (ii) if x is added, z_2 is not a k -nearest neighbour of z_1 and z_1 is not a k -nearest neighbour of z_2 . Thus, the fact that the edge between z_1 and z_2 is deleted after adding x only depends on the configuration of the points contained in the set $(B^d(z_1, \|z_1 - z_2\|) \cup B^d(z_2, \|z_1 - z_2\|)) \cap \mu$ and x , which concludes the proof. \square

Proof of Lemma 7.2 In the sequel, we will use the fact that there are constants $D_{d,k}$ such that the vertices of a k -nearest neighbour graph in \mathbb{R}^d have at most degree $D_{d,k}$ (for an argument for the planar case, which can be generalized to higher dimensions, we refer to the proof of Lemma 6.1 in [43]).

Now one can choose $m \in \mathbb{N}$ and $z_1, \dots, z_m \in \mathbb{R}^d$ with $1/2 \leq \|z_i\| \leq 1, i \in \{1, \dots, m\}$, such that

$$\left| \left\{ i \in \{1, \dots, m\} : \|z_i - y\| < \max\{\|y\|, \inf_{x \in \partial B^d} \|y - x\|\} \right\} \right| \geq k + 1, \quad y \in B^d, \quad (7.1)$$

and such that all pairwise distances between z_1, \dots, z_m and the origin are different.

For $x \in \text{int}(H)$ (where $\text{int}(H)$ stands for the interior of H), $\tau > 0$ such that $B^d(x, \tau) \subset H$ and a point configuration $\mu \in \mathbf{N}_H$ the expression

$$l_1^{(\alpha)}\left(\mu + \sum_{i=1}^m \delta_{x+\tau z_i} + \delta_x\right) - l_1^{(\alpha)}\left(\mu + \sum_{i=1}^m \delta_{x+\tau z_i}\right)$$

only depends on the points of μ that are in $B^d(x, 3\tau)$ and is not affected by changes of μ outside of $B^d(x, 3\tau)$. This follows from Lemma 7.3 since (7.1) implies that $R(x, \mu + \sum_{i=1}^m \delta_{x+\tau z_i})$ is determined by the points of μ in $B^d(x, 2\tau)$ and that $R(x, \mu + \sum_{i=1}^m \delta_{x+\tau z_i}) \leq \tau$.

If $\mu(B^d(x, \tau)) = 0$, we obtain

$$l_1^{(\alpha)}\left(\mu + \sum_{i=1}^m \delta_{x+\tau z_i} + \delta_x\right) - l_1^{(\alpha)}\left(\mu + \sum_{i=1}^m \delta_{x+\tau z_i}\right) \geq k(\tau/2)^\alpha.$$

Indeed, adding the point x generates k new edges to points of z_1, \dots, z_m , whose length is at least $\tau/2$, and does not delete any edges since by (7.1) all other points keep their k -nearest neighbours. If $\mu(B^d(x, \tau)) \neq 0$, we have

$$\left| l_1^{(\alpha)}\left(\mu + \sum_{i=1}^m \delta_{x+\tau z_i} + \delta_x\right) - l_1^{(\alpha)}\left(\mu + \sum_{i=1}^m \delta_{x+\tau z_i}\right) \right| \leq D_{d,k} \tau^\alpha$$

since the balance of generated and deleted edges originating from the addition of the point x equals at most $D_{d,k}$, and each edge has length at most τ . Consequently, we have

$$\begin{aligned} & \left| \mathbb{E} \left[l_1^{(\alpha)}\left(\eta_1 + \sum_{i=1}^m \delta_{x+\tau z_i} + \delta_x\right) - l_1^{(\alpha)}\left(\eta_1 + \sum_{i=1}^m \delta_{x+\tau z_i}\right) \right] \right| \\ & \geq \exp(-\kappa_d \tau^d) k(\tau/2)^\alpha - (1 - \exp(-\kappa_d \tau^d)) D_{d,k} \tau^\alpha. \end{aligned}$$

Now it is easy to see that the right-hand side is positive if $\tau \leq \tau_0$ with some $\tau_0 > 0$. Putting $\hat{x}_1 = x$ and $\hat{x}_{i+1} = x + \tilde{\tau} z_i, i \in \{1, \dots, m\}$, with $\tilde{\tau} > 0$ sufficiently small, we have $\hat{x}_1, \dots, \hat{x}_{m+1} \in \text{int}(H)$ with $\tilde{\tau}/2 \leq \|\hat{x}_i - \hat{x}_1\| \leq \tilde{\tau}, i \in \{2, \dots, m+1\}$, different pairwise distances, $B^d(\hat{x}_1, 4\tilde{\tau}) \subset H$ and

$$\left| \mathbb{E} \left[l_1^{(\alpha)}\left(\eta_1 + \sum_{i=1}^{m+1} \delta_{\hat{x}_i}\right) - l_1^{(\alpha)}\left(\eta_1 + \sum_{i=2}^{m+1} \delta_{\hat{x}_i}\right) \right] \right| > 0.$$

We define $g_t : H^{m+1} \rightarrow \mathbb{R}, t \geq 1$, by

$$g_t(x_1, \dots, x_{m+1}) := \left| \mathbb{E} \left[l_t^{(\alpha)}\left(\eta_t + \sum_{i=1}^{m+1} \delta_{x_i}\right) - l_t^{(\alpha)}\left(\eta_t + \sum_{i=2}^{m+1} \delta_{x_i}\right) \right] \right|.$$

Obviously, we have $g_1(\hat{x}_1, \dots, \hat{x}_{m+1}) > 0$, and the different pairwise distances imply that g_1 is continuous in $(\hat{x}_1, \dots, \hat{x}_{m+1})$. Since the expectation in the definition of $g_1(\hat{x}_1, \dots, \hat{x}_{m+1})$ only depends on the points of η_1 in $B^d(\hat{x}_1, 3\tilde{\tau})$ and, by Lemma 7.3, such a property still holds if $\hat{x}_1, \dots, \hat{x}_{m+1}$ are slightly disturbed, there are a set $A \subset \mathbb{R}^d$ with $\ell_d(A) > 0$ and a constant $\varepsilon > 0$ such that

$$g_1(\hat{x}_1 + z, \hat{x}_2 + y_2 + z, \dots, \hat{x}_{m+1} + y_{m+1} + z) = g_1(\hat{x}_1, \hat{x}_2 + y_2, \dots, \hat{x}_{m+1} + y_{m+1})$$

for all $z \in A$ and $y_2, \dots, y_{m+1} \in B_\varepsilon^d$. By the scaling property of a homogeneous Poisson process and the definition of $l_t^{(\alpha)}$ we have

$$g_t(\hat{x}_1 + z, \hat{x}_1 + z + t^{-1/d}(\hat{x}_2 - \hat{x}_1 + y_2), \dots, \hat{x}_1 + z + t^{-1/d}(\hat{x}_{m+1} - \hat{x}_1 + y_{m+1})) = g_1(\hat{x}_1, \hat{x}_2 + y_2, \dots, \hat{x}_{m+1} + y_{m+1})$$

for all $z \in A \cup \{0\}$, $y_2, \dots, y_{m+1} \in B_\varepsilon^d$ and $t \geq 1$. Now Corollary 5.4 concludes the proof. \square

Moreover, we will use the following Lemma, which is shown in the proof of Lemma 2.5 in [28].

Lemma 7.4 *Let $H \subset \mathbb{R}^d$ be a compact convex set with non-empty interior. Then there is a constant $c_H > 0$ depending on H such that*

$$\ell_d(B^d(x, r) \cap H) \geq c_H r^d$$

for all $x \in H$ and $0 < r \leq \max_{z_1, z_2 \in H} \|z_1 - z_2\|$.

Proof of Theorem 7.1 We aim at applying Proposition 1.4 with $p_1 = p_2 = 1$. Note that, for $x \in H$ and $0 < r \leq \max_{z_1, z_2 \in H} \|z_1 - z_2\|$,

$$\mathbb{P}(\eta_t(B^d(x, r)) < k) \leq \sum_{i=0}^{k-1} \frac{t^i \kappa_d^i r^{id}}{i!} \exp(-t c_H r^d) \leq C \exp(-t c r^d)$$

with suitable constants $C, c > 0$, where we have used Lemma 7.4 in the first inequality. For $x, y \in H$ this implies that

$$\mathbb{P}(y \in N(x, \eta_t + \delta_y) \text{ or } x \in N(y, \eta_t + \delta_x)) \leq \tilde{C} \exp(-t \tilde{c} \|x - y\|^d) \tag{7.2}$$

with suitable constants $\tilde{C}, \tilde{c} > 0$.

For $x_1, x_2 \in H$ we put $r = \|x_1 - x_2\|$. If we assume that

- (A) $\eta_t(B^d(x_1, r/8) \cap H) \geq k + 1$,
- (B) $y \in B^d(x_1, r/8) \cap H$ for all $y \in \eta_t$ with $x_1 \in N(y, \eta_t + \delta_{x_1})$,

we have $R(x_1, \eta_t) \leq r/4$. Under the additional assumption

- (C) $x_1 \notin N(x_2, \eta_t + \delta_{x_1})$

we see that $R(x_1, \eta_t + \delta_{x_2}) = R(x_1, \eta_t)$. Consequently, Lemma 7.3 implies that $D_{x_1, x_2}^2 F_t = 0$ if the conditions (A)–(C) are satisfied. Obviously, we have

$$\mathbb{P}(x_1 \in N(x_2, \eta_t + \delta_{x_1})) \leq \tilde{C} \exp(-t \tilde{c} r^d)$$

and

$$\mathbb{P}(\eta_t(B^d(x_1, r/8) \cap H) < k + 1) \leq \bar{C} \exp(-t \bar{c} r^d)$$

with suitable constants $\bar{C}, \bar{c} > 0$. Using the Mecke formula, (7.2) and spherical coordinates, we see that

$$\begin{aligned} &\mathbb{P}(\exists y \in \eta_t \setminus B^d(x_1, r/8) : x_1 \in N(y, \eta_t + \delta_{x_1})) \\ &\leq \mathbb{E} \sum_{y \in \eta_t \setminus B^d(x_1, r/8)} \mathbf{1}(x_1 \in N(y, \eta_t + \delta_{x_1})) \\ &\leq t \int_{\mathbb{R}^d \setminus B^d(x_1, r/8)} \mathbb{P}(x_1 \in N(y, \eta_t + \delta_{x_1})) \, dy \\ &\leq t \int_{\mathbb{R}^d \setminus B^d(x_1, r/8)} \tilde{C} \exp(-t\tilde{c}\|y - x_1\|^d) \, dy \leq \hat{C} \exp(-t\hat{c}r^d) \end{aligned}$$

with suitable constants $\hat{C}, \hat{c} > 0$. Altogether, we see that there are constants $C^*, c^* > 0$ such that

$$\mathbb{P}(D_{x_1, x_2}^2 F_t \neq 0) \leq C^* \exp(-tc^*\|x_1 - x_2\|^d).$$

Combining this with spherical coordinates shows that

$$\sup_{x \in H, t \geq 1} t \int \mathbb{P}(D_{x,y}^2 F_t \neq 0)^{1/20} \, dy < \infty.$$

Let $M_1(x, \eta_t)$ be the length of the longest edge that is generated by adding the point x and let $M_2(x, \eta_t)$ be the length of the longest edge that is removed by adding the point x . It follows from the Mecke formula and (7.2) that

$$\begin{aligned} \mathbb{P}(M_1(x, \eta_t) \geq s) &\leq \mathbb{E} \sum_{y \in \eta_t \setminus B^d(x, s)} \mathbf{1}\{y \in N(x, \eta_t) \text{ or } x \in N(y, \eta_t + \delta_x)\} \\ &\leq \tilde{C}t \int_{\mathbb{R}^d \setminus B^d(x, s)} \exp(-t\tilde{c}\|x - y\|^d) \, dy \leq \tilde{C}_1 \exp(-t\tilde{c}_1 s^d) \end{aligned}$$

and

$$\begin{aligned} &\mathbb{P}(M_2(x, \eta_t) \geq s) \\ &\leq \mathbb{E} \sum_{(y_1, y_2) \in \eta_t^2, \neq} \mathbf{1}\{\|y_1 - y_2\| \geq s, y_2 \in N(y_1, \eta_t), x \in N(y_1, \eta_t + \delta_x)\} \\ &\leq t^2 \int \mathbf{1}\{\|y_1 - y_2\| \geq s\} \mathbb{P}(x \in N(y_1, \eta_t + \delta_x), y_2 \in N(y_1, \eta_t + \delta_{y_2})) \, d(y_1, y_2) \\ &\leq t^2 \int \mathbf{1}\{\|y_1 - y_2\| \geq s\} \mathbb{P}(x \in N(y_1, \eta_t + \delta_x))^{1/2} \mathbb{P}(y_2 \in N(y_1, \eta_t + \delta_{y_2}))^{1/2} \, d(y_1, y_2) \\ &\leq t^2 \tilde{C}^2 \int \int_{\mathbb{R}^d \setminus B^d(y_1, s)} \exp(-t\tilde{c}\|y_1 - y_2\|^d/2) \, dy_2 \exp(-t\tilde{c}\|y_1 - x\|^d/2) \, dy_1 \\ &\leq \tilde{C}_2 \exp(-t\tilde{c}_2 s^d) \end{aligned}$$

with suitable constants $\tilde{C}_1, \tilde{c}_1, \tilde{C}_2, \tilde{c}_2 > 0$. In a similar way we obtain that

$$\mathbb{P}(M_1(x_1, \eta_t + \delta_{x_2}) \geq s) \leq \tilde{C}_3 \exp(-t\tilde{c}_3s^d) \text{ and } \mathbb{P}(M_2(x_1, \eta_t + \delta_{x_2}) \geq s) \leq \tilde{C}_4 \exp(-t\tilde{c}_4s^d)$$

for all $x_1, x_2 \in H$ with constants $\tilde{C}_3, \tilde{c}_3, \tilde{C}_4, \tilde{c}_4 > 0$. Since at most $D_{d,k}$ edges are generated and removed by adding the point $x \in H$, we have

$$|D_x F_t| \leq D_{d,k} t^{\alpha/d} \max\{M_1(x, \eta_t), M_2(x, \eta_t)\}^\alpha$$

and, for $x_1, x_2 \in H$,

$$|D_{x_1, x_2}^2 F_t| \leq 2D_{d,k} t^{\alpha/d} \max\{M_1(x_1, \eta_t), M_2(x_1, \eta_t), M_1(x_1, \eta_t + \delta_{x_2}), M_2(x_1, \eta_t + \delta_{x_2})\}^\alpha.$$

Because of the exponential tail probabilities for the expressions in the maxima, there are constants c_1 and c_2 such that

$$\mathbb{E}|D_x F_t|^5 \leq c_1, \quad x \in H, \quad \text{and} \quad \mathbb{E}|D_{x_1, x_2}^2 F_t|^5 \leq c_2, \quad x_1, x_2 \in H.$$

Now Proposition 1.4 concludes the proof. □

7.2 Poisson-Voronoi tessellation

Let η_t be a stationary Poisson process in \mathbb{R}^d whose intensity measure is t times the Lebesgue measure. Now we can divide the whole \mathbb{R}^d into cells

$$C(x, \eta_t) = \{y \in \mathbb{R}^d : \|x - y\| \leq \|z - y\|, z \in \eta_t\}, \quad x \in \eta_t,$$

i.e. the cell with nucleus x contains all points of \mathbb{R}^d such that x is the closest point of η_t . The collection of all these cells is called Poisson-Voronoi tessellation. All its cells are (almost surely) bounded polytopes, and we let $X_t^k, k \in \{0, \dots, d\}$, denote the system of all k -faces of these polytopes. For an introduction to some fundamental mathematical properties of such tessellations, as well as for relevant definitions, see [48, Chapter 10].

Let H be a compact convex set with non-empty interior. We are interested in the normal approximation of the Poisson functionals

$$V_t^{(k,i)} := \sum_{G \in X_t^k} V_i(G \cap H),$$

where $k \in \{0, \dots, d\}, i \in \{0, \dots, \min\{k, d - 1\}\}$ and $V_i(\cdot)$ is the i th intrinsic volume (see [48]). In particular $V_t^{(d-1, d-1)}$ is the total surface content (edge length in case $d = 2$) of all cells within H , while $V_t^{(k, 0)}$ is the total number of all k -faces intersecting H . We do not allow $k = i = d$ since $V_t^{(d, d)} = \ell_d(H)$ is constant.

Central limit theorems for the functionals $V_t^{(k,i)}$ are implied by the mixing properties of the Poisson Voronoi tessellation derived by Heinrich in [18]. The Voronoi tessellation within the observation window can be also constructed with respect to a finite Poisson process in the observation window and not with respect to a stationary Poisson process. For this slightly different situation, which has the same asymptotic behaviour as the setting described above, central limit theorems were derived by stabilization techniques in [3, 6, 41, 43, 45]. Quantitative bounds on the Kolmogorov distance for the edge length in the planar case were proved by Avram and Bertsimas [3] and improved by Penrose and Yukich in [45]. These bounds of the orders of $(\log t)^{1+3/4}t^{-1/4}$ and $(\log t)^{3d}t^{-1/2}$, respectively, can be further improved as the following theorem shows.

Theorem 7.5 *Let N be a standard Gaussian random variable. Then there are constants $c_{i,k}$, $k \in \{0, \dots, d\}$, $i \in \{0, \dots, \min\{k, d - 1\}\}$, such that*

$$d_K \left(\frac{V_t^{(k,i)} - \mathbb{E}V_t^{(k,i)}}{\sqrt{\text{Var } V_t^{(k,i)}}}, N \right) \leq c_{k,i}t^{-1/2}, \quad t \geq 1.$$

Let $k \in \{0, \dots, d\}$ and $i \in \{0, \dots, \min\{k, d - 1\}\}$ be fixed in the following. In order to prove the previous theorem we consider the Poisson functionals $\tilde{V}_t^{(k,i)} = t^{i/d}V_t^{(k,i)}$ and denote by $\tilde{v}_t^{(k,i)} : \mathbb{N}_{\mathbb{R}^d} \rightarrow \mathbb{R}$ a representative of $\tilde{V}_t^{(k,i)}$. We start by proving the following lemma for the variance. More details on the asymptotic covariance structure of these random variables are provided in [26].

Lemma 7.6 *There are constants $\sigma_{k,i} > 0$, $k \in \{0, \dots, d\}$, $i \in \{0, \dots, \min\{k, d - 1\}\}$, such that*

$$\text{Var } V_t^{(k,i)} \geq \sigma_{k,i}t^{1-2i/d}, \quad t \geq 1.$$

Proof Let \hat{x}_1 be in the interior of H and let $\varepsilon = \inf_{y \in \partial H} \|\hat{x}_1 - y\|$ and $H_\varepsilon = \{x \in H : \inf_{y \in \partial H} \|y - x\| \geq \varepsilon/2\}$. Now we choose $l \in \mathbb{N}$ and points $\hat{x}_2, \dots, \hat{x}_l \in B^d(\hat{x}_1, \varepsilon/2)$ such that

$$\sup_{y \in \partial B^d(\hat{x}_1, \varepsilon/2)} \min_{i=2, \dots, l} \|y - \hat{x}_i\| < \frac{\varepsilon}{4}$$

and $\hat{x}_1, \dots, \hat{x}_l$ are in general position (see [48, p. 472]). Now we can choose a point $\hat{x}_{l+1} \in B^d(\hat{x}_1, \varepsilon/2)$ such that $\hat{x}_1, \dots, \hat{x}_{l+1}$ are still in general position and such that

$$\mathbb{E} \left[\tilde{v}_1^{(k,i)} \left(\eta_1 + \sum_{i=1}^{l+1} \delta_{\hat{x}_i} \right) - \tilde{v}_1^{(k,i)} \left(\eta_1 + \sum_{i=1}^l \delta_{\hat{x}_i} \right) \right] > 0, \tag{7.3}$$

as can be seen from the following argument. For an arbitrary $w \in \mathbb{R}^d$ with $\|w\| = 1$ we have almost surely that

$$\lim_{r \rightarrow 0} \tilde{v}_1^{(k,i)} \left(\eta_1 + \sum_{i=1}^l \delta_{\hat{x}_i} + \delta_{\hat{x}_1+rw} \right) - \tilde{v}_1^{(k,i)} \left(\eta_1 + \sum_{i=1}^l \delta_{\hat{x}_i} \right) > 0.$$

This is the case since adding $\hat{x}_1 + rw$ for sufficiently small r means that we split the cell around \hat{x}_1 in two cells which generates new faces, whereas the old faces are slightly moved (here we use the fact that the points of η_1 and the additional points are in general position almost surely). Now the dominated convergence theorem implies the same for the expectations. Putting $\hat{x}_{l+1} = \hat{x}_1 + rw$ with w such that $\hat{x}_1, \dots, \hat{x}_{l+1}$ are in general position and r sufficiently small yields (7.3). We define $g_t : H^{l+1} \rightarrow \mathbb{R}, t \geq 1$, by

$$g_t(x_1, \dots, x_{l+1}) := \mathbb{E} \left[\tilde{v}_t^{(k,i)} \left(\eta_t + \sum_{i=1}^{l+1} \delta_{x_i} \right) - \tilde{v}_t^{(k,i)} \left(\eta_t + \sum_{i=1}^l \delta_{x_i} \right) \right].$$

By (7.3) we have that $g_1(\hat{x}_1, \dots, \hat{x}_{l+1}) > 0$. Since the points $\hat{x}_1, \dots, \hat{x}_{l+1}$ are by choice in general position, g_1 is continuous in $(\hat{x}_1, \dots, \hat{x}_{l+1})$.

For $y_2, \dots, y_{l+1} \in B_{\varepsilon/4}^d, z \in H_\varepsilon - \hat{x}_1$ and $t \geq 1$ we have that

$$\begin{aligned} g_t \left(\hat{x}_1 + z, \hat{x}_1 + z + t^{-1/d}(\hat{x}_2 + y_2 - \hat{x}_1), \dots, \hat{x}_1 + z + t^{-1/d}(\hat{x}_{l+1} + y_{l+1} - \hat{x}_1) \right) \\ = g_1(\hat{x}_1, \hat{x}_2 + y_2, \dots, \hat{x}_{l+1} + y_{l+1}), \end{aligned}$$

which follows from the stationarity of η_t , the scaling property $\eta_t \stackrel{d}{=} t^{-1/d} \eta_1$ and the i -homogeneity of the i th intrinsic volume. Moreover, we have used that on both sides all changes of the tessellations occur within H , which is a consequence of the construction of $\hat{x}_1, \dots, \hat{x}_{l+1}$ and of the choice of y_2, \dots, y_{l+1} and z . Now Corollary 5.4 yields the assertion. \square

For $\mu \in \mathbb{N}_{\mathbb{R}^d}$ and $x \in \mathbb{R}^d$ we denote by $R(x, \mu)$ the maximal distance of a vertex of $C(x, \mu + \delta_x)$ to x . We define $d(x, A) = \inf_{z \in A} \|x - z\|$ for $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$.

Lemma 7.7 *There are constants $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$ such that*

$$\mathbb{P}(R(x, \eta_t) \geq s) \leq \tilde{C}_1 \exp(-t\tilde{c}_1 s^d), \quad s \geq 0, \quad x \in \mathbb{R}^d, \quad t \geq 1, \quad (7.4)$$

$$\mathbb{P}(C(x, \eta_t + \delta_x) \cap H \neq \emptyset) \leq \tilde{C}_2 \exp(-t\tilde{c}_2 d(x, H)^d), \quad x \in \mathbb{R}^d, \quad t \geq 1, \quad (7.5)$$

and

$$\begin{aligned} \mathbb{P}(C(x_1, \eta_t + \delta_{x_1}) \cap C(x_2, \eta_t + \delta_{x_2}) \neq \emptyset) \\ \leq \tilde{C}_3 \exp(-t\tilde{c}_3 \|x_1 - x_2\|^d), \quad x_1, x_2 \in \mathbb{R}^d, \quad t \geq 1. \end{aligned}$$

Proof The inequality (7.4) follows from Theorem 2 in [20]. The other bounds can be deduced from (7.4). \square

Proof of Theorem 7.5 For $x \in \mathbb{R}^d$ and $\mu \in \mathbf{N}_{\mathbb{R}^d}$ let $A_{x,\mu} = \{y \in \mu : C(x, \mu + \delta_x) \cap C(y, \mu) \neq \emptyset\}$, which is the set of all neighbour points of the cell around x . It is easy to see that all these points must be included in $B^d(x, 2R(x, \mu))$ so that

$$|A_{x,\mu}| \leq \mu(B^d(x, 2R(x, \mu))).$$

By adding the point x to μ , some k -faces of the Voronoi tessellation are changed or removed. Since each of these faces is associated with $d + 1 - k$ neighbours of x (because the tessellation is normal, see [48, Theorem 10.2.3]), at most $\mu(B^d(x, 2R(x, \mu)))^{d+1-k} k$ -faces are changed or removed. By the monotonicity of the intrinsic volumes the i -th intrinsic volume of each of these k -faces is reduced by $V_i(B^d(x, R(x, \mu)))$ at most. On the other hand, adding the point x generates some new k -faces. Each of them is associated with $d - k$ neighbours of x and their i -th intrinsic volumes are bounded by $V_i(B^d(x, R(x, \mu)))$. Altogether we see that, for $x \in \mathbb{R}^d$,

$$\begin{aligned} |D_x \tilde{v}_t^{(k,i)}(\mu)| &\leq t^{i/d} V_i(B^d(x, R(x, \mu))) \mu(B^d(x, 2R(x, \mu)))^{d+1-k} \\ &= t^{i/d} V_i(B^d(0, 1)) R(x, \mu)^i \mu(B^d(x, 2R(x, \mu)))^{d+1-k}. \end{aligned}$$

By iterating this argument and using the monotonicity of $R(x_1, \mu)$, we see that, for $x_1, x_2 \in \mathbb{R}^d$,

$$\begin{aligned} |D_{x_1} \tilde{v}_t^{(k,i)}(\mu + \delta_{x_2})| &\leq t^{i/d} V_i(B^d(0, 1)) R(x_1, \mu + \delta_{x_2})^i (\mu + \delta_{x_2}) \\ &\quad \times (B^d(x, 2R(x_1, \mu + \delta_{x_2})))^{d+1-k} \\ &\leq t^{i/d} V_i(B^d(0, 1)) R(x_1, \mu)^i (\mu(B^d(x, 2R(x_1, \mu))) + 1)^{d+1-k}. \end{aligned}$$

This implies for $x_1, x_2 \in \mathbb{R}^d$ that

$$|D_{x_1, x_2}^2 \tilde{v}_t^{(k,i)}(\mu)| \leq 2t^{i/d} V_i(B^d(0, 1)) R(x_1, \mu)^i (\mu(B^d(x, 2R(x_1, \mu))) + 1)^{d+1-k}.$$

Together with the stationarity of η_t , we obtain that the fifth moments of $|D_x \tilde{V}_t^{(k,i)}|$ and $|D_{x_1, x_2}^2 \tilde{V}_t^{(k,i)}|$ are bounded by linear combinations of the expectations

$$\mathbb{E} t^{5i/d} R(0, \eta_t)^{5i} \sum_{(y_1, \dots, y_m) \in \eta_t^m, \neq} \mathbf{1}\{y_1, \dots, y_m \in B^d(0, 2R(0, \eta_t))\},$$

for $m \in \{0, \dots, 5d + 5 - 5k\}$.

Using the Mecke formula and the monotonicity of $R(0, \mu)$, we see that the latter expression can be bounded by

$$\begin{aligned} &t^{m+5i/d} \int_{(\mathbb{R}^d)^m} \mathbb{E} R(0, \eta_t + \delta_{y_1} + \dots + \delta_{y_m})^{5i} \\ &\quad \times \mathbf{1}\{y_1, \dots, y_m \in B^d(0, 2R(0, \eta_t + \delta_{y_1} + \dots + \delta_{y_m}))\} \mathrm{d}(y_1, \dots, y_m) \end{aligned}$$

$$\begin{aligned} &\leq t^{m+5i/d} \int_{(\mathbb{R}^d)^m} \mathbb{E}R(0, \eta_t)^{5i} \mathbf{1}\{y_1, \dots, y_m \in B^d(0, 2R(0, \eta_t))\} d(y_1, \dots, y_m) \\ &= 2^{dm} \kappa_d^m t^{m+5i/d} \mathbb{E}R(0, \eta_t)^{md+5i}. \end{aligned}$$

Now (7.4) yields that the right-hand side is uniformly bounded for $t \geq 1$, whence the fifth absolute moments of the first and the second difference operator are also uniformly bounded for $t \geq 1$.

Let B_H and R_H be the circumball and the circumradius of H , respectively. We have that $D_x \tilde{V}_t^{(k,i)} = 0$ if $C(x, \eta_t + \delta_x) \cap H = \emptyset$ since in this case the tessellation in H is the same for η_t and $\eta_t + \delta_x$. Together with (7.5) and spherical coordinates, we see that

$$\begin{aligned} t \int \mathbb{P}(D_x \tilde{V}_t^{(k,i)} \neq 0)^{1/20} dx &\leq t \kappa_d R_H^d + t \int_{\mathbb{R}^d \setminus B_H} \tilde{C}_2^{1/20} \exp(-t \tilde{c}_2 d(x, H)^d / 20) dx \\ &\leq t \kappa_d R_H^d + d \kappa_d t \int_{R_H}^\infty \tilde{C}_2^{1/20} \exp(-t \tilde{c}_2 (r - R_H)^d / 20) r^{d-1} dr. \end{aligned}$$

Here, the right-hand side is bounded by a constant times t for $t \geq 1$.

Observe that $D_{x,y}^2 \tilde{V}_t^{(k,i)} = 0$ if $C(x, \eta_t + \delta_x) \cap H = \emptyset$ or $C(x, \eta_t + \delta_x) \cap C(y, \eta_t + \delta_y) = \emptyset$ since in both cases we have $D_y \tilde{v}_t^{(k,i)}(\eta_t) = D_y \tilde{v}_t^{(k,i)}(\eta_t + \delta_x)$. For $x \in \mathbb{R}^d$ combining this with Lemma 7.7 implies that

$$\begin{aligned} t \int \mathbb{P}(D_{x,y}^2 \tilde{V}_t^{(k,i)} \neq 0)^{1/20} dy &\leq \tilde{C}_2^{1/20} t \kappa_d d(x, H)^d \exp(-t \tilde{c}_2 d(x, H)^d / 20) \\ &\quad + \tilde{C}_3^{1/20} t \int_{\mathbb{R}^d \setminus B^d(x, d(x, H))} \exp(-t \tilde{c}_3 \|x - y\|^d / 20) dy. \end{aligned}$$

By using polar coordinates and estimating the first summand, we obtain that there are constants $\hat{C}, \hat{c} > 0$ such that

$$t \int \mathbb{P}(D_{x,y}^2 \tilde{V}_t^{(k,i)} \neq 0)^{1/20} dy \leq \hat{C} \exp(-t \hat{c} d(x, H)^d).$$

Now a similar calculation as above shows that

$$t \int \left(t \int \mathbb{P}(D_{x_1, x_2}^2 \tilde{V}_t^{(k,i)} \neq 0)^{1/20} dx_2 \right)^2 dx_1 \quad \text{and} \quad t^2 \int \mathbb{P}(D_{x_1, x_2}^2 \tilde{V}_t^{(k,i)} \neq 0)^{1/20} d(x_1, x_2)$$

are of order t . Now Theorem 6.1 with $p_1 = p_2 = 1$ and Lemma 7.6 conclude the proof. □

8 Functionals of Poisson shot noise random fields

We will now describe a further application of our results, dealing with non-linear functionals of stochastic functions that are obtained as integrals of a deterministic kernel with respect to a Poisson measure. As anticipated in the Introduction, when

specialised to the case of moving averages (see Sect. 8.2) our results provide substantial extensions of the findings contained in [17,37,40], which only considered linear and quadratic functionals.

8.1 General results

In this section we consider non-linear functionals of first order Wiener-Itô integrals depending on a parameter $t \in \mathbb{Y}$. For this purpose, we fix a measurable space $(\mathbb{X}, \mathcal{X})$, as well as a Poisson measure on \mathbb{X} with σ -finite intensity measure λ . We let $(\mathbb{Y}, \mathcal{Y})$ be a measurable space and let $f_t : \mathbb{X} \rightarrow \mathbb{R}, t \in \mathbb{Y}$, be a family of functions such that $(x, t) \rightarrow f_t(x)$ is jointly measurable and

$$\int |f_t(x)|^p \lambda(dx) < \infty, \quad p \in \{1, 2\}, \quad t \in \mathbb{Y}. \tag{8.1}$$

We consider the random field $(X_t)_{t \in \mathbb{Y}}$ defined by $X_t = I_1(f_t), t \in \mathbb{Y}$, where I_1 indicates the Wiener-Itô integral with respect to $\hat{\eta} = \eta - \lambda$. Using the pathwise representation

$$X_t = \int f_t(x) \eta(dx) - \int f_t(x) \lambda(dx),$$

we see that $(\omega, t) \rightarrow X_t(\omega)$ can be assumed to be jointly measurable. We are interested in the normal approximation of the random variable

$$F = \int \varphi(X_t) \varrho(dt), \tag{8.2}$$

where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a twice differentiable function and ϱ is a finite measure on \mathbb{Y} . For obvious reasons, we require that λ, f_t and φ are such that F is not almost surely constant.

A random variable of the type (8.2) is the quintessential example of a non-linear functional of the field $(X_t)_{t \in \mathbb{Y}}$. Such fields are crucial for applications, for instance: when $(\mathbb{X}, \mathcal{X}) = (\mathbb{Y}, \mathcal{Y}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)), f_t(x) = e^{i\langle t, x \rangle}$ (where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product and $i^2 = -1$) and $\eta - \lambda$ is adequately complexified, then the field $(X_t)_{t \in \mathbb{Y}}$ represents the prototypical example of a centred stationary field on \mathbb{R}^d (see e.g. [1, Sect. 5.4], [31, Sects. 5.3 and 5.4]); when $(\mathbb{X}, \mathcal{X}) = (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)), (\mathbb{Y}, \mathcal{Y}) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and $f_t(u, x) = uf(t - x)$, then $(X_t)_{t \in \mathbb{Y}}$ is a so-called moving average Lévy process. Moving average Lévy processes have gained momentum in a number of fields: for example, starting from the path-breaking paper [5], they have become relevant for the mathematical modeling of stochastic volatility in continuous-time financial models; they are also used in nonparametric Bayesian survival analysis (where they play the role of random hazard rates, see e.g. [14,36]). For some recent applications of CLTs involving linear and quadratic functionals of moving average Lévy processes in a statistical context, see e.g. [13] and the references therein. We refer the reader to

[2,54,55] for a survey of recent examples and applications of limit theorems for non-linear functionals of random fields, as well as to [16] for a collection of CLTs involving functionals of a Poisson field on the sphere, with applications to cosmological data analysis.

We assume that

$$|\varphi(r)| \leq h(r), \quad r \in \mathbb{R}, \tag{8.3}$$

$$|\varphi'(r_1 + r_2)| \leq h(r_1) + h(r_2), \quad r_1, r_2 \in \mathbb{R}, \tag{8.4}$$

$$|\varphi''(r_1 + r_2 + r_3)| \leq h(r_1) + h(r_2) + h(r_3), \quad r_1, r_2, r_3 \in \mathbb{R}, \tag{8.5}$$

with a measurable function $h : \mathbb{R} \rightarrow [0, \infty)$. We also assume that

$$C_2 := \max \left\{ \sup_{t \in \mathbb{Y}} \mathbb{E}h(X_t)^4, 1 \right\} < \infty \tag{8.6}$$

and

$$\iint \psi_t(x)^2 + \psi_t(x)^4 \lambda(dx) \varrho(dt) < \infty, \tag{8.7}$$

where

$$\psi_t(x) = C_2^{1/4} |f_t(x) + H(f_t(x))|, \quad x \in \mathbb{X}, \quad t \in \mathbb{Y},$$

and

$$H(r) := \mathbf{1}\{r \leq 0\} \int_r^0 h(s) ds + \mathbf{1}\{r > 0\} \int_0^r h(s) ds, \quad r \in \mathbb{R}.$$

For $g_1, g_2 \in L^2(\lambda)$ we write $\langle g_1, g_2 \rangle := \int g_1 g_2 d\lambda$.

Theorem 8.1 *Let F be given by (8.2) and assume that (8.1) and (8.3)–(8.7) hold and let N be a standard Gaussian random variable. Then, $F \in L^2_\eta$ and*

$$\begin{aligned} & d_K \left(\frac{F - \mathbb{E}F}{\sqrt{\text{Var } F}}, N \right) \\ & \leq \frac{36}{\text{Var } F} \left[\int \langle \psi_{t_1}, \psi_{t_3} \rangle \langle \psi_{t_2}, \psi_{t_4} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle + \langle \psi_{t_1}, \psi_{t_2} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \right. \\ & \quad \left. + \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle + \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle^2 \varrho^4(d(t_1, t_2, t_3, t_4)) \right]^{1/2} \\ & \quad + \left(\int \langle \psi_{t_1}, \psi_{t_2} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle + \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \varrho^4(d(t_1, t_2, t_3, t_4)) / (\text{Var } F)^2 + 2 \right)^{1/4} \\ & \quad \times \frac{16}{(\text{Var } F)^{3/2}} \int \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \rangle \varrho^3(d(t_1, t_2, t_3)) \\ & \quad + \frac{2\sqrt{2}}{\text{Var } F} \left[\int \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \varrho^4(d(t_1, t_2, t_3, t_4)) \right]^{1/2}. \end{aligned}$$

Proof We can of course assume that the right-hand side of the inequality in the statement is finite (otherwise, there is nothing to prove). Combining (8.3) and (8.6) with the Cauchy-Schwarz inequality, yields that $\mathbb{E} \int |\varphi(X_t)| \varrho(dt) < \infty$ and $F \in L^2_\eta$ (recall that ϱ is finite). Moreover, the subsequent calculations show that, for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$,

$$\mathbb{E} \int |\varphi(X_t + f_t(x_1))| + |\varphi(X_t + f_t(x_1) + f_t(x_2))| \varrho(dt) < \infty.$$

Therefore we obtain from (8.1) that, \mathbb{P} -a.s. and for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$,

$$D_{x_1} F = \int \varphi(X_t + f_t(x_1)) - \varphi(X_t) \varrho(dt) = \int \int_0^{f_t(x_1)} \varphi'(X_t + a) da \varrho(dt)$$

and

$$\begin{aligned} D^2_{x_1, x_2} F &= \int \varphi(X_t + f_t(x_1) + f_t(x_2)) - \varphi(X_t + f_t(x_1)) \\ &\quad - \varphi(X_t + f_t(x_2)) + \varphi(X_t) \varrho(dt) \\ &= \int \int_0^{f_t(x_1)} \int_0^{f_t(x_2)} \varphi''(X_t + a + b) da db \varrho(dt). \end{aligned}$$

Now (8.4) and (8.5) imply that, for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$,

$$|D_{x_1} F| \leq \int h(X_t) |f_t(x_1)| + H(f_t(x_1)) \varrho(dt)$$

and

$$|D^2_{x_1, x_2} F| \leq \int h(X_t) |f_t(x_1)| |f_t(x_2)| + |f_t(x_2)| H(f_t(x_1)) + |f_t(x_1)| H(f_t(x_2)) \varrho(dt).$$

Using Hölder’s inequality, Jensen’s inequality and $(r + s)^{1/4} \leq r^{1/4} + s^{1/4}$, $r, s \geq 0$, we obtain, for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$,

$$\begin{aligned} \mathbb{E}(D_{x_1} F)^4 &\leq \int \mathbb{E} \prod_{i=1}^4 (h(X_{t_i}) |f_{t_i}(x_1)| + H(f_{t_i}(x_1))) \varrho^4(d(t_1, t_2, t_3, t_4)) \\ &\leq \int \prod_{i=1}^4 \left[\mathbb{E} (h(X_{t_i}) |f_{t_i}(x_1)| + H(f_{t_i}(x_1)))^4 \right]^{1/4} \varrho^4(d(t_1, t_2, t_3, t_4)) \\ &\leq 8 \int \prod_{i=1}^4 \left[\mathbb{E} h(X_{t_i})^4 |f_{t_i}(x_1)|^4 + H(f_{t_i}(x_1))^4 \right]^{1/4} \varrho^4(d(t_1, t_2, t_3, t_4)) \\ &\leq 8 \int \prod_{i=1}^4 (C_2^{1/4} |f_{t_i}(x_1)| + H(f_{t_i}(x_1))) \varrho^4(d(t_1, t_2, t_3, t_4)) \end{aligned}$$

$$\begin{aligned} &\leq 8 \left(\int C_2^{1/4} |f_t(x_1)| + H(f_t(x_1)) \varrho(dt) \right)^4 \\ &\leq 8 \left(\int \psi_t(x_1) \varrho(dt) \right)^4 \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}(D_{x_1, x_2}^2 F)^4 &\leq \int \mathbb{E} \prod_{i=1}^4 (h(X_{t_i}) |f_{t_i}(x_1)| |f_{t_i}(x_2)| + |f_{t_i}(x_2)| H(f_{t_i}(x_1)) \\ &\quad + |f_{t_i}(x_1)| H(f_{t_i}(x_2))) \varrho^4(d(t_1, t_2, t_3, t_4)) \\ &\leq 27 \int \prod_{i=1}^4 (C_2^{1/4} |f_{t_i}(x_1)| |f_{t_i}(x_2)| + |f_{t_i}(x_2)| H(f_{t_i}(x_1)) \\ &\quad + |f_{t_i}(x_1)| H(f_{t_i}(x_2))) \varrho^4(d(t_1, t_2, t_3, t_4)) \\ &\leq 27 \left(\int \psi_t(x_1) \psi_t(x_2) \varrho(dt) \right)^4. \end{aligned}$$

We aim at applying Theorem 1.2 to $(F - \mathbb{E}F)/\sqrt{\text{Var } F}$. First of all, we exploit Lemma 4.2 to show that

$$\begin{aligned} &\frac{\mathbb{E}(F - \mathbb{E}F)^4}{(\text{Var } F)^2} \\ &\leq \max \left\{ \frac{256}{(\text{Var } F)^2} \left[\int [\mathbb{E}(D_z F)^4]^{1/2} \lambda(dz) \right]^2, \frac{4}{(\text{Var } F)^2} \int \mathbb{E}(D_z F)^4 \lambda(dz) + 2 \right\} \\ &\leq \max \left\{ 2048 \left[\iint \psi_{t_1}(x) \psi_{t_2}(x) \varrho^2(d(t_1, t_2)) \lambda(dx) \right]^2 / (\text{Var } F)^2, \right. \\ &\quad \left. 32 \iint \psi_{t_1}(x) \psi_{t_2}(x) \psi_{t_3}(x) \psi_{t_4}(x) \varrho^4(d(t_1, t_2, t_3, t_4)) \lambda(dx) / (\text{Var } F)^2 + 2 \right\} \\ &\leq \max \left\{ 2048 \int \langle \psi_{t_1}, \psi_{t_2} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle \varrho^4(d(t_1, t_2, t_3, t_4)) / (\text{Var } F)^2, \right. \\ &\quad \left. 32 \int \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \varrho^4(d(t_1, t_2, t_3, t_4)) / (\text{Var } F)^2 + 2 \right\} < \infty, \end{aligned}$$

showing, in particular, that $F \in \text{dom}D$. Using the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} \gamma_1^2 &\leq \frac{24\sqrt{6}}{(\text{Var } F)^2} \iint \psi_{t_1}(x_1) \psi_{t_2}(x_2) \psi_{t_3}(x_1) \\ &\quad \times \psi_{t_3}(x_3) \psi_{t_4}(x_2) \psi_{t_4}(x_3) \varrho^4(d(t_1, t_2, t_3, t_4)) \lambda^3(d(x_1, x_2, x_3)) \\ &= \frac{24\sqrt{6}}{(\text{Var } F)^2} \int \langle \psi_{t_1}, \psi_{t_3} \rangle \langle \psi_{t_2}, \psi_{t_4} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle \varrho^4(d(t_1, t_2, t_3, t_4)). \end{aligned}$$

Analogously, we have

$$\begin{aligned}
 \gamma_2^2 &\leq \frac{27}{(\text{Var } F)^2} \iint \psi_{t_1}(x_1)\psi_{t_1}(x_3)\psi_{t_2}(x_1)\psi_{t_2}(x_3) \\
 &\quad \times \psi_{t_3}(x_2)\psi_{t_3}(x_3)\psi_{t_4}(x_2)\psi_{t_4}(x_3) \varrho^4(d(t_1, t_2, t_3, t_4)) \lambda^3(d(x_1, x_2, x_3)) \\
 &= \frac{27}{(\text{Var } F)^2} \int \langle \psi_{t_1}, \psi_{t_2} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \varrho^4(d(t_1, t_2, t_3, t_4)), \\
 \gamma_3 &\leq \frac{8^{3/4}}{(\text{Var } F)^{3/2}} \iint \psi_{t_1}(x_1)\psi_{t_2}(x_1)\psi_{t_3}(x_1) \varrho^3(d(t_1, t_2, t_3)) \lambda(dx_1) \\
 &= \frac{8^{3/4}}{(\text{Var } F)^{3/2}} \int \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \rangle \varrho^3(d(t_1, t_2, t_3)), \\
 \gamma_4 &\leq \frac{[\mathbb{E}(F - \mathbb{E}F)^4]^{1/4}}{2(\text{Var } F)^2} \int \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \rangle \varrho^3(d(t_1, t_2, t_3)), \\
 \gamma_5^2 &\leq \frac{8}{(\text{Var } F)^2} \iint \psi_{t_1}(x_1)\psi_{t_2}(x_1)\psi_{t_3}(x_1)\psi_{t_4}(x_1) \varrho^4(d(t_1, t_2, t_3, t_4)) \lambda(dx_1) \\
 &= \frac{8}{(\text{Var } F)^2} \int \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \varrho^4(d(t_1, t_2, t_3, t_4)), \\
 \gamma_6^2 &\leq \frac{36\sqrt{6}}{(\text{Var } F)^2} \iint \psi_{t_1}(x_1)\psi_{t_2}(x_1)\psi_{t_3}(x_1) \\
 &\quad \times \psi_{t_3}(x_2)\psi_{t_4}(x_1)\psi_{t_4}(x_2) \varrho^4(d(t_1, t_2, t_3, t_4)) \lambda^2(d(x_1, x_2)) \\
 &\quad + \frac{81}{(\text{Var } F)^2} \int \psi_{t_1}(x_1)\psi_{t_1}(x_2)\psi_{t_2}(x_1)\psi_{t_2}(x_2)\psi_{t_3}(x_1)\psi_{t_3}(x_2)\psi_{t_4}(x_1)\psi_{t_4}(x_2) \\
 &\quad \times \varrho^4(d(t_1, t_2, t_3, t_4)) \lambda^2(d(x_1, x_2)) \\
 &\leq \frac{36\sqrt{6}}{(\text{Var } F)^2} \int \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle \varrho^4(d(t_1, t_2, t_3, t_4)) \\
 &\quad + \frac{81}{(\text{Var } F)^2} \int \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \varrho^4(d(t_1, t_2, t_3, t_4)).
 \end{aligned}$$

Now an application of Theorem 1.2 completes the proof. □

8.2 Moving averages

We illustrate Theorem 8.1 by focussing on stationary random fields $(X_t)_{t \in \mathbb{R}^d}$ defined in the following way. We let η be a Poisson process on $\mathbb{R} \times \mathbb{R}^d$ with intensity measure $\lambda(du, dx) = \nu(du) dx$, where the measure ν on \mathbb{R} satisfies

$$\int |u|^j \nu(du) < \infty, \quad j = 1, 2.$$

We further let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be such that

$$\int |f(x)| + f(x)^2 dx < \infty$$

and define

$$X_t = \int u f(t - x) \hat{\eta}(d(u, x)), \quad t \in \mathbb{R}^d,$$

where $\hat{\eta}(du, dx) = \eta(du, dx) - \lambda(du, dx)$. This means that $X_t = I_1(f_t)$, where $f_t(u, x) := u f(t - x)$. We are interested in the normal approximation of the random variables

$$F_T := \int_{W_T} \varphi(X_t) dt, \quad T > 0,$$

where $W_T := T^{1/d}[0, 1]^d$ is a cube with volume T .

Theorem 8.2 *Let $(X_t)_{t \in \mathbb{R}^d}$ and F_T be as above and let N be a standard Gaussian random variable. Assume that*

$$\text{Var } F_T \geq \sigma T, \quad T \geq t_0, \tag{8.8}$$

with $\sigma, t_0 > 0$ and that there are finite constants $p, \tilde{C} > 0$ such that

$$|\varphi(r)| + |\varphi'(r)| + |\varphi''(r)| \leq \tilde{C}(1 + |r|^p), \quad r \in \mathbb{R}, \tag{8.9}$$

and

$$m := 16 \int u^2 v(du) + 16 \int |u|^{4+4p} v(du) < \infty \quad \text{and} \quad C_2 := \max\{\mathbb{E}|X_0|^{4+4p}, 1\} < \infty.$$

Moreover, assume that the function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ given by

$$g(y) = (C_2^{1/4} + \tilde{C})(|f(y)| + 3^p |f(y)|^{1+p})$$

satisfies

$$M := \max \left\{ \int g(z) dz, \int \left(\int g(y - z)g(y) dy \right)^4 dz \right\} < \infty.$$

Then, there is a finite constant $C > 0$ depending on σ, m, M and t_0 such that

$$d_K \left(\frac{F_T - \mathbb{E}F_T}{\sqrt{\text{Var } F_T}}, N \right) \leq \frac{C}{\sqrt{T}}, \quad T \geq t_0.$$

Remark 8.3 Standard computations show that the assumption $M < \infty$ is satisfied in the following two standard cases: (i) f is a bounded function with compact support, and (ii) $f(x) = c \exp(-\langle v, x \rangle) \mathbf{1}\{x_i \geq 0, i = 1, \dots, d\}$, where $c \in \mathbb{R}$ and $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ with $v_i > 0$ for $i = 1, \dots, d$. In the case $d = 1$ the process X_t is called an *Ornstein-Uhlenbeck Lévy process*. See [5] and [14,36], respectively,

for applications of these processes in mathematical finance and Bayesian statistics. See [37] for a number of related CLTs involving linear and quadratic functionals of Ornstein-Uhlenbeck Lévy processes.

Proof It follows from (8.9) that the Assumptions (8.3)–(8.5) are satisfied with

$$h(r) = \tilde{C}(1 + 3^p|r|^p), \quad r \in \mathbb{R}.$$

Together with the special structure of f_t we see that

$$\begin{aligned} \psi_t(u, x) &\leq C_2^{1/4}|f_t(u, x)| + \tilde{C}(1 + 3^p|f_t(u, x)|^p)|f_t(u, x)| \\ &\leq (C_2^{1/4} + \tilde{C})(|u| + |u|^{1+p})(|f(t-x)| + 3^p|f(t-x)|^{1+p}) \\ &= (|u| + |u|^{1+p})g(t-x). \end{aligned}$$

Using $\psi_t(u, x) = \psi_{t-s}(u, x-s)$, $u \in \mathbb{R}$, $s, t, x \in \mathbb{R}^d$, this estimate and the product form of λ , we obtain that

$$\begin{aligned} &\int_{W_T^4} \langle \psi_{t_1}, \psi_{t_3} \rangle \langle \psi_{t_2}, \psi_{t_4} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle \mathbf{d}(t_1, t_2, t_3, t_4) \\ &\leq \int_{W_T} \int_{(\mathbb{R}^d)^3} \langle \psi_{t_1-t_4}, \psi_{t_3-t_4} \rangle \langle \psi_{t_2-t_4}, \psi_0 \rangle \langle \psi_{t_3-t_4}, \psi_0 \rangle \mathbf{d}(t_1, t_2, t_3) \mathbf{d}t_4 \\ &= T \int \langle \psi_{t_1}, \psi_{t_3} \rangle \langle \psi_{t_2}, \psi_0 \rangle \langle \psi_{t_3}, \psi_0 \rangle \mathbf{d}(t_1, t_2, t_3) \\ &\leq m^3 T \int g(t_1-x_1)g(t_3-x_1)g(t_2-x_2)g(-x_2)g(t_3-x_3)g(-x_3) \\ &\quad \times \mathbf{d}(x_1, x_2, x_3, t_1, t_2, t_3) \\ &= m^3 T \left(\int g(z) \mathbf{d}z \right)^6. \end{aligned}$$

In a similar way, we deduce that

$$\begin{aligned} &\int_{W_T^4} \langle \psi_{t_1}, \psi_{t_2} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \mathbf{d}(t_1, t_2, t_3, t_4) \\ &\leq T \int \langle \psi_{t_1}, \psi_{t_2} \rangle \langle \psi_{t_3}, \psi_0 \rangle \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_0 \rangle \mathbf{d}(t_1, t_2, t_3) \\ &\leq m^3 T \int g(t_1-x_1)g(t_2-x_1)g(t_3-x_2)g(-x_2) \\ &\quad \times g(t_1-x_3)g(t_2-x_3)g(t_3-x_3)g(-x_3) \mathbf{d}(x_1, x_2, x_3, t_1, t_2, t_3) \\ &= m^3 T \int g(t_1)g(t_2)g(t_3)g(-x_2)g(t_1+x_1-x_3) \\ &\quad \times g(t_2+x_1-x_3)g(t_3+x_2-x_3)g(-x_2+x_2-x_3) \mathbf{d}(x_1, x_2, x_3, t_1, t_2, t_3) \\ &= m^3 T \left(\int \left(\int g(y-z)g(y) \mathbf{d}y \right)^2 \mathbf{d}z \right)^2, \end{aligned}$$

and then also

$$\begin{aligned} & \int_{W_T^4} \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle \, d(t_1, t_2, t_3, t_4) \\ & \leq m^2 T \left(\int g(z) \, dz \right)^2 \int \left(\int g(y - z) g(y) \, dy \right)^2 \, dz, \\ & \int_{W_T^4} \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle^2 \, d(t_1, t_2, t_3, t_4) \leq m^2 T \int \left(\int g(y - z) g(y) \, dy \right)^4 \, dz, \\ & \int_{W_T^4} \langle \psi_{t_1}, \psi_{t_2} \rangle \langle \psi_{t_3}, \psi_{t_4} \rangle \, d(t_1, t_2, t_3, t_4) \leq m^2 T^2 \left(\int g(z) \, dz \right)^4, \\ & \int_{W_T^4} \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \psi_{t_4} \rangle \, d(t_1, t_2, t_3, t_4) \leq m T \left(\int g(z) \, dz \right)^4 \end{aligned}$$

and

$$\int_{W_T^3} \langle \psi_{t_1} \psi_{t_2}, \psi_{t_3} \rangle \, d(t_1, t_2, t_3) \leq m T \left(\int g(z) \, dz \right)^3.$$

Now the assertion follows from Theorem 8.1. □

The following lemma helps to check Assumption (8.8).

Lemma 8.4 *Let the assumptions of Theorem 8.2 hold and let*

$$\tilde{\varphi}(r) = \mathbb{E}[\varphi(X_0 + r) - \varphi(X_0)], \quad r \in \mathbb{R}.$$

Then,

$$\liminf_{T \rightarrow \infty} \frac{\text{Var } F_T}{T} \geq \int \left(\int \tilde{\varphi}(uf(t)) \, dt \right)^2 \nu(du) =: \sigma$$

and (8.8) is satisfied whenever $\sigma \in (0, \infty)$.

Proof We have that, for $u \in \mathbb{R}$ and $x \in \mathbb{R}^d$,

$$\mathbb{E} D_{(u,x)} F_T = \mathbb{E} \int_{W_T} \varphi(X_t + uf(t - x)) - \varphi(X_t) \, dt = \int_{W_T} \tilde{\varphi}(uf(t - x)) \, dt,$$

where we have used the stationarity of $(X_t)_{t \in \mathbb{R}^d}$. Together with (2.2), we obtain that

$$\begin{aligned} \frac{\text{Var } F_T}{T} & \geq \frac{1}{T} \iint \mathbf{1}\{t_1, t_2 \in W_T\} \tilde{\varphi}(uf(t_1 - x)) \tilde{\varphi}(uf(t_2 - x)) \, d(t_1, t_2, x) \nu(du) \\ & = \iint \tilde{\varphi}(uf(t_1)) \tilde{\varphi}(uf(t_2)) \frac{\ell_d(W_T \cap (W_T + t_1 - t_2))}{T} \, d(t_1, t_2) \nu(du). \end{aligned}$$

By the assumptions of Theorem 8.2, we see that

$$\begin{aligned} & \iint |\tilde{\varphi}(uf(t_1))\tilde{\varphi}(uf(t_2))| \mathbf{d}(t_1, t_2) \nu(\mathbf{d}u) \\ & \leq \tilde{C}^2 \iint (1 + 2^{p-1}\mathbb{E}|X_0|^p + 2^{p-1}|u|^p|f(t_1)|^p) |u| |f(t_1)| \\ & \quad \times (1 + 2^{p-1}\mathbb{E}|X_0|^p + 2^{p-1}|u|^p|f(t_2)|^p) |u| |f(t_2)| \mathbf{d}(t_1, t_2) \nu(\mathbf{d}u) < \infty. \end{aligned}$$

Now the dominated convergence theorem concludes the proof. □

In the special case where φ is strictly increasing, f satisfies $f \geq 0$ and $f \not\equiv 0$ and ν is not concentrated at the origin, we have that

$$\int \tilde{\varphi}(uf(t)) \mathbf{d}t \neq 0$$

for all $u \neq 0$ so that the previous lemma implies that $\text{Var } F_T \geq \sigma T, t \geq t_0$, with constants $\sigma, t_0 > 0$. For the Ornstein-Uhlenbeck process discussed in the introduction (see Proposition 1.5), we have that

$$\int_{-\infty}^{\infty} \tilde{\varphi}(u\mathbf{1}\{t \geq 0\}) \exp(-t) \mathbf{d}t = \int_0^{\infty} \tilde{\varphi}(u \exp(-t)) \mathbf{d}t = \int_0^u \frac{1}{r} \tilde{\varphi}(r) \mathbf{d}r.$$

Consequently, the Assumption (8.8) is satisfied if $\int_0^u \tilde{\varphi}(r) \mathbf{d}r \neq 0$ for some u in the support of ν .

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