Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 915 (2017) 285–334

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Gauge coupling field, currents, anomalies and $\mathcal{N} = 1$ super-Yang–Mills effective actions

Nicola Ambrosetti [a], Daniel Arnold [a], Jean-Pierre Derendinger [a,*], Jelle Hartong [b]

[a] *Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland*

[b] *Physique Théorique et Mathématique and International Solvay Institutes, Université Libre de Bruxelles, C.P. 231, 1050 Brussels, Belgium*

Received 14 August 2016; accepted 12 December 2016

Available online 15 December 2016

Editor: Stephan Stieberger

Abstract

Working with a gauge coupling field in a linear superfield, we construct effective Lagrangians for $\mathcal{N} = 1$ super-Yang–Mills theory fully compatible with the expected all-order behavior or physical quantities. Using the one-loop dependence on its ultraviolet cutoff and anomaly matching or cancellation of R and dilatation anomalies, we obtain the Wilsonian effective Lagrangian. With similar anomaly matching or cancellation methods, we derive the effective action for gaugino condensates, as a function of the real coupling field. Both effective actions lead to a derivation of the NSVZ β function from algebraic arguments only. The extension of results to $\mathcal{N} = 2$ theories or to matter systems is briefly considered. The main tool for the discussion of anomalies is a generic supercurrent structure with $16_B + 16_F$ operators (the $\mathcal{S}$ multiplet), which we derive using superspace identities and field equations for a fully general gauge theory Lagrangian with the linear gauge coupling superfield, and with various $U(1)_R$ currents. As a byproduct, we show under which conditions the $\mathcal{S}$ multiplet can be improved to contain the Callan–Coleman–Jackiw energy-momentum tensor whose trace measures the breaking of scale invariance.

* Corresponding author.
E-mail address: derendinger@itp.unibe.ch (J.-P. Derendinger).

http://dx.doi.org/10.1016/j.nuclphysb.2016.12.011
0550-3213/

1. Introduction

The approach which identifies coupling constants with background values of fields and superfields has proved, following Seiberg [1], a useful and powerful tool in the study of perturbative and nonperturbative properties of supersymmetric gauge theories. It has been particularly successful for $\mathcal{N}=2$ theories [2,3], using the factorization "theorem" of hypermultiplet and vector multiplet scalars and special Kähler geometry formulated in terms of a holomorphic prepotential. It is also an important ingredient in the study of perturbative and nonperturbative moduli spaces of $\mathcal{N}=1$ theories, described in terms of holomorphic invariants [4].

The situation changes if one introduces a field, the *gauge coupling field*, to describe the gauge coupling constant in an $\mathcal{N}=1$ supersymmetric gauge theory, in agreement with the fact that, as shown for instance in [5–7], the holomorphic dependence on the gauge coupling in $\mathcal{N}=1$ super Yang–Mills theory is anomalous. This anomaly is reflected in the discrepancy between the all-order running of the gauge coupling and the absence of perturbative corrections to the vacuum angle.

In other words, one cannot in general use a chiral superfield to describe the gauge coupling in $\mathcal{N}=1$ super Yang–Mills theory. We will show that the correct description is obtained using a real linear superfield which includes in its 4_B+4_F components a real scalar, the coupling field, and an antisymmetric tensor with gauge invariance. Such a tensor is in general dual to a pseudoscalar with axionic symmetry, and the linear superfield to a chiral superfield. We will also show that the anomalous dependence on the gauge coupling creates an obstruction to *analytically* perform the duality transformation, that it provides the adequate information to write all-order effective actions with the linear superfield and also how the obstruction disappears with extended $\mathcal{N}=2$ supersymmetry, where holomorphicity is relevant.

When writing effective actions, gauge-invariant operators are needed. The linear superfield introduces, besides the familiar chiral $\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}$, a second real, dimension two, operator $\hat{L}=L-2\Omega$, where Ω is the Chern–Simons superfield. With these two operators, anomaly matching or cancellation of the R and dilatation (rescaling) anomalies can be performed. As a tool, we use the appropriate supercurrent superfield equation. In the first part of this work, we construct supercurrent structures for supersymmetric gauge theories coupled to the linear superfield and study their currents and anomalies. These structures naturally involve 16_B+16_F fields, as in the $\mathcal{S}$ structure described by Komargodski and Seiberg, [8] and include both chiral and linear anomaly sources in the supercurrent superfield equation $\overline{D}^{\dot{\alpha}}J_{\alpha\dot{\alpha}}=D_\alpha X+\chi_\alpha$.

This construction of the supercurrent structure for an arbitrary simple gauge group and matter content extends our previous work [9]. We find again that the supercurrent superfield including the Belinfante energy-momentum tensor (obtained when coupling the theory to a background space-time metric) also includes the $U(1)_{\widetilde{R}}$ current with zero R charge for the chiral multiplets. We then derive supercurrent structures with arbitrary R charges for these superfields and discuss the corresponding improvement of the energy-momentum tensor. In these supercurrent structures, the sources X and χ_α depend classically on the superfields controlling in the Lagrangian the breaking of $U(1)_R$ in terms of the chiral superfield R charges and the breaking of scale invariance with scale dimensions equal to the R charges, as would be required by the superconformal algebra.

In general, the divergence of the dilatation current, which is not present in the supercurrent superfield, is the sum of the divergence of a virial current and of the trace of the energy-momentum tensor. While the sum is of course unchanged, both contributions are sensitive to improvements of the energy-momentum tensor. In particular, if there exists a Callan–Coleman–Jackiw (CCJ)

[10,11] energy-momentum tensor which cancels the virial current, a scale-invariant theory is also conformal invariant. The CCJ tensor exists for all renormalizable Lagrangians but many theories have an irreducible virial current: this is the case whenever a linear superfield is coupled to chiral and gauge superfields. This has implications for us: supercurrent structures specify the on-shell value of the energy-momentum tensor trace only. To get the divergence of the dilatation current, a specific virial current, which we derive, is needed, except if the theory would be scale invariant.

Both source superfields X and χ_α are supplemented by quantum contributions from chiral $U(1)_R$ and dilatation anomalies. These quantum corrections use both superfields $\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}$ and $L-2\Omega$. The source superfields determine the divergence of the $U(1)_R$ current and the trace $T^\mu{}_\mu$ of the energy-momentum tensor in $J_{\alpha\dot{\alpha}}$, which is not in general the divergence of the dilatation current, a point which we also carefully discuss. This is of importance since a non-trivial coupling of the linear superfield always breaks (classically) scale invariance.

We then establish two effective Lagrangians with the gauge coupling field in the linear superfield: the all-order perturbative Wilsonian Lagrangian for super-Yang–Mills theory and the effective action determining the gaugino condensate. In both cases, anomaly matching or compensation is sufficient to derive the all-order renormalization-group (RG) equation and β function originally found by Novikov, Shifman, Vainshtein and Zakharov (NSVZ) [12].

The local Wilsonian effective action is obtained from a microscopic theory by integrating short-distance physics up to distance μ^{-1}. The energy scale μ which explicitly appears in the (loop-corrected) Wilsonian action acts then as a UV cutoff. When expressed in terms of physical quantities, the Wilsonian action also depends on a second energy scale, M, the scale at which quantities like the gauge coupling are normalized. Since both μ and M are arbitrary,[1] two RG equations follow. The dependence on the scale μ is fixed by the fact that the Wilsonian effective action depends holomorphically on μ and therefore runs only to one-loop [6]. By supersymmetry (and chirality), rescaling μ is equivalent to an anomalous $U(1)_R$ transformation, or to an anomalous scale transformation. However, there is a residual dilatation anomaly which must be canceled, by RG invariance. Since it involves a non-holomorphic dependence on the coupling, it requires the use of the gauge-invariant real superfield $\hat{L}$. The corresponding anomaly counterterm encodes the dependence of the effective action on the physical coupling $g^2(M)$ identified as the background value of the lowest scalar component C of $\hat{L}$. While arbitrariness of μ leads to the expected one-loop behavior of the Wilsonian action, arbitrariness of M leads to the all-order NSVZ β function [12]. The content of the NSVZ β function is thus entirely described by the cancellation of the dilatation anomaly and the one-loop μ-dependence of the Wilsonian action.[2]

Similar anomaly matching/cancellation arguments can be used to derive an effective Lagrangian describing gaugino condensates in $\mathcal{N}=1$ super-Yang–Mills theory, as a function of the real gauge coupling field C.[3] It actually provides the effective Lagrangian version of the derivation performed by NSVZ using instanton methods [12]. The theory has two superfields, the familiar chiral $U=\langle\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}\rangle$ and the real $V=\langle\hat{L}\rangle$, related by $U=-\frac{1}{2}\overline{DD}V$ as a consequence of $\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}=-\frac{1}{2}\overline{DD}\,\hat{L}$. The effective Lagrangian is again derived by anomaly matching of the $U(1)_R$ one-loop anomaly by a chiral (F-term) counterterm using U, and anomaly cancellation of the residual dilatation anomaly by a real (D-term) counterterm using V. Since the fundamental condensate field V, which also includes the coupling field C as its lowest component, is real, the

---

[1] In general however, $M>\mu$.

[2] A supergravity based derivation of the NSVZ β function of pure super-Yang–Mills using similar anomaly matching arguments has been given long ago [13].

[3] Following and extending ref. [14].

effective scalar potential determines the modulus $|\langle\widetilde{\mathrm{Tr}}\lambda\lambda\rangle|$ of the gaugino condensate as a function of C or $g^2(M)$: perturbative anomaly arguments are not able to discretize the R-symmetry spontaneously broken by the condensate. Discretization to Z_{2N} (with $SU(N)$ gauge group) can be easily expressed in a non-perturbative superpotential in U where each allowed term can be interpreted as a k-instanton contribution. Arbitrariness of M in the effective condensate Lagrangian leads again to the all-order NSVZ β function [12].

The outline of the paper is as follows. In Section 2, we define the gauge coupling field as the lowest component C of the real linear superfield L and we introduce the gauge-invariant coupling L to the Chern–Simons superfield Ω, in the combination $\hat{L} = L - 2\Omega$. The next Section 3 discusses chiral-linear duality in $\mathcal{N} = 1$ superspace, repeating for completeness long-known arguments [15]. At this point, the main result is that the dependence on the gauge coupling field C of the super-Yang–Mills Lagrangian is not restricted by supersymmetry, that holomorphicity is not relevant and also that the vacuum angle does not depend on C. Section 4 presents the supercurrent structures for theories with linear, chiral and gauge superfields. We first derive a *natural* $16_B + 16_F$ structure including the Belinfante improved energy-momentum tensor. Tools in the derivation are superfield identities and field equations. We then show how to improve this structure to a supercurrent making the scale properties of the theory manifest and consider the case where the superpotential would be a generic function of the super-Yang–Mills superfield $\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}$. Section 4 also provides a detailed discussion of scale transformation properties and of the existence (or nonexistence) of the Callan–Coleman–Jackiw (CCJ) energy-momentum tensor [10,11]. Appendices A, B and C are in support of this Section. With this understanding of the supercurrent structures, we next consider the incorporation of anomalies. We focus on the source or anomaly superfields X and χ_α appearing in the on-shell conservation laws of the supercurrent multiplet $J_{\alpha\dot{\alpha}}$. The study of the $U(1)_R$ and scale perturbative anomalies is the subject of section 5.

Section 6 discusses the Wilsonian effective Lagrangian for pure $\mathcal{N} = 1$ super-Yang–Mills and the effective Lagrangian for gaugino condensates. In both cases, the all-order NSVZ β function is derived, using anomaly matching/cancellation only. For completeness, it also briefly shows how $\mathcal{N} = 2$ theories escape corrections beyond one-loop.

Finally, we have added a number of appendices. Appendix A reviews the properties of the supercurrent structure and its improvements in component language. Appendices B and C give relevant background information on scaling properties of the theory, on the very particular properties of a certain scale superfield denoted by Δ and on improvements of the canonical (Noether) energy-momentum tensor to the Belinfante and CCJ energy-momentum tensors. Appendix D provides the link between our supercurrent structures and the better known Ferrara–Zumino [16] structure. Finally, in Appendix E, we collect some useful formulas for the Legendre transformation which appears in linear-chiral duality.

2. The gauge coupling field

Consider the Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{g^2}\,\mathcal{L}_{\mathrm{SYM}}, \\
\mathcal{L}_{\mathrm{SYM}} &= -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \frac{i}{2}\lambda^a\sigma^\mu(D_\mu\bar{\lambda})^a - \frac{i}{2}(D_\mu\lambda)^a\sigma^\mu\bar{\lambda}^a + \frac{1}{2}D^aD^a,
\end{aligned}
\tag{2.1}
$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \frac{1}{2} f^{abc} A^b_\mu A^c_\nu\,, \tag{2.2}$$

$$(D_\mu\lambda)^a = \partial_\mu \lambda^a - \frac{1}{2} f^{abc} A^b_\mu \lambda^c\,, \tag{2.3}$$

with f^{abc} the structure constants of some simple gauge group with generators T^a, i.e.

$$[T^a, T^b] = i f^{abc} T^c\,. \tag{2.4}$$

One wants to replace the coupling g^2 by a function of a real scalar field C,

$$g^2 \qquad \longrightarrow \qquad h(C),$$

or simply by a real scalar field C. It is then easy to see that $\mathcal{N} = 1$ supersymmetry does not provide any information or condition on the function $h(C)$. The argument is as follows. Since

$$\mathcal{L}_{\mathrm{SYM}} = \frac{1}{4} \int d^2\theta\, \widetilde{\mathrm{Tr}}\, \mathcal{W}\mathcal{W} + \frac{1}{4} \int d^2\overline{\theta}\, \widetilde{\mathrm{Tr}}\, \overline{\mathcal{W}\mathcal{W}}, \tag{2.5}$$

where $\mathcal{W}_\alpha(\mathcal{A}) = -\frac{1}{4}\overline{DD}\, e^{-\mathcal{A}} D_\alpha e^{\mathcal{A}}$ is the chiral superfield of gauge curvatures,[4] one first observes that there exists a Chern–Simons real superfield Ω defined by[5]

$$\widetilde{\mathrm{Tr}}\, \mathcal{W}\mathcal{W} = \overline{DD}\,\Omega, \qquad\qquad \widetilde{\mathrm{Tr}}\, \overline{\mathcal{W}\mathcal{W}} = DD\,\Omega \tag{2.7}$$

such that its gauge variation is linear, $\overline{DD}\,\delta_{\mathrm{gauge}}\Omega = 0$. One then introduces a real linear superfield L,

$$\overline{DD}\, L = DD\, L = 0, \tag{2.8}$$

one postulates that L has gauge variation

$$\delta_{\mathrm{gauge}} L = 2\,\delta_{\mathrm{gauge}}\Omega \tag{2.9}$$

and one forms the gauge-invariant real superfield

$$\hat{L} = L - 2\,\Omega. \tag{2.10}$$

The lowest component of L is a real scalar field C and the gauge-invariant supersymmetric Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\, \mathcal{H}(\hat{L}) \tag{2.11}$$

---

4 To be precise, $\mathcal{A}$ is the Lie algebra-valued real superfield of gauge potentials, $\mathcal{A} = \mathcal{A}^a T^a_r$, with generators in some representation r normalized by $\mathrm{Tr}(T^a_r T^b_r) = T(r)\delta^{ab}$ and we use the notation

$$\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} \equiv T(r)^{-1}\,\mathrm{Tr}\,\mathcal{W}^\alpha \mathcal{W}_\alpha\,.$$

For the components of $\mathcal{A}^a$ in Wess–Zumino gauge we write

$$\mathcal{A}^a_{\mathrm{WZ}} = \theta\sigma^\mu\overline{\theta} A^a_\mu + i\theta\theta\overline{\theta}\overline{\lambda}^a - i\overline{\theta}\overline{\theta}\theta\lambda^a + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta} D^a\,. \tag{2.6}$$

We also write $A = A^a T^a_r$, $F_{\mu\nu} = F^a_{\mu\nu} T^a_r$, $D = D^a T^a_r$ and $\lambda = \lambda^a T^a_r$.

If needed, the factors $1/2$ in gauge curvatures (2.2) and covariant derivatives (2.3) can be eliminated by the rescalings $\mathcal{A} \to 2\mathcal{A}$ and $\mathcal{W}_\alpha(\mathcal{A}) \to \frac{1}{2}\mathcal{W}_\alpha(2\mathcal{A})$.

5 For a detailed study of this superfield, see ref. [17].

includes in its component expansion[6]

$$\mathcal{L} = \mathcal{H}_C(C)\,\mathcal{L}_{\rm SYM} + \ldots, \qquad \mathcal{H}_C(C) = \frac{d}{dC}\mathcal{H}(C). \tag{2.12}$$

Since the function $\mathcal{H}$ is arbitrary we have a gauge coupling field

$$\frac{1}{g^2} = \mathcal{H}_C(C) \tag{2.13}$$

and $\mathcal{N} = 1$ supersymmetry does not provide information or constraints on the gauge coupling field. Since[7]

$$\int d^2\theta d^2\overline{\theta}\,\hat{L} = -\frac{1}{8}\int d^2\theta\,\overline{DD}\hat{L} + \text{h.c.} + \text{total deriv.} = \mathcal{L}_{\rm SYM} + \text{total deriv.}, \tag{2.14}$$

the linear superfield decouples in a term linear in $\hat{L}$.

Hence, since theory (2.11) does not have a scalar potential, the field equations of the linear superfield have a (supersymmetric) solution $\hat{L}$ = constant, which allows us to identify this background value of $\hat{L}$ with the gauge coupling constant.

We will use in this work three components of the superfield $\hat{L}$:

$$\hat{L} = C + \theta\sigma^\mu\overline{\theta}\left[\frac{1}{6}\epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma} + \widetilde{\rm Tr}\,\lambda\sigma_\mu\overline{\lambda}\right] + \theta\theta\overline{\theta\theta}\left[\frac{1}{4}\Box C + \mathcal{L}_{SYM}\right] + \ldots \tag{2.15}$$

Note the presence of a gaugino axial current besides the tensor field

$$H_{\mu\nu\rho} = 3\,\partial_{[\mu}B_{\nu\rho]} - \omega_{\mu\nu\rho}, \tag{2.16}$$

where $\omega_{\mu\nu\rho}$ is the gauge Chern–Simons form, in the $\theta\sigma^\mu\overline{\theta}$ component. The Lagrangian has then kinetic terms $\sim H_{\mu\nu\rho}H^{\mu\nu\rho}$. This interaction of gauge fields with an antisymmetric tensor with gauge symmetry is a standard occurrence in higher-dimensional global and local supersymmetry and in superstring theories. It is only in four dimensions that the antisymmetric tensor can be transformed into an axion scalar coupled to $\widetilde{\rm Tr}F_{\mu\nu}\widetilde{F}^{\mu\nu}$. It seems then a natural approach to use $\hat{L}$, as we do here, to introduce a gauge coupling field since in addition it does not introduce any dependence on the background value of an axion scalar, *i.e.* any explicit dependence on the vacuum θ angle of the Yang–Mills theory.

In the context of four-dimensional effective supergravity descriptions of superstring compactifications, the role of the linear supermultiplet as the string loop-counting dilaton field has been originally shown by Cecotti, Ferrara and Villasante [17]. Its role in anomaly cancellation and in the four-dimensional Green–Schwarz mechanism [18] has been displayed in many examples, following the effective description [19] of one-loop gauge threshold corrections in simple orbifolds [20].

3. The linear superfield and chiral-linear duality

Like the chiral superfield, the linear superfield [21,15] describes four bosonic and four fermionic $(4_B + 4_F)$ off-shell field components. We use the expansion

6 All terms have at most two derivatives.

7 When dealing with Lagrangians we will sometimes omit total derivative terms when writing equalities.

$$L = C + i\theta\chi_L - i\overline{\theta}\overline{\chi}_L + \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}\theta\sigma^\mu\bar{\theta}\,h^{\nu\rho\sigma} + \frac{1}{2}\theta\theta\,\partial_\mu\chi_L\sigma^\mu\bar{\theta}$$
$$+ \frac{1}{2}\overline{\theta\theta}\,\theta\sigma^\mu\partial_\mu\overline{\chi}_L + \frac{1}{4}\theta\theta\overline{\theta\theta}\,\Box C\,, \tag{3.1}$$

where $h_{\mu\nu\rho} = 3\,\partial_{[\mu}B_{\nu\rho]}$, to solve the linearity condition (2.8). Since $B_{\mu\nu}$ with its gauge invariance $\delta B_{\mu\nu} = 2\,\partial_{[\mu}\Lambda_{\nu]}$ describes three bosons, the linear superfield does not have any scalar auxiliary field and does not generate a specific contribution to the scalar potential in a supersymmetric Lagrangian. When coupled to Ω, as in expression (2.10), or in conformal supergravity, the linear superfield L (and its bosonic components C and $B_{\mu\nu}$) has canonical scale dimension two.

In four space-time dimensions, an antisymmetric tensor with gauge invariance, as described in the linear superfield, is dual to a real scalar with axionic shift symmetry. At the Lagrangian level, the supersymmetric version exchanges a chiral and a linear superfield, and this *chiral-linear* duality corresponds to the following chain of equalities [15]:

$$\begin{aligned}
\mathcal{L} \;=\; \int d^2\theta d^2\bar{\theta}\,\mathcal{H}(\hat{L}) \;&=\; \int d^2\theta d^2\bar{\theta}\,\mathcal{H}(V) \\
&\quad + \frac{1}{8}\int d^2\theta\, S\overline{DD}(V+2\Omega) + \frac{1}{8}\int d^2\bar{\theta}\,\bar{S}DD(V+2\Omega) \\
&= \int d^2\theta d^2\bar{\theta}\left[\mathcal{H}(V) - \frac{1}{2}(S+\bar{S})V\right] + \text{derivative} \\
&\quad + \frac{1}{4}\int d^2\theta\, S\,\widetilde{\mathrm{Tr}}\,\mathcal{W}\mathcal{W} + \frac{1}{4}\int d^2\bar{\theta}\,\bar{S}\,\widetilde{\mathrm{Tr}}\,\overline{\mathcal{W}\mathcal{W}} \\
&= \int d^2\theta d^2\bar{\theta}\,\mathcal{K}(S+\bar{S}) + \text{derivative} \\
&\quad + \frac{1}{4}\int d^2\theta\, S\,\widetilde{\mathrm{Tr}}\,\mathcal{W}\mathcal{W} + \frac{1}{4}\int d^2\bar{\theta}\,\bar{S}\,\widetilde{\mathrm{Tr}}\,\overline{\mathcal{W}\mathcal{W}}.
\end{aligned} \tag{3.2}$$

In the first equality, the Lagrange multiplier chiral superfield S imposes that $V + 2\Omega$ is linear. The third equality (3.2) defines the Kähler potential of the dual theory in terms of the Legendre transformation

$$\mathcal{K}(S+\bar{S}) = \mathcal{H}(V) - \frac{1}{2}(S+\bar{S})V \tag{3.3}$$

exchanging variables V and $S+\bar{S}$, *i.e.* with V expressed as a function of $S+\bar{S}$ by solving the usual relations

$$\frac{d\mathcal{H}}{dV} = \frac{1}{2}(S+\bar{S}), \qquad\qquad \frac{d\mathcal{K}}{d(S+\bar{S})} = -\frac{1}{2}V. \tag{3.4}$$

The resulting chiral theory has axionic shift symmetry $\delta S = ia$ (a is a real constant).

Some comments are appropriate. Firstly, all information on the function $\mathcal{H}$ goes into the Kähler potential $\mathcal{K}$. The dual holomorphic gauge kinetic function is always S and the dual gauge coupling constant is [8]

[8] We can replace S by a (non constant) function $f(S)$ in equalities (3.2).

$$\frac{1}{g^2} = \mathrm{Re}\, s \tag{3.5}$$

for all functions $\mathcal{H}$. Secondly, the Legendre transformation exchanges a *real* with a *chiral* superfield, with axionic symmetry on S dual to the gauge invariance of $B_{\mu\nu}$. The shift symmetry has an important consequence. Defining the Yang–Mills vacuum angle as

$$\langle \mathrm{Im}\, s\rangle = -\frac{\theta}{8\pi^2}, \tag{3.6}$$

its contribution to the Lagrangian

$$-\frac{\theta}{32\pi^2}\,\widetilde{\mathrm{Tr}}[F_{\mu\nu}\widetilde{F}^{\mu\nu} - 2\,\partial_\mu(\lambda\sigma^\mu\bar{\lambda})]$$

is a derivative irrespective of $\mathcal{H}$. Hence, the all-order dependence on the gauge coupling and the absence of θ-dependence in perturbation theory are fully compatible with supersymmetry. Thirdly, the linear superfield does not have an auxiliary field: $B_{\mu\nu}$ describes three off-shell fields and one on-shell helicity zero state. In the dual chiral version, S has a complex auxiliary field f_S which vanishes in theory (3.2). In theories with additional matter chiral superfields Φ, the auxiliary field f_S is a well-defined linear combination of the auxiliary f_Φ in Φ. Hence, if S is dual to a linear superfield, its auxiliary f_S does not generate an independent contribution to the scalar potential and this has clearly implications on the vacuum properties.[9]

Finally, notice that we may also add a term proportional to $\mathcal{L}_{SYM}$ (and then independent from L or S) to theory (3.2). Doing this adds a constant term to g^{-2} which is then a one-loop correction. Hence, there is no information in the holomorphic coupling S, it is naturally defined up to a one-loop correction only and its relation to the original coupling field C is fully included in the Legendre transformation (3.3).

Since Ω has canonical scale dimension two, this is also the case for L and V in the equalities (3.2). Then, the natural canonical dimension of the chiral S is zero. The quantity

$$\Delta \equiv 2V\mathcal{H}_V - 2\mathcal{H}$$

measures the violation of scale invariance in the original linear multiplet theory. But, according to the Legendre transformation (3.3) and (3.4),

$$\Delta = -2\mathcal{K}$$

as expected if the scale dimension of S is zero. Hence, imposing scale invariance $\Delta = 0$ leads to $\mathcal{H}(\hat{L}) \propto \hat{L}$ which is super-Yang–Mills theory with a constant coupling, *i.e.* in which L or S are absent. Clearly, this restriction is the obvious statement that there is no scale-invariant propagating gauge coupling field, in the absence of another dimensionful field. Hence, we expect to always find a classical contribution induced by the gauge coupling field to the divergence of the dilatation current.

4. Supercurrent superfields

In this Section, we consider a $\mathcal{N} = 1$ theory for chiral superfields Φ in some representation r of the gauge group,[10] gauge superfields $\mathcal{A}$, $\mathcal{W}_\alpha$, Ω, as defined earlier, and the linear gauge coupling superfield L. These superfields carry linear representations of Poincaré supersymmetry, but

9 Reference [22] discusses this point.

10 We suppress i indices on Φ^i and $\bar{\Phi}_i$.

they actually carry representations of the full $\mathcal{N}=1$ superconformal algebra $SU(2,2|1)$ even if dynamical equations respect in general only Poincaré supersymmetry. In other words, fields in the theory have well-defined transformation properties under the superconformal algebra, the variation of the action under these transformations is well-defined, but the invariance of the theory is in general generated by the super-Poincaré subalgebra only. Since the bosonic subalgebra of $SU(2,2|1)$ is

$$SU(2,2)\times U(1)_R \supset SO(1,3)_{\text{Lorentz}}\times SO(1,1)_{\text{dil}}\times U(1)_R\,,$$

we may then assign two abelian quantum numbers to all fields, superspace coordinates and superfields, a chiral charge q for $U(1)_R$ transformations, and a scale dimension w for dilatations $SO(1,1)_{\text{dil}}$. As far as the super-Poincaré symmetry is concerned, q and w are arbitrary. But the superconformal algebra introduces further constraints: $w=q$ for chiral superfields[11] and canonical scale dimensions for gauge superfields.

In addition, unitarity of the quantum theory would introduce further constraints (unitarity bounds) [23]. We are not concerned with them as long as we consider the theory as classical.

The assigned chiral and scale charges are then as follows[12]:

$$\begin{aligned}&\Phi:(q,w), \qquad \overline{\Phi}:(-q,w), \qquad L:(0,2), \qquad \mathcal{A}:(0,0),\\ &\mathcal{W}_\alpha:(3/2,3/2), \qquad \Omega:(0,2).\end{aligned}$$

The charges of L are as required by $\hat{L}=L-2\Omega$. If the representation of the chiral superfield is reducible, $r=\oplus_i r_i$, charges (q_i,w_i) are assigned. The Lagrangian describing the dynamics of these superfields includes in general $U(1)_R$ and scale symmetry violating terms. In addition, a non-R abelian chiral algebra may act with charge t or t_i on the chiral superfields.[13]

Since we will later on be concerned with quantum anomalies in $U(1)_R$ and dilatation transformations, the natural setup is to establish a supercurrent structure, *i.e.* a supercurrent superfield [16] $J_{\alpha\dot{\alpha}}$, anomaly superfields and the associated supercurrent equation. The supercurrent superfield is primarily defined to include the conserved supercurrent and energy-momentum tensor. It is defined up to improvement transformations. In this section, our goal is first to construct supercurrent structures for theories with a coupling field and then to establish how these transformations encode the relation of the supercurrent structure with the assigned chiral and dilatation weights. This will be done for generic super-Poincaré theories with scale-invariant, conformal or R-symmetric theories appearing as particular cases.

We begin with a detailed discussion of the supercurrent structures in a theory with chiral, gauge and linear superfields. Some aspects have been studied by Magro, Sachs and Wolf [24].[14] Supplementary formulas are provided in Appendix A.

4.1. A superfield identity

Consider the gauge-invariant real superfield

[11] In our convention.

[12] In our convention, the Grassmann coordinates have weights $(q,w)=(3/2,-1/2)$ while for gauginos $(q,w)=(3/2,3/2)$.

[13] Non-abelian chiral groups will be mostly irrelevant to us.

[14] Our discussion in this section generalizes some of the results of our earlier article [9], which can be recovered by decoupling the linear superfield. Identical notations are used.

$$\mathcal{H} = \mathcal{H}(\hat{L}, Y) \qquad\qquad Y = \overline{\Phi} e^{\mathcal{A}} \Phi. \tag{4.1}$$

In Y, the real gauge superfield is Lie algebra-valued, $\mathcal{A} = \mathcal{A}^a T_r^a$, with generators T_r^a in the representation r of the matter chiral superfield Φ. Gauge transformations are

$$\Phi \quad \longrightarrow \quad e^{\Lambda}\,\Phi, \qquad \overline{\Phi} \quad \longrightarrow \quad \overline{\Phi}\, e^{\overline{\Lambda}}, \qquad e^{\mathcal{A}} \quad \longrightarrow \quad e^{-\overline{\Lambda}} e^{\mathcal{A}} e^{-\Lambda} \tag{4.2}$$

with $\Lambda = \Lambda^a T_r^a$ and $\overline{D}_{\dot\alpha}\Lambda = 0$. Gauge-covariant superspace derivatives read

$$\mathcal{D}_\alpha \Phi = e^{-\mathcal{A}}(D_\alpha e^{\mathcal{A}} \Phi), \qquad\qquad \overline{\mathcal{D}}_{\dot\alpha} \overline{\Phi} = (\overline{D}_{\dot\alpha} \overline{\Phi} e^{\mathcal{A}}) e^{-\mathcal{A}} \tag{4.3}$$

and

$$(\overline{\mathcal{D}}_{\dot\alpha} \overline{\Phi}) e^{\mathcal{A}} (\mathcal{D}_\alpha \Phi) = (\overline{D}_{\dot\alpha} \overline{\Phi} e^{\mathcal{A}}) e^{-\mathcal{A}} (D_\alpha e^{\mathcal{A}} \Phi)$$

is gauge invariant.[15]

By direct calculation of, for instance, $\overline{DD}D_\alpha(\mathcal{H} - \hat{L}\mathcal{H}_L)$, the following identity can be derived:

$$\begin{aligned} 2\overline{D}^{\dot\alpha}\Big[(\overline{\mathcal{D}}_{\dot\alpha} \overline{\Phi}) \mathcal{H}_{\Phi\overline{\Phi}} (\mathcal{D}_\alpha \Phi) - \mathcal{H}_{LL} (\overline{D}_{\dot\alpha} \hat{L})(D_\alpha \hat{L}) \Big] \\ = -\hat{L}\, \overline{DD} D_\alpha \mathcal{H}_L - (\overline{DD}\mathcal{H}_\Phi)\, \mathcal{D}_\alpha \Phi - \overline{DD} D_\alpha (\mathcal{H} - \hat{L}\mathcal{H}_L) \\ -2\,\widetilde{\mathrm{Tr}}\, \mathcal{W}\mathcal{W}\, D_\alpha \mathcal{H}_L - 4\,\mathcal{H}_Y\, \overline{\Phi} e^{A} \mathcal{W}_\alpha \Phi, \end{aligned} \tag{4.4}$$

where subscripts indicate derivatives of $\mathcal{H}$ with respect to either Φ, $\overline{\Phi}$, L or Y. We stress that eq. (4.4) is merely an identity, without any information content. The next step is to consider a theory for $\hat{L}$ and Φ and to use its field equations to rearrange identity (4.4) into a supercurrent equation.

4.2. *The natural (Belinfante) supercurrent structure*

Let us hence consider the theory

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\, \mathcal{H}(\hat{L}, Y) + \int d^2\theta\, W(\Phi) + \int d^2\overline{\theta}\, \overline{W}(\overline{\Phi}). \tag{4.5}$$

Gauge invariance of the holomorphic superpotential $W(\Phi)$, i.e.

$$W_{\Phi^i} (T_r^a)^i{}_j \Phi^j = 0\,, \tag{4.6}$$

implies $W_\Phi \mathcal{D}_\alpha \Phi = D_\alpha W$. The $\mathcal{H}$ term in the Lagrangian has in general several chiral symmetries. In particular, since $\mathcal{H}$ satisfies

$$\mathcal{H}_\Phi \Phi = \overline{\Phi} \mathcal{H}_{\overline{\Phi}} = \mathcal{H}_Y Y, \tag{4.7}$$

it is always invariant under the non-R $U(1)$ symmetry rotating all chiral superfields Φ by the same phase.[16] Its chiral symmetries also include the R symmetry (that we call $\widetilde{R}$) which transforms Grassmann coordinates and leaves all superfields in $\hat{L}$ or Y inert. These chiral symmetries are in general broken by the superpotential.

---

15 In general, the gauge invariant function $\mathcal{H}$ can depend on variables Y_i if the representation of the chiral superfields is reducible, $r = \oplus_i r_i$. This generalization is straightforward. It may also depend on other gauge invariant quantities, such as holomorphic invariants, which we do not consider here.

16 If the representation of the matter superfields is reducible, each irreducible component has an associated $U(1)$ global symmetry. It extends to $U(n)$ factors if the matter superfields include n copies of an irreducible component.

For completeness, the component expansion of theory (4.5) is as follows[17]:

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{2}\mathcal{H}_{CC}\Big[\tfrac{1}{2}(\partial_\mu C)(\partial^\mu C) + \tfrac{1}{12}H_{\mu\nu\rho}H^{\nu\mu\rho}\Big] + \mathcal{H}_{z\bar{z}}\Big[(D_\mu \bar{z})(D^\mu z) + \overline{f}f\Big] \\ &+ \mathcal{H}_C\Big[-\tfrac{1}{4}\widetilde{\mathrm{Tr}}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\widetilde{\mathrm{Tr}}DD\Big] + \tfrac{1}{2}\mathcal{H}_z Dz - W_z f - \overline{f}\,\overline{W}_{\bar{z}} \\ &+ \tfrac{i}{12}\epsilon_{\mu\nu\rho\sigma}H^{\mu\nu\rho}\Big[\mathcal{H}_{Cz}D^\sigma z - \mathcal{H}_{C\bar{z}}D^\sigma \bar{z}\Big] \\ &+ \text{fermion terms}\,,\end{aligned} \tag{4.8}$$

where

$$(D_\mu z)^i = \partial_\mu z^i + \frac{i}{2}A^a_\mu (T^a_r)^i{}_j z^j\,, \tag{4.9}$$

$$H_{\mu\nu\rho} = h_{\mu\nu\rho} - \omega_{\mu\nu\rho}\,, \tag{4.10}$$

in which ω is the Chern–Simons form normalized such that

$$dH = -\widetilde{\mathrm{Tr}}F\wedge F\,. \tag{4.11}$$

The kinetic metrics are then $\mathcal{H}_{z\bar{z}}$, $-\frac{1}{2}\mathcal{H}_{CC}$ and $\mathcal{H}_C$ for the components of superfields Φ, L and $\mathcal{W}_\alpha$ respectively.

The field equations for theory (4.5) are[18]

$$\begin{aligned} L: &\quad \overline{DD}D_\alpha\mathcal{H}_L = 0, \\ \Phi: &\quad \overline{DD}\mathcal{H}_\Phi = 4\,W_\Phi, \\ \mathcal{A}: &\quad \overline{D}^{\dot\alpha}\Big[\mathcal{H}_L\,e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}}\Big] = \mathcal{W}^\alpha\,D_\alpha\mathcal{H}_L - T(r)\,\mathcal{H}_Y\,\Phi\overline{\Phi}e^{\mathcal{A}},\end{aligned} \tag{4.12}$$

with index $\mathrm{Tr}(T^a_r T^b_r) = T(r)\delta^{ab}$.

To derive the field equation for the gauge superfield $\mathcal{A}$, it is indeed easier to use the dual chiral version of the theory,[19]

$$\begin{aligned}\mathcal{L} = &\int d^2\theta d^2\bar\theta\,\mathcal{K}(S+\bar{S},Y) + \int d^2\theta\left[W(\Phi) + \frac{1}{4}S\,\widetilde{\mathrm{Tr}}\,\mathcal{W}\mathcal{W}\right] \\ &+ \int d^2\bar\theta\left[\overline{W}(\overline{\Phi}) + \frac{1}{4}\bar{S}\,\widetilde{\mathrm{Tr}}\,\overline{\mathcal{W}\mathcal{W}}\right],\end{aligned} \tag{4.13}$$

and to apply on the resulting field equation the Legendre transformation into the linear version. Variation of eq. (4.13) and use of the Bianchi identity

$$D^\alpha(e^{\mathcal{A}}\mathcal{W}_\alpha e^{-\mathcal{A}}) = e^{\mathcal{A}}\overline{D}_{\dot\alpha}(e^{-\mathcal{A}}\overline{\mathcal{W}}^{\dot\alpha}e^{\mathcal{A}})e^{-\mathcal{A}} \tag{4.14}$$

gives then the field equation

$$(S+\bar{S})\,\overline{D}_{\dot\alpha}(e^{-\mathcal{A}}\overline{\mathcal{W}}^{\dot\alpha}e^{\mathcal{A}}) = -(D^\alpha S)\,\mathcal{W}_\alpha - (\overline{D}_{\dot\alpha}\bar{S})\,e^{-\mathcal{A}}\overline{\mathcal{W}}^{\dot\alpha}e^{\mathcal{A}} + 2\,T(r)\,\mathcal{K}_Y\,\Phi\overline{\Phi}e^{\mathcal{A}}. \tag{4.15}$$

---

[17] Gauge invariance of $\mathcal{H}$ implies $\mathcal{H}_z Dz = \bar{z}D\mathcal{H}_{\bar{z}}$.

[18] We use the convention $\overline{\mathcal{W}}_{\dot\alpha} = \frac{1}{4}DDe^{\mathcal{A}}\overline{D}_{\dot\alpha}e^{-\mathcal{A}}$, with $\overline{\mathcal{W}}_{\dot\alpha} = -(\mathcal{W}_\alpha)^\dagger$.

[19] To avoid dealing with the complicated non-Abelian Chern–Simons superfield [17].

It can be rewritten

$$\overline{D}^{\dot\alpha}\Big[(S+\overline{S})\,e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}}\Big]=D^{\alpha}(S+\overline{S})\,\mathcal{W}_{\alpha}-2\,T(r)\,\mathcal{K}_Y\,\Phi\overline{\Phi}e^{\mathcal{A}}. \tag{4.16}$$

Multiplying by $\mathcal{W}_\beta$ and taking the trace gives

$$\overline{D}^{\dot\alpha}\Big[(S+\overline{S})\,\mathrm{Tr}(\mathcal{W}_\beta e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}})\Big]=\frac{1}{2}D_\beta(S+\overline{S})\;\mathrm{Tr}\,\mathcal{W}\mathcal{W}+2\,T(r)\,\mathcal{K}_Y\,\overline{\Phi}e^{\mathcal{A}}\mathcal{W}_\beta\Phi. \tag{4.17}$$

The Legendre transformation indicates then that $\mathcal{K}_Y=\mathcal{H}_Y$ and $S+\overline{S}=2\mathcal{H}_L$, which in turn implies the field equation (4.12) for $\mathcal{A}$ and the relation

$$\overline{D}^{\dot\alpha}\Big[\mathcal{H}_L\,\mathrm{Tr}(\mathcal{W}_\beta e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}})\Big]=\frac{1}{2}D_\beta\mathcal{H}_L\;\mathrm{Tr}\,\mathcal{W}\mathcal{W}+T(r)\mathcal{H}_Y\,\overline{\Phi}e^{\mathcal{A}}\mathcal{W}_\beta\Phi. \tag{4.18}$$

With field equations (4.12) and relation (4.18), identity (4.4) finally leads to the supercurrent structure[20]

$$\begin{aligned}
\overline{D}^{\dot\alpha}J_{\alpha\dot\alpha}&=D_\alpha X+\chi_\alpha,\\
J_{\alpha\dot\alpha}&=-2\big[(\overline{\mathcal{D}}_{\dot\alpha}\overline{\Phi})\mathcal{H}_{\Phi\overline{\Phi}}(\mathcal{D}_\alpha\Phi)-\mathcal{H}_{LL}(\overline{D}_{\dot\alpha}\hat{L})(D_\alpha\hat{L})+2\,\mathcal{H}_L\widetilde{\mathrm{Tr}}(\mathcal{W}_\alpha e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}})\big],\\
X&=4\,W,\\
\chi_\alpha&=\overline{DD}D_\alpha(\mathcal{H}-\hat{L}\mathcal{H}_L).
\end{aligned} \tag{4.19}$$

This supercurrent structure can be considered as natural for theory (4.5). It actually also applies if $\mathcal{H}$ is simply a gauge-invariant function of $\hat{L}$, Φ and $\overline{\Phi}e^{\mathcal{A}}$, instead of a function of $\hat{L}$ and Y.

In the supercurrent structure (4.19), field equations have not been used to generate from identity (4.4) the source superfield χ_α and the chiral or linear contributions to the supercurrent superfield $J_{\alpha\dot\alpha}$. Field equations for $\mathcal{A}$, Φ and L have been respectively used to generate the gauge supercurrent term,[21] the chiral source X and to eliminate the first term in the right-hand side of identity (4.4).

Using expansion (A.5) for the superfield $J_\mu=(\overline{\sigma}_\mu)^{\dot\alpha\alpha}J_{\alpha\dot\alpha}$, we find that the supercurrent superfield (4.19) contains the following lowest component:

$$j_\mu^{\widetilde{R}}\equiv\frac{3}{8}(\overline{\sigma}_\mu)^{\dot\alpha\alpha}J_{\alpha\dot\alpha}\big|_{\theta=0}=-\frac{3}{2}\mathcal{H}_{z\bar z}\,\psi\sigma_\mu\overline{\psi}+\frac{3}{4}\mathcal{H}_{CC}\,\chi\sigma_\mu\overline{\chi}+\frac{3}{2}\mathcal{H}_C\,\widetilde{\mathrm{Tr}}\lambda\sigma_\mu\bar\lambda\,, \tag{4.20}$$

where we use the expansions

$$\hat{L}=C+i\theta\chi-i\overline{\theta}\overline{\chi}+\ldots,\qquad \Phi=z+\sqrt{2}\theta\psi-\theta\theta f+\ldots,\qquad \mathcal{W}_\alpha=-i\lambda_\alpha+\ldots$$

(and $\overline{\mathcal{W}}_{\dot\alpha}=-i\bar\lambda_{\dot\alpha}+\ldots$). This is the Noether current of $\widetilde{R}$-transformations with chiral charges $-3/2$, $-3/2$ and $3/2$ for χ, ψ and λ respectively. The chiral charges of superfields Φ, L and $\mathcal{W}_\alpha$ for this $U(1)_{\widetilde{R}}$ are then $q=0,\,0,\,3/2$ in this supercurrent structure and $U(1)_{\widetilde{R}}$ only acts on the Grassmann coordinates. It is an automatic symmetry of D-term Lagrangians and, according to the second eq. (A.6), the $\widetilde{R}$ current is conserved if the superpotential vanishes, $\partial^\mu j_\mu^{\widetilde{R}}=-\frac{3}{2}\,\mathrm{Im}\,f_X$.

20 The superfields $J_{\alpha\dot\alpha}$, X and χ_α can be calculated directly from the Lagrangian. They are then defined off-shell, but field equations (4.12) can be used to reformulate them since the superfield equation $\overline{D}^{\dot\alpha}J_{\alpha\dot\alpha}=D_\alpha X+\chi_\alpha$ only holds on-shell.

21 Field equations are needed to derive the Yang–Mills Belinfante energy-momentum tensor from the canonical tensor.

The supercurrent superfield $J_{\alpha\dot{\alpha}}$ of eqs. (4.19) contains the Belinfante improved (symmetric, gauge-invariant) energy-momentum tensor $T_{\mu\nu}$ for theory (4.5). Omitting fermions and gauge fields, its expression is

$$\begin{aligned} T_{\mu\nu} &= -\tfrac{1}{2}\mathcal{H}_{CC}(\partial_\mu C)(\partial_\nu C) - \tfrac{1}{4}\mathcal{H}_{CC}h_{\mu\rho\sigma}h_\nu{}^{\rho\sigma} + \mathcal{H}_{z\bar{z}}[(\partial_\mu z)(\partial_\nu \bar{z}) + (\partial_\nu z)(\partial_\mu \bar{z})] \\ &\quad - \eta_{\mu\nu}\Big(-\tfrac{1}{4}\mathcal{H}_{CC}(\partial_\rho C)(\partial^\rho C) - \tfrac{1}{24}\mathcal{H}_{CC}h_{\rho\sigma\lambda}h^{\rho\sigma\lambda} + \mathcal{H}_{z\bar{z}}[(\partial_\rho z)(\partial^\rho \bar{z}) + \overline{f}f]\Big) \\ &\quad + \tfrac{1}{2}\eta_{\mu\nu}\mathcal{H}_C\widetilde{\mathrm{Tr}}(D^2) + \tfrac{1}{2}\eta_{\mu\nu}\,\mathrm{Re}\, f_X, \end{aligned} \tag{4.21}$$

with auxiliary fields[22]

$$f_X = 4W_z f, \qquad \overline{f}\mathcal{H}_{\bar{z}z} = W_z, \qquad D^a = -\frac{1}{2}\mathcal{H}_C^{-1}\mathcal{H}_z T_r^a z = -\frac{1}{2}\mathcal{H}_C^{-1}\mathcal{H}_Y \bar{z} T_r^a z.$$

Notice that terms depending on $\mathcal{H}_{Cz}$ or $\mathcal{H}_{C\bar{z}}$ present in the Lagrangian do not appear in the Belinfante tensor $T_{\mu\nu}$. If the superpotential vanishes, as we will often assume, we have $f = f_X = 0$.

4.3. Scale transformations

The supercurrent superfield $J_{\alpha\dot{\alpha}}$ includes the $U(1)_{\widetilde{R}}$ current and the Belinfante energy-momentum tensor which can then be viewed as partners under Poincaré supersymmetry. The superconformal algebra, besides $U(1)_R$ transformations, also includes scale transformations, but the dilatation current is not present in $J_{\alpha\dot{\alpha}}$.

To discuss the behavior of the theory under scale transformations, we first use that the source superfield χ_α contributes to the trace of the Belinfante energy-momentum tensor, according to the first eq. (A.6). We then define the real superfield

$$\Delta_{(0)} = 2\hat{L}\mathcal{H}_L - 2\mathcal{H}, \qquad\qquad \chi_\alpha = -\frac{1}{2}\overline{DD}D_\alpha\Delta_{(0)}. \tag{4.22}$$

Then, using the field equation for C, the quantity

$$\delta_{(0)} \equiv \frac{\partial\mathcal{L}}{\partial C}2C + \frac{\partial\mathcal{L}}{\partial\partial^\mu C}3\partial^\mu C + \frac{\partial\mathcal{L}}{\partial h^{\mu\nu\rho}}3h^{\mu\nu\rho} + \frac{\partial\mathcal{L}}{\partial\partial^\mu z}\partial^\mu z + \frac{\partial\mathcal{L}}{\partial\partial^\mu \bar{z}}\partial^\mu \bar{z} - 4\mathcal{L}, \tag{4.23}$$

which is the variation of the bosonic Lagrangian under a scale transformation with scale dimensions $w = 2$ for L and $w = 0$ for Φ, verifies

$$\begin{aligned} \delta_{(0)} &= -\partial^\mu[C\mathcal{H}_{CC}\partial_\mu C] + T^\mu{}_\mu \\ &= -\frac{1}{2}\partial^\mu\left[\frac{\partial}{\partial C}\Delta_{(0)}\Big|_{\theta=0}\partial_\mu C\right] + T^\mu{}_\mu. \end{aligned} \tag{4.24}$$

The expression for $\delta_{(0)}$ is the Lagrangian variation under a dilatation of the fields at fixed coordinates x (or at $x = 0$), and this last equation suggests to define a dilatation current

[22] The auxiliary field contribution to $T_{\mu\nu}$ is $\eta_{\mu\nu}V$, where V is the usual scalar potential

$$V(C, z, \bar{z}) = \frac{1}{2}\mathcal{H}_C\widetilde{\mathrm{Tr}}D^2 + \mathcal{H}_{\bar{z}z}\overline{f}f.$$

$$j^D_\mu = -\frac{1}{2}\left[\frac{\partial}{\partial C}\Delta_{(0)}\Big|_{\theta=0}\partial_\mu C\right] + x^\nu T_{\mu\nu} \tag{4.25}$$

verifying $\partial^\mu j^D_\mu = \delta_{(0)}$ as it should. Even if it does not appear in the natural supercurrent structure (4.19), this dilatation current is naturally associated by Poincaré supersymmetry with the $\widetilde{R}$ current (4.20) present in the supercurrent superfield $J_{\alpha\dot\alpha}$. Both currents correspond to zero $U(1)_{\widetilde{R}}$ charge q and scale dimension w for the chiral superfield Φ, the equality $w = q$ following from the superfield supercurrent equations of Poincaré supersymmetry and of the underlying superconformal character of the supermultiplets.

If the theory would be scale-invariant, $W = \Delta_{(0)} = 0$ and the anomaly source superfields $X = 4W$ and $\chi_\alpha = -\frac{1}{2}\overline{DD}D_\alpha\Delta_{(0)}$ would also vanish. An example is $\mathcal{H} = \hat{L}$ which leads to the superconformal super-Yang–Mills Lagrangian. Then, $\partial^\mu j^D_\mu = T^\mu{}_\mu = 0$. If however $\Delta_{(0)} \neq 0$, the divergence of the dilatation current is not given by the nonzero trace of the Belinfante energy-momentum tensor: with the linear superfield, there is a virial current. With scale dimension zero chiral superfields a (two derivative[23]) scale-invariant theory is generated by $\mathcal{H} = \hat{L}\,\mathcal{F}(\Phi, \overline{\Phi})$. The first field equation (4.12) for $\hat{L}$ only makes sense if $\mathcal{F} = f(\Phi) + \overline{f}(\overline{\Phi})$, in which case the linear superfield disappears from the dynamical Lagrangian which simply couples the holomorphic $f(\Phi)$ to $\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}$.[24]

We now want to generalize this discussion to the case of a nonzero scale dimension w of the chiral fields, in view of a supersymmetric improvement of the natural (Belinfante) supercurrent structure.

With respect to a system with chiral and gauge superfields only, the presence of the linear superfield introduces some technical subtleties[25] which play a role when discussing the behavior of the theory under scale transformations. Since these subtleties involve scalar fields only, we omit fermions and gauge fields in this subsection. Assigning scale dimensions w and two to the superfields Φ and $\hat{L}$, the bosonic quantity which measures the breaking of scale invariance is

$$\begin{aligned}\delta_{(w)} \;=\; & \frac{\partial\mathcal{L}}{\partial C}2C + \frac{\partial\mathcal{L}}{\partial z}wz + \frac{\partial\mathcal{L}}{\partial\bar z}w\bar z + \frac{\partial\mathcal{L}}{\partial\partial_\mu C}3\,\partial_\mu C + \frac{\partial\mathcal{L}}{\partial h_{\mu\nu\rho}}3\,h_{\mu\nu\rho}\\ & + \frac{\partial\mathcal{L}}{\partial\partial_\mu z}(w+1)\partial_\mu z + \frac{\partial\mathcal{L}}{\partial\partial_\mu\bar z}(w+1)\partial_\mu\bar z - 4\mathcal{L}.\end{aligned} \tag{4.26}$$

Using the field equations, it can be written as

$$\delta_{(w)} = \partial^\mu\mathcal{V}_{(w)\mu} + T^\mu{}_\mu \tag{4.27}$$

in terms of the trace of the Belinfante gauge-invariant energy-momentum tensor $T_{\mu\nu}$ and the virial current

$$\mathcal{V}_{(w)\mu} = -C\mathcal{H}_{CC}\partial_\mu C + wz\mathcal{H}_{z\bar z}\partial_\mu\bar z + w\bar z\mathcal{H}_{z\bar z}\partial_\mu z - \frac{i}{12}w\epsilon_{\mu\nu\rho\sigma}h^{\nu\rho\sigma}(z\mathcal{H}_{Cz} - \bar z\mathcal{H}_{C\bar z}). \tag{4.28}$$

This in turn indicates that the dilatation current is

$$\delta_{(w)} = \partial^\mu j^D_\mu \qquad\qquad j^D_\mu = \mathcal{V}_{(w)\mu} + x^\nu T_{\mu\nu} \tag{4.29}$$

[23] The real scale-invariant variable $\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}\,\widetilde{\mathrm{Tr}}\overline{\mathcal{W}\mathcal{W}}\hat{L}^{-3}$ leads to four-derivative terms.

[24] Chiral-linear duality as described in Section 3 cannot be performed.

[25] See Appendix B.

up maybe to a conserved current. Notice that

$$\mathcal{V}_{(w)\mu} = \frac{\partial \mathcal{L}}{\partial \partial^\mu C} 2C + \frac{\partial \mathcal{L}}{\partial \partial^\mu z} wz + \frac{\partial \mathcal{L}}{\partial \partial^\mu \bar{z}} w\bar{z} \tag{4.30}$$

is gauge-invariant and does not include a term related to the variation of the antisymmetric tensor.[26] Notice also that the contribution quadratic in $h_{\mu\nu\rho}$ in the energy-momentum tensor (4.21) would be traceless in six dimensions. This follows from a general result [25]: in $2(p+1)$ dimensions, the kinetic Lagrangian of a p-form field with gauge invariance is scale and conformal invariant with canonical dimension $w = p$.

Defining the superfields

$$\begin{aligned} \Delta_{(w)}(L, \Phi, \overline{\Phi}e^{\mathcal{A}}) &= 2\hat{L}\mathcal{H}_L + w\mathcal{H}_\Phi \Phi + w\overline{\Phi}\mathcal{H}_{\overline{\Phi}} - 2\mathcal{H}, &\qquad (\Delta_{(w)} \text{ real}), \\ \widetilde{\Delta}_{(w)}(\Phi) &= \tfrac{w}{4}\overline{DD}(\mathcal{H}_\Phi \Phi) - 3W, &\qquad (\overline{D}_{\dot\alpha}\widetilde{\Delta}_{(w)} = 0), \end{aligned} \tag{4.31}$$

leads to the relation

$$\begin{aligned} \delta_{(w)} &= \Delta_{(w)}|_{\theta\theta\bar\theta\bar\theta} + \tfrac{1}{4}\Box\Delta_{(w)}(C, z, \bar z) - \tfrac{1}{2}\partial^\mu\Big[\Delta_{(w)C}(C, z, \bar z)\partial_\mu C\Big] + \widetilde{\Delta}_{(w)}|_{\theta\theta} + \overline{\widetilde{\Delta}}_{(w)}|_{\bar\theta\bar\theta} \\ &= \tfrac{1}{2}D_{\Delta_{(w)}} - \tfrac{1}{2}\partial^\mu\Big[\Delta_{(w)C}(C, z, \bar z)\partial_\mu C\Big] - f_{\widetilde{\Delta}_{(w)}} - \overline{f}_{\overline{\widetilde{\Delta}}_{(w)}}, \end{aligned} \tag{4.32}$$

with $D_{\Delta_{(w)}}$ as defined in the appendices [eqs. (A.4) or (B.3)] and $\Delta_{(w)C} = \frac{\partial}{\partial C}\Delta_{(w)}$. Using equations (B.1) and (B.3) it can be shown that $\delta_{(w)}$ takes the same functional form as the bosonic Lagrangian (4.8) but with the substitutions: $\mathcal{H}$ replaced by $\Delta_{(w)}$ and W replaced by $\widetilde{\Delta}_{(w)}$. Note the appearance of a supplementary derivative term in $\delta_{(w)}$ whenever a linear superfield is present. This equation remains true in the fully supersymmetric theory with fermion and gauge fields: the supplementary derivative depends on scalar fields only. We then have:

$$\begin{aligned} \mathcal{V}_{(w)\mu} &= -\tfrac{1}{2}\Delta_{(w)C}\partial_\mu C + \tfrac{w}{2}\partial_\mu(z\mathcal{H}_z + \bar z\mathcal{H}_{\bar z}) \\ &\qquad - \tfrac{1}{2}\left[\tfrac{i}{6}\epsilon_{\mu\nu\rho\sigma}h^{\nu\rho\sigma}\tfrac{\partial}{\partial C} + (\partial_\mu z)\tfrac{\partial}{\partial z} - (\partial_\mu \bar z)\tfrac{\partial}{\partial \bar z}\right](wz\mathcal{H}_z - w\bar z\mathcal{H}_{\bar z}). \end{aligned} \tag{4.33}$$

This equality is true for an arbitrary function $\mathcal{H}(C, z, \bar z)$. Since the choice $w = 0$ has been discussed earlier, we consider now $w \neq 0$.

Two cases then exist. Firstly, if the function $\mathcal{H}$ has a $U(1)$ symmetry with charges proportional to the scale dimension w, then $wz\mathcal{H}_z = w\bar z\mathcal{H}_{\bar z}$ and

$$\mathcal{V}_{(w)\mu} = -\frac{1}{2}\Delta_{(w)C}\,\partial_\mu C + \frac{w}{2}\partial_\mu(z\mathcal{H}_z + \bar z\mathcal{H}_{\bar z}). \tag{4.34}$$

As shown explicitly in the next subsection, the second term can be eliminated by an improvement to a new energy-momentum tensor $\Theta_{\mu\nu}$ and to a new virial current $\widehat{\mathcal{V}}_\mu$ for which, in view of eqs. (4.27) and (4.32),

$$\partial^\mu j^D_\mu = \partial^\mu\widehat{\mathcal{V}}_\mu + \Theta^\mu{}_\mu = -\frac{1}{2}\partial^\mu[\Delta_{(w)C}\partial_\mu C] + \Theta^\mu{}_\mu \tag{4.35}$$

and

---

[26] This result only holds if scale dimension two is assigned to the linear superfield.

$$\Theta^{\mu}{}_{\mu} = \frac{1}{2} D_{\Delta_{(w)}} - 2\,\mathrm{Re}\, f_{\widetilde{\Delta}_{(w)}}. \tag{4.36}$$

Notice that the $U(1)$ symmetry of $\mathcal{H}$ does not need to be an R-symmetry.[27] In this first case, if the theory is scale-invariant, *i.e.* if we have $\Delta_{(w)} = \widetilde{\Delta}_{(w)} = 0$, then it follows that

$$\widehat{\mathcal{V}}_{\mu} = \Theta^{\mu}{}_{\mu} = 0 \tag{4.37}$$

and the theory is conformal since the currents

$$K^{\alpha}_{\mu} = (2x^{\alpha}x^{\nu} - \eta^{\alpha\nu}x^2)\,\Theta_{\mu\nu} \tag{4.38}$$

are conserved, $\partial^{\mu} K^{\alpha}_{\mu} = 0$. If $\mathcal{H}$ has a $U(1)$ symmetry but scale invariance is broken, $\Theta^{\mu}{}_{\mu}$ is given by the highest components of the superfields $\Delta_{(w)}$ and $\widetilde{\Delta}_{(w)}$ which measure the breaking of scale invariance, according to eqs. (4.35) and (4.36). But if $\Delta_{(w)C} \neq 0$, the divergence of the dilatation current is not given by the trace $\Theta^{\mu}{}_{\mu}$. This discussion includes the case $w = 0$ considered earlier. Since we restrict ourselves to $\mathcal{H}(\hat{L}, Y)$, the $U(1)$ symmetry exists and the improvement transformation will be performed at the superfield level in the next subsection.

In the second option, $\mathcal{H}$ does not have the global $U(1)$ symmetry, $wz\mathcal{H}_z \neq w\bar{z}\mathcal{H}_{\bar{z}}$ ($\forall w \neq 0$). The chiral superfield interactions provide then an example of a classical theory where scale invariance ($\Delta_{(w)} = \widetilde{\Delta}_{(w)} = 0$) does not imply conformal invariance because the virial current in (4.33) cannot be transformed away by an improvement transformation. This case is briefly discussed in Appendix C.

In any case, the message of this subsection is that even when $\mathcal{H}$ has a $U(1)$ symmetry but the theory is not scale invariant because $\Delta_{(w)} \neq 0$ and there is a non-trivial coupling of a linear superfield to chiral superfields such that $\Delta_{(w)C}\partial_{\mu}C$ is not a derivative, one cannot construct an energy-momentum tensor $\Theta_{\mu\nu}$ which is such that $\partial^{\mu} j^{D}_{\mu} = \Theta^{\mu}{}_{\mu}$. Whenever there does exist an energy-momentum tensor $\Theta_{\mu\nu}$ such that $\partial^{\mu} j^{D}_{\mu} = \Theta^{\mu}{}_{\mu}$ we will refer to it as the Callan, Coleman, Jackiw (CCJ) [10,11] energy-momentum tensor (see Appendix C).

4.4. Improved supercurrent structure: making scale (non-)invariance manifest

Just like in the previous subsection we assume that the chiral superfields Φ have an arbitrary scale dimension(s) w. The canonical value is $w = 1$, but dimensions can be anomalous. The dimension of $\hat{L}$ is always two.[28] In terms of the superfields $\Delta_{(w)}$ and $\widetilde{\Delta}_{(w)}$ defined in eqs. (4.31), the anomaly superfields of the natural supercurrent structure read

$$X = -\frac{4}{3}\widetilde{\Delta}_{(w)} + \frac{4}{3} w\, W_{\Phi}\Phi, \qquad \chi_{\alpha} = -\frac{1}{2}\overline{DD}D_{\alpha}\Delta_{(w)} + \frac{w}{2}\overline{DD}D_{\alpha}(\mathcal{H}_{\Phi}\Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}}). \tag{4.39}$$

We may then improve the supercurrent structure using transformation (A.8) with

$$\mathcal{G} = -\frac{w}{6}(\mathcal{H}_{\Phi}\Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}}) \tag{4.40}$$

to eliminate the second term in χ_{α}. The resulting chiral anomaly superfield is

---

[27] This observation extends results stated in ref. [26].

[28] The dimension of Ω is canonical. Notice that L contains then a dimension-three vector field $v_{\mu} = \epsilon_{\mu\nu\rho\sigma}\partial^{\nu} b^{\rho\sigma}$ which is conserved or transverse, $\partial^{\mu} v_{\mu} = 0$.

$$\widetilde{X} = -\frac{4}{3}\widetilde{\Delta}_{(w)} + \frac{4}{3}w\, W_\Phi \Phi - \frac{w}{6}\overline{DD}(\mathcal{H}_\Phi \Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}}) \tag{4.41}$$

and the field equation of Φ leads then to the supercurrent structure

$$\begin{aligned}
\overline{D}^{\dot\alpha}\widetilde{J}_{\alpha\dot\alpha} &= D_\alpha \widetilde{X} + \widetilde{\chi}_\alpha, \\
\widetilde{J}_{\alpha\dot\alpha} &= -2\Big[(\overline{\mathcal{D}}_{\dot\alpha}\overline{\Phi})\mathcal{H}_{\Phi\overline{\Phi}}(\mathcal{D}_\alpha\Phi) - \mathcal{H}_{LL}(\overline{D}_{\dot\alpha}\hat{L})(D_\alpha\hat{L}) + 2\mathcal{H}_L \widetilde{\mathrm{Tr}}(\mathcal{W}_\alpha e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}})\Big] \\
&\qquad - \tfrac{w}{3}[D_\alpha, \overline{D}_{\dot\alpha}](\mathcal{H}_\Phi\Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}}), \\
\widetilde{X} &= -\tfrac{4}{3}\widetilde{\Delta}_{(w)} + \tfrac{w}{6}\overline{DD}(\mathcal{H}_\Phi\Phi - \overline{\Phi}\mathcal{H}_{\overline{\Phi}}), \\
\widetilde{\chi}_\alpha &= -\tfrac{1}{2}\overline{DD}D_\alpha \Delta_{(w)}.
\end{aligned} \tag{4.42}$$

The extension of these formulas to a reducible matter content, with independent scale dimensions $w_{(i)}$ for each irreducible component $\Phi_{(i)}$ is straightforward.

In the case of the canonical Wess–Zumino model, $\mathcal{H} = \overline{\Phi}\Phi$, the improved supercurrent superfield reduces to

$$\widetilde{J}_{\alpha\dot\alpha} = \frac{4}{3}\left[\left(w - \frac{3}{2}\right)(\overline{D}_{\dot\alpha}\overline{\Phi})(D_\alpha\Phi) - i w\,(\sigma^\mu)_{\alpha\dot\alpha}\,\overline{\Phi}\,\overset{\leftrightarrow}{\partial}_\mu\,\Phi\right] \tag{4.43}$$

with R-current

$$j_\mu = \left(w - \frac{3}{2}\right)\psi\sigma_\mu\overline{\psi} - i w \bar{z}\,\overset{\leftrightarrow}{\partial}_\mu\, z, \tag{4.44}$$

two results often used in the literature with canonical scale dimension $w = 1$.

These formulas hold for a function $\mathcal{H}(\hat{L}, \Phi, \overline{\Phi}e^{\mathcal{A}})$. They simplify if $\mathcal{H}(\hat{L}, Y)$, as in our theory (4.5):

$$\begin{aligned}
\overline{D}^{\dot\alpha}\widehat{J}_{\alpha\dot\alpha} &= D_\alpha \widehat{X} + \widehat{\chi}_\alpha, \\
\widehat{J}_{\alpha\dot\alpha} &= -2\Big[(\overline{\mathcal{D}}_{\dot\alpha}\overline{\Phi})\mathcal{H}_{\Phi\overline{\Phi}}(\mathcal{D}_\alpha\Phi) - \mathcal{H}_{LL}(\overline{D}_{\dot\alpha}\hat{L})(D_\alpha\hat{L}) + 2\mathcal{H}_L \widetilde{\mathrm{Tr}}(\mathcal{W}_\alpha e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}})\Big] \\
&\qquad - \tfrac{2}{3}[D_\alpha, \overline{D}_{\dot\alpha}](w\mathcal{H}_Y Y), \\
\widehat{X} &= -\tfrac{4}{3}\widetilde{\Delta}_{(w)}, \\
\widehat{\chi}_\alpha &= -\tfrac{1}{2}\overline{DD}D_\alpha \Delta_{(w)}.
\end{aligned} \tag{4.45}$$

In $\widehat{J}_{\alpha\dot\alpha}$, the energy-momentum tensor $\Theta_{\mu\nu}$ is related to the Belinfante tensor by the improvement

$$\begin{aligned}
\Theta_{\mu\nu} &= T_{\mu\nu} - \tfrac{1}{6}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box) w(\mathcal{H}_z z + \bar{z}\mathcal{H}_{\bar{z}}) \\
&= T_{\mu\nu} - \tfrac{1}{3}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box) w\mathcal{H}_y y, \qquad\qquad y = \bar{z}z
\end{aligned} \tag{4.46}$$

and the corresponding improved (scalar) virial current is

$$\widehat{\mathcal{V}}_\mu = \mathcal{V}_{(w)\mu} - w\,\partial_\mu(\mathcal{H}_y y) = -\frac{1}{2}\Delta_{(w)C}\partial_\mu C \tag{4.47}$$

as explained in the previous subsection. Using (A.6) we find that $\Theta^\mu{}_\mu$ satisfies eq. (4.36) while the dilatation current verifies eq. (4.35).

The superfield improvement transformation can also be understood in the following way. The field equation $\overline{DD}\mathcal{H}_\Phi = 4W_\Phi$ implies

$$\frac{w}{2}\overline{DD}(\mathcal{H}_\Phi \Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}}) = 4wW_\Phi \Phi - \frac{w}{2}\overline{DD}(\mathcal{H}_\Phi \Phi - \overline{\Phi}\mathcal{H}_{\overline{\Phi}}). \tag{4.48}$$

The right-hand side vanishes if the theory is invariant under phase rotations of Φ. In this case,

$$\mathcal{Z} = \frac{w}{2}(\mathcal{H}_\Phi \Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}}) = -3\,\mathcal{G} \tag{4.49}$$

includes the Noether current of the $U(1)_Z$ symmetry (with charge w on Φ) in its $\theta\sigma^\mu\overline{\theta}$ component and $\overline{DD}\,\mathcal{Z} = 0$ is the supersymmetric extension of its conservation equation. Taking the derivative D_α of eq. (4.48), using identity (A.7) and the field equation leads then to the improvement transformation to the supercurrent structure (4.42). Hence, the supercurrent superfield $\widehat{\mathcal{J}}_{\alpha\dot{\alpha}}$ includes in its lowest component the current of the R transformation with R charges 0 and w for $\hat{L}$ and Φ respectively. Gauginos, and fermions ψ in Φ and χ in L have chiral weights $3/2$, $w - 3/2$ and $-3/2$ respectively. Notice that w has been originally introduced as the scale dimension of Φ and it here also plays the role of an R charge. This is reminiscent of the chirality condition in a superconformal theory, in which the scale dimension and the $U(1)_R$ charge are identified.

We may further improve the structure (4.45) to a Ferrara–Zumino supercurrent with $\chi_\alpha = 0$. This second improvement would lead to a supercurrent depending on the superfield $\Delta_{(w)}$,

$$\widehat{\mathcal{J}}_{\alpha\dot{\alpha}} \quad \longrightarrow \quad \widehat{\mathcal{J}}_{\alpha\dot{\alpha}} + \frac{1}{3}[D_\alpha, \overline{D}_{\dot{\alpha}}]\Delta_{(w)}. \tag{4.50}$$

The content of the supercurrent structure (4.45) is however more intuitive, with the Lagrangian superfield $\mathcal{H}$ defining the supercurrent superfield $\widehat{\mathcal{J}}_{\alpha\dot{\alpha}}$ and the scale- and R-breaking superfields $\Delta_{(w)}$ and $\widetilde{\Delta}_{(w)}$ defining the source superfields $\widehat{X}$ and $\widehat{\chi}_\alpha$. In the following, we will use the improved supercurrent structure (4.45) as a starting point and we will be mostly concerned with the case $W(\Phi) = 0 = \widehat{X}$. This structure is both $\widetilde{R}$- and R-symmetric and is then naturally related to new-minimal supergravity.

If we wish to cancel the virial current completely, we need that $\mathcal{V}_\mu$ is a derivative, and then Δ_C should be a function of C only. This is the case if

$$\mathcal{H}(\hat{L}, Y) = \mathcal{F}(\hat{L}) + \mathcal{K}(Y) + \mathcal{I}(\hat{L}, Y), \qquad\qquad wY\mathcal{I}_Y + \hat{L}\mathcal{I}_L = \mathcal{I}. \tag{4.51}$$

Then,

$$\Delta_{(w)} = 2\hat{L}\mathcal{F}_L - 2\mathcal{F} + 2wY\mathcal{K}_Y - 2\mathcal{K}, \qquad\qquad \Delta_{(w)LY} = 0.$$

The second equation (4.51) has a very simple significance: with dimensions w and two for Φ and $\hat{L}$, the interaction term must be a dimension-two function. Hence, the interaction Lagrangian is scale invariant:

$$\mathcal{I}(\hat{L}, Y) = \hat{L}\,\widetilde{\mathcal{I}}(X), \qquad\qquad X = Y\hat{L}^{-w}. \tag{4.52}$$

Only in this case can we find an energy-momentum tensor such that $\partial^\mu j^D_\mu = \Theta^\mu{}_\mu$.

For reasons explained in Section 5, it will be natural to write the anomaly source superfields as a sum of a classical contribution and an anomalous term as follows

$$\Delta_{(w)} = \Delta_{\text{classical}} + \Delta_{\text{anom.}}, \qquad\qquad \Delta_{\text{anom.}} = B\,\hat{L}, \tag{4.53}$$

and

$$\widetilde{\Delta}_{(w)} = \widetilde{\Delta}_{\text{classical}} + \widetilde{\Delta}_{\text{anom.}}, \qquad \widetilde{\Delta}_{\text{anom.}} = A\,\widetilde{\text{Tr}}\mathcal{W}\mathcal{W}, \tag{4.54}$$

with some numerical coefficients A and B to be discussed below. An anomalous contribution $\Delta_{\text{anom.}} = B\hat{L}$ arises if an anomaly term

$$\mathcal{H}_{\text{anom.}}(\hat{L}) = \frac{B}{2}(\hat{L}\ln\hat{L} - \hat{L}) \tag{4.55}$$

is added to $\mathcal{H}$. Similarly, an anomalous contribution $\widetilde{\Delta}_{\text{anom.}}$ may be obtained if the superpotential is allowed to depend on $\widetilde{\text{Tr}}\mathcal{W}\mathcal{W}$: this is the subject of the next subsection.

4.5. Adding a dependence to the superpotential on $\widetilde{\text{Tr}}\mathcal{W}\mathcal{W}$

The chiral superfield $\widetilde{\text{Tr}}\mathcal{W}\mathcal{W}$ has a fermionic lowest component. In principle, the superpotential W could also be a function of $\widetilde{\text{Tr}}\mathcal{W}\mathcal{W}$, $W(\Phi, \widetilde{\text{Tr}}\mathcal{W}\mathcal{W})$, but this dependence does not play any role in the bosonic Lagrangian and for the gauge coupling field, except for a linear term which is already included in $\mathcal{H}(\hat{L}, \Phi, \overline{\Phi})$ since[29]

$$\frac{1}{2}\int d^2\theta\, f(\Phi)\widetilde{\text{Tr}}\mathcal{W}\mathcal{W} + \text{h.c.} = \int d^2\theta d^2\bar{\theta}\,[f(\Phi) + \overline{f}(\overline{\Phi})]\hat{L} + \text{total deriv.}$$

It however plays a role in effective Lagrangians like, for instance, in the description of gaugino condensates. Defining the variable $\mathcal{U} = \widetilde{\text{Tr}}\mathcal{W}\mathcal{W}$, the field equation for the gauge superfield $\mathcal{A}$ is now

$$\overline{D}^{\dot{\alpha}}\Big[(\mathcal{H}_L + 2W_{\mathcal{U}} + 2\overline{W}_{\overline{\mathcal{U}}})\,e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot{\alpha}}e^{\mathcal{A}}\Big] = \mathcal{W}^{\alpha}\,D_{\alpha}(\mathcal{H}_L + 2W_{\mathcal{U}}) - T(r)\,\mathcal{H}_Y\,\Phi\overline{\Phi}e^{\mathcal{A}} \tag{4.56}$$

with $W_{\mathcal{U}} = \frac{\partial}{\partial\mathcal{U}}W(\Phi, \mathcal{U})$, instead of the third eq. (4.12). Following the same steps, we obtain the "natural" supercurrent structure

$$\begin{aligned}
\overline{D}^{\dot{\alpha}}J_{\alpha\dot{\alpha}} &= D_{\alpha}X + \chi_{\alpha}, \\
J_{\alpha\dot{\alpha}} &= -2\Big[(\overline{\mathcal{D}}_{\dot{\alpha}}\overline{\Phi})\mathcal{H}_{\Phi\overline{\Phi}}(\mathcal{D}_{\alpha}\Phi) - \mathcal{H}_{LL}(\overline{D}_{\dot{\alpha}}\hat{L})(D_{\alpha}\hat{L}) \\
&\qquad + 2(\mathcal{H}_L + 2W_{\mathcal{U}} + 2\overline{W}_{\overline{\mathcal{U}}})\widetilde{\text{Tr}}(\mathcal{W}_{\alpha}e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot{\alpha}}e^{\mathcal{A}})\Big], \\
X &= 4\,(W - \mathcal{U}W_{\mathcal{U}}), \\
\chi_{\alpha} &= \overline{DD}D_{\alpha}(\mathcal{H} - \hat{L}\mathcal{H}_L)
\end{aligned} \tag{4.57}$$

instead of expressions (4.19). A violation of scale invariance in the chiral density is measured by the superfield

$$\widetilde{\Delta}_{(w)} = wW_{\Phi}\Phi + 3W_{\mathcal{U}}\mathcal{U} - 3W \tag{4.58}$$

since $\mathcal{U} = \widetilde{\text{Tr}}\mathcal{W}\mathcal{W}$ has canonical scale dimension three. Relations (4.39) are then unaffected and the same improvement transformation leads to the improved supercurent structure $\overline{D}^{\dot{\alpha}}\widetilde{J}_{\alpha\dot{\alpha}} = D_{\alpha}\widetilde{X} + \widetilde{\chi}_{\alpha}$ with a modified supercurrent superfield

[29] But then $\mathcal{H}$ is not a function of Y but instead it depends on the (gauge-invariant) function $f(\Phi)$.

$$\begin{aligned}\widetilde{J}_{\alpha\dot\alpha} &= -2(\overline{\mathcal{D}}_{\dot\alpha}\overline{\Phi})\mathcal{H}_{\Phi\overline{\Phi}}(\mathcal{D}_\alpha\Phi) + 2\mathcal{H}_{LL}(\overline{D}_{\dot\alpha}\hat{L})(D_\alpha\hat{L}) - \tfrac{w}{3}[D_\alpha, \overline{D}_{\dot\alpha}](\mathcal{H}_\Phi\Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}}) \\ &\quad -4\,(\mathcal{H}_L + 2W_{\mathcal{U}} + 2\overline{W}_{\overline{\mathcal{U}}})\widetilde{\mathrm{Tr}}(\mathcal{W}_\alpha e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}})\end{aligned} \tag{4.59}$$

and anomaly superfields as defined in eqs. (4.42) but with $\widetilde{\Delta}_{(w)}$ as given in (4.58). An anomalous contribution to $\widetilde{\Delta}_{(w)}$ as in (4.54) follows then, using (4.58), from a Veneziano–Yankielowicz [27] "gauge superpotential"

$$W(\mathcal{U}) = \frac{A}{3}(\mathcal{U}\ln\mathcal{U} - \mathcal{U}). \tag{4.60}$$

Since $\int d^2\theta d^2\overline{\theta}\,\hat{L} = \frac{1}{4}\int d^2\theta\, U +$ derivative, a theory defined by functions $\mathcal{H} + (A + \overline{A})\hat{L}$ and W is equivalent to a theory defined by $\mathcal{H}$ and $W + \frac{1}{2}AU$ (A is chiral). All expressions in this section respect this equivalence.

5. Perturbative anomalies

The improved supercurrent structure (4.45) with scale dimension w for chiral superfields Φ includes the Noether current of the $U(1)_R$ acting with charges $3/2$ on gauginos and $w - 3/2$ on chiral fermions in representation r. This $U(1)_R$ group combines the natural $\widetilde{R}$ transformation described in the natural (Belinfante) supercurrent structure (4.19)[30] and the non-R $U(1)_Z$ acting with charges w on superfields Φ. As explained earlier, the Noether current $j_\mu^{(Z)}$ associated with $U(1)_Z$ is in the $\theta\sigma^\mu\overline{\theta}$ component of superfield (4.49),

$$\mathcal{Z} = \frac{w}{2}(\mathcal{H}_\Phi\Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}}) = wY\mathcal{H}_Y.$$

Using field equations, its superfield conservation equation is of the form

$$\overline{DD}\mathcal{Z} = \Delta_{\mathcal{Z}}, \tag{5.1}$$

with a chiral source superfield $\Delta_{\mathcal{Z}}$ given in eq. (4.48) at the classical level and including in general quantum anomalies. With identity (A.7), this conservation equation can always be turned into an equivalent "supercurrent equation"

$$\begin{aligned}\overline{D}^{\dot\alpha}J_{\alpha\dot\alpha} &= D_\alpha\Delta_{\mathcal{Z}} + 3\overline{DD}D_\alpha\mathcal{Z}, \\ J_{\alpha\dot\alpha} &= 2[D_\alpha, \overline{D}_{\dot\alpha}]\mathcal{Z},\end{aligned} \tag{5.2}$$

where the Noether current $j_\mu^{(Z)}$ is now in $J_{\alpha\dot\alpha}|_{\theta=0}$. In the $\theta\sigma^\mu\overline{\theta}$ component, the "energy-momentum" tensor is merely a trivial improvement term, according to transformations (A.10), its trace is $-3\Box C_{\mathcal{Z}}$ and this corresponds to a formal contribution[31]

$$J_\mu^D = -3\,\partial_\mu C_{\mathcal{Z}} = -\frac{3}{2}w\,\partial_\mu(\mathcal{H}_z z + \bar{z}\mathcal{H}_{\bar{z}}) = -\frac{3}{2}\,\partial_\mu[\Delta_{(w)} - \Delta_{(0)}]_{\theta=0}$$

to the dilatation current, in terms of the source superfield (4.31).

---

30 Obtained with $w = 0$ in expressions (4.45).

31 Omitting as earlier fermions and gauge fields.

5.1. Mixed “internal” anomalies

In this paragraph, we repeatedly use

$$\begin{aligned}\widetilde{\mathrm{Tr}}\,\mathcal{W}\mathcal{W}|_{\theta\theta} &= -\frac{1}{2}\widetilde{\mathrm{Tr}}\,F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\widetilde{\mathrm{Tr}}\,F_{\mu\nu}\widetilde{F}^{\mu\nu} + \ldots \\ &= 2\,\mathcal{L}_{SYM} - \frac{i}{2}\,\widetilde{\mathrm{Tr}}\Big[F_{\mu\nu}\widetilde{F}^{\mu\nu} - 2\,\partial_\mu(\lambda\sigma^\mu\bar{\lambda})\Big], \end{aligned} \tag{5.3}$$

$$\overline{DD}\mathcal{Z}|_{\theta\theta} = -2i\,\partial^\mu j_\mu^{(\mathcal{Z})} + \ldots, \qquad \hat{L}|_{\theta\theta\bar\theta\bar\theta} = -\frac{1}{4}\widetilde{\mathrm{Tr}}\,F_{\mu\nu}F^{\mu\nu} + \ldots = \mathcal{L}_{SYM},$$

and $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

Since a global symmetry $U(1)_Z$ has $U(1)_Z$-gauge–gauge mixed anomaly[32]

$$\partial^\mu j_\mu^{(\mathcal{Z})} = \frac{1}{16\pi^2}\,wT(r)\,\widetilde{\mathrm{Tr}}\,F^{\mu\nu}\widetilde{F}_{\mu\nu} + \ldots, \tag{5.4}$$

the corresponding superfield anomaly equation is

$$\overline{DD}\,\mathcal{Z} = \frac{1}{4\pi^2}\,wT(r)\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}. \tag{5.5}$$

At this point, w is the $\mathcal{Z}$-charge of the superfield Φ and of its fermionic components. Identity (A.7) with $\mathcal{G} = -\mathcal{Z}/3$ leads to

$$-\frac{2}{3}\overline{D}^{\dot\alpha}[D_\alpha, \overline{D}_{\dot\alpha}]\mathcal{Z} = -\frac{1}{12\pi^2}\,wT(r)\,D_\alpha\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} - \overline{DD}D_\alpha\mathcal{Z}, \qquad \mathcal{Z} = wY\mathcal{H}_Y. \tag{5.6}$$

Comparing with the improved supercurrent structure (4.45), the anomaly adds a contribution to the chiral source superfield $\widehat{X}$,

$$\widehat{X} \quad\longrightarrow\quad \widehat{X} - \frac{1}{12\pi^2}wT(r)\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}, \tag{5.7}$$

and by supersymmetry a contribution to the energy-momentum trace.

Similarly, the natural $\widetilde{R}$-symmetry has $U(1)_{\widetilde{R}}$-gauge–gauge anomaly[33]

$$\partial^\mu j_\mu^{(\widetilde{R})} = \frac{1}{16\pi^2}\,\frac{3}{2}[C(G) - T(r)]\,\widetilde{\mathrm{Tr}}(F^{\mu\nu}\widetilde{F}_{\mu\nu}) + \ldots \tag{5.8}$$

According to the second eq. (A.6), this also leads to an anomalous shift of the source superfield $\widehat{X}$:

$$\widehat{X} \quad\longrightarrow\quad \widehat{X} - \frac{1}{8\pi^2}\,[C(G) - T(r)]\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}. \tag{5.9}$$

Combining both anomalies leads to

$$\widehat{X}_{(anomaly)} = -\frac{1}{24\pi^2}\,b(w)\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} \tag{5.10}$$

with coefficient

[32] Dots indicate terms generated by supersymmetry.

[33] $C(G) = T(Adj\ G)$ is the quadratic Casimir, $C(G)\delta^{ab} = f^{acd}f^{bcd}$ in terms of structure constants.

$$b(w) = b_0 + 2(w-1)T(r), \qquad b_0 = 3C(G) - T(r). \tag{5.11}$$

This is of course the anomaly of the R-symmetry with current described by the lowest component of the improved supercurrent (4.45). Writing instead

$$b(w) = 3C(G) - T(r)(1-\gamma), \qquad \gamma = 2(w-1), \qquad w = 1 + \frac{\gamma}{2}, \tag{5.12}$$

the number γ is now the anomalous dimension and $b(w)$ is the numerator of the NSVZ β function [12,28].[34]

Now, according to the first eq. (A.6), the energy-momentum tensor in this supercurrent superfield verifies

$$\Theta^\mu{}_\mu = \frac{1}{4}D + \frac{3}{2}\,\mathrm{Re}\, f_{\widehat{X}_{anomaly}} = \frac{1}{4}D - \frac{1}{32\pi^2}\, b(w)\, \widetilde{\mathrm{Tr}} F^{\mu\nu}F_{\mu\nu} + \ldots \tag{5.13}$$

Since gauginos and chiral fermions have scale dimensions $3/2$ and $w + 1/2$ respectively, we expect that the dilatation current has dilatation-gauge–gauge anomaly

$$\partial^\mu j^D_\mu = -\frac{1}{48\pi^2}\, c(w)\, \widetilde{\mathrm{Tr}} F^{\mu\nu}F_{\mu\nu} + \ldots, \qquad c(w) = \frac{3}{2}[C(G) + T(r)] + (w-1)T(r). \tag{5.14}$$

As a consequence,

$$D = \frac{1}{4\pi^2}\, d(w)\, \widetilde{\mathrm{Tr}} F^{\mu\nu}F_{\mu\nu} + \ldots, \qquad d(w) = C(G) - T(r) + \frac{2}{3}(w-1)T(r). \tag{5.15}$$

This residual dilatation anomaly is introduced in the supercurrent structure by a quantum contribution

$$\chi_{\alpha(anomaly)} = -\frac{1}{4}\overline{DD}D_\alpha U_{(anomaly)}, \qquad U_{(anomaly)} = -\frac{1}{2\pi^2}\, d(w)\, \hat{L} \tag{5.16}$$

added to the source superfield $\widehat{\chi}_\alpha$. Hence, with the chiral contribution (5.10), the improved supercurrent structure including the anomalies is

$$\begin{aligned}
\overline{D}^{\dot\alpha}\widehat{J}_{\alpha\dot\alpha} &= D_\alpha \widehat{X} + \widehat{\chi}_\alpha, \\
\widehat{J}_{\alpha\dot\alpha} &= -2\Big[(\overline{\mathcal{D}}_{\dot\alpha}\overline{\Phi})\mathcal{H}_{\Phi\overline{\Phi}}(\mathcal{D}_\alpha\Phi) - \mathcal{H}_{LL}(\overline{D}_{\dot\alpha}\hat{L})(D_\alpha\hat{L}) + 2\mathcal{H}_L\widetilde{\mathrm{Tr}}(\mathcal{W}_\alpha e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}})\Big] \\
&\qquad - \frac{2}{3}[D_\alpha, \overline{D}_{\dot\alpha}](w\mathcal{H}_Y Y), \\
\widehat{X} &= -\frac{4}{3}\left(\widetilde{\Delta}_{(w)} + \frac{1}{32\pi^2}\, b(w)\, \widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}\right), \\
\widehat{\chi}_\alpha &= -\frac{1}{2}\overline{DD}D_\alpha\left(\Delta_{(w)} - \frac{1}{4\pi^2}\, d(w)\, \hat{L}\right),
\end{aligned} \tag{5.17}$$

with $\widetilde{\Delta}_{(w)} = 0$ if the superpotential vanishes. In the case of pure $\mathcal{N} = 2$ super-Yang–Mills theory in which $r = Adj(G)$, $C(G) = T(r)$ and $w = 1$ (since both gauginos have same R charge),

$$b(w) = b_0 = 2C(G), \qquad d(w) = 0, \qquad \chi_{\alpha(anomaly)} = 0. \tag{5.18}$$

[34] While a conserved current has dimension three, see for instance ref. [29], anomalous currents have in general $\gamma > 0$.

5.2. Matching and canceling anomalies

Following the discussion of the previous section, we may use in an effective or phenomenological Lagrangian local counterterms which, depending on the context, either match an anomaly of the microscopic theory or compensate an anomaly generated in perturbation theory of the effective theory in order to restore a quantum symmetry of the underlying theory. An example of the first situation is the familiar axial current chiral anomaly. An example of the second case would be the cancellation of target-space T-duality (Kähler) anomalies in the effective supergravity of string compactifications as originally described in refs. [18,19].

Consider

$$\Delta\mathcal{H}_{corr.}(\hat{L},Y) = -\frac{\epsilon}{8\pi^2}\,d(w)\,\hat{L}(\ln\hat{L}-1), \tag{5.19}$$

$$\Delta W_{corr.}(\Phi,\mathcal{U}) = \frac{\epsilon}{96\pi^2}\,b(w)\,\mathcal{U}(\ln\mathcal{U}-1), \tag{5.20}$$

where $\mathcal{U}=\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}$ as in Subsection 4.5 and $\epsilon=\pm 1$. The corresponding scale and R-breaking superfields are then

$$\begin{aligned} \Delta_{corr.} &= -\frac{\epsilon}{4\pi^2}\,d(w)\,\hat{L}, \\ \widetilde{\Delta}_{corr.} &= \frac{\epsilon}{32\pi^2}\,b(w)\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}. \end{aligned} \tag{5.21}$$

These counterterms are used to obtain effective Lagrangians with "classical" anomalous behavior by modifying the currents in the supercurrent structure. For $\epsilon=1$, when added as quantum corrections to the function $\mathcal{H}$ defining an effective Lagrangian, they would match the microscopic anomaly terms in expressions (5.17). For $\epsilon=-1$, they would cancel or compensate these quantum anomalies to describe an exact symmetry, as for instance the renormalization-group does with scale transformations.

If we expand the anomaly counterterm in expression (5.19) around a constant background value

$$\hat{L} \qquad \longrightarrow \qquad g^2+\hat{L}, \tag{5.22}$$

it can be rewritten

$$\begin{aligned} &-\epsilon\frac{d(w)}{8\pi^2}\int d^2\theta d^2\bar{\theta}\,\left(g^2+\hat{L}\right)\left[\ln g^2+\ln\left(1+\frac{\hat{L}}{g^2}\right)-1\right] \\ &\qquad = -\epsilon\frac{d(w)}{8\pi^2}\ln g^2\int d^2\theta d^2\bar{\theta}\,\hat{L}+\ldots = -\epsilon\frac{d(w)}{8\pi^2}\ln g^2\,\frac{1}{4}\int d^2\theta\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}+\mathrm{c.c.}+\ldots, \end{aligned} \tag{5.23}$$

omitting terms of higher orders in $\hat{L}$. Hence, with a constant coupling, it can be expressed as a chiral integral. This is the rescaling anomaly calculated by Arkani-Hamed and Murayama [7].[35] But in terms of the gauge coupling field, it is included in the full superspace integral of the real superfield (5.19). Actually, ref. [7] evaluates the anomaly induced by the rescaling of the gauge

35 Their eqs. (2.8) and (2.9) for super-Yang–Mills fields.

superfield $\mathcal{A} \to g\mathcal{A}$ which brings the gauge kinetic terms from $-\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu}$ to the canonical normalization $-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$. This rescaling corresponds to $\hat{L} \to g^2\hat{L}$ in our context. When applied to the anomaly-matching term (5.19), it produces the correct anomaly variation.

Notice that the chiral anomaly-matching superpotential in expression (5.20) generates

$$\begin{aligned} \frac{\epsilon}{96\pi^2} b(w) \int d^2\theta\, \mathcal{U}(ln\,\mathcal{U} - 1) + \text{h.c.} &= \frac{\epsilon}{48\pi^2} b(w)\, \ln(\bar{u}u)\mathcal{L}_{SYM} \\ &+ \frac{i\epsilon}{48\pi^2} b(w)\, \ln(u/\bar{u}) \left[-\frac{1}{4}\widetilde{\text{Tr}} F_{\mu\nu}\widetilde{F}^{\mu\nu} + \frac{1}{2}\partial^\mu(\widetilde{\text{Tr}}\lambda\sigma_\mu\bar{\lambda}) \right] + \dots \end{aligned} \tag{5.24}$$

which, since $u = -\widetilde{\text{Tr}}\lambda\lambda$, is a correction to the gauge coupling in a fermionic background $\langle\widetilde{\text{Tr}}\lambda\lambda\rangle \neq 0$ only.[36]

6. Effective Lagrangians

We now apply our formalism to two types of effective descriptions of a supersymmetric gauge theory, the Wilson effective Lagrangian and the effective action, as defined in quantum field theory, for the description of gaugino condensates. This section is a development of refs. [13,14].

In this section, it is important to clearly distinguish the scale and the mass dimensions. As defined earlier, the scale dimension encodes the behavior under dilatation of coordinates and fields (with scale dimensions w_i). The mass dimension follows from simple dimensional analysis (in energy units) and allows for a mass dimension of parameters (which have zero scale dimension). A Lagrangian has mass dimension four since the action is dimensionless, it does not have a well-defined scale dimension in general.[37] Gauge fields and superfields have identical canonical scale and mass dimensions: this is the case of superfields $\mathcal{A}$ ($w = 0$), L and Ω ($w = 2$), $\mathcal{W}$ ($w = 3/2$). Chiral superfields have in general anomalous scale dimensions $w = 1 + \gamma/2$. The distinction between scale and mass dimensions disappears if dilatation would be a symmetry: in this case, the Lagrangian has scale dimension four and all parameters have then vanishing mass dimension.

6.1. Wilson Lagrangian

The Wilson effective Lagrangian $\mathcal{L}_{W,\mu}$ explicitly depends on a mass parameter $\mu > 0$, which acts as an ultraviolet cutoff. Schematically, it is obtained from a fundamental microscopic quantum field theory by integrating modes with frequencies larger than μ. In perturbation theory, the loop expansion in the microscopic theory is matched by the combination of a perturbative expansion of the Wilson Lagrangian,

$$\mathcal{L}_{W,\mu} = \mathcal{L}_W^{(0)} + \sum_{n>0} \mathcal{L}_{W,\mu}^{(n)}$$

(n is the loop order in the microscopic theory) and loops generated from $\mathcal{L}_{W,\mu}$, with cutoff μ. If the microscopic theory includes only fields with masses lighter than μ, the classical $\mathcal{L}_W^{(0)}$ coincides with the microscopic quantum field theory Lagrangian. If the microscopic theory includes

36 See next Section.

37 As a consequence, in a supersymmetric theory, the Kähler potential, the superpotential and $\widetilde{\text{Tr}}\mathcal{W}\mathcal{W}$ have mass dimensions two, three and three.

fields with masses heavier than μ, $\mathcal{L}_W^{(0)}$ also includes higher-dimensional operators controlled by these mass parameters. The Wilson Lagrangian is local and the scale μ is arbitrary. Its dependence on μ is then dictated by a specific renormalization-group (RG) equation.[38]

We wish to consider the Wilson effective Lagrangian of a microscopic $N = 1$ gauge theory with zero superpotential:

$$\mathcal{L}_{micro.} = \sum_i Z_i \int d^2\theta d^2\bar{\theta}\, \overline{\Phi}_i e^{\mathcal{A}_i} \Phi_i + \frac{1}{4\bar{g}^2} \int d^2\theta\, \widetilde{\mathrm{Tr}} \mathcal{W}\mathcal{W} + \mathrm{h.c.} \tag{6.1}$$

where $\bar{g}$ is the bare coupling of the gauge group assumed simple and the sum is over irreducible representations. The wave function renormalization matrix Z is diagonal with zero superpotential. As we will see later on, it is not always wise to assume that $Z_i \to 1$ in the limit $\bar{g} \to 0$. The Lagrangian is classically scale invariant with canonical scale dimensions $w = 1$ and $w = 3/2$ for Φ and $\mathcal{W}$ respectively, and $\bar{g}$ has mass dimension zero.

We are interested in the Wilson effective Lagrangian expressed with a supersymmetrized background field C for the gauge coupling. Hence, the background value $\langle C \rangle$ will be identified with the physical gauge coupling $g^2(M)$ at a reference energy scale M. This scale can be viewed as defining the renormalization scheme in the microscopic theory, or as the scale used to normalize quantities in the renormalized theory. For instance, there exists in general subtraction schemes (in the microscopic theory) where $g^2(M) = \bar{g}^2$ (for a given M). It can also be regarded as a physical quantity like a unification scale. The Wilson Lagrangian depends on the reference scale M implicitly via $g^2(M)$ or C and explicitly via the ratio μ/M and RG equations reflect the arbitrariness of these mass parameters.

Our goal in this section is to algebraically derive some of the all-order results of Novikov, Shifman, Vainshtein and Zakharov (NSVZ) [12][39] with the gauge coupling background or propagating field C which actually plays a central role in the understanding of the higher order contributions to the β function. In spirit, our discussion is very close or identical to the interpretation of Shifman and Vainshtein and to the anomaly argument of Arkani-Hamed and Murayama for constant coupling parameters [7]. Using then the formulation presented in Section 4, it immediately follows that the presence in the β function of these higher-order contributions is fully compatible with the supercurrent superfield structure expected in $\mathcal{N} = 1$ theories.

6.1.1. $\mathcal{N} = 1$ *super-Yang–Mills theory*

At tree-level, or in the microscopic theory, we would certainly use

$$\mathcal{H}_{(0)} = m^2 \ln(\hat{L}/m^2) \tag{6.2}$$

with a mass parameter m to keep track of the mass dimensions of the function $\mathcal{H}$ and of C. The bosonic Lagrangian is then[40]

$$\begin{aligned} \mathcal{L}_{(0)} &= \int d^2\theta d^2\bar{\theta}\, \mathcal{H}_{(0)} \\ &= \frac{m^2}{C}\left[-\frac{1}{4}\widetilde{\mathrm{Tr}} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\widetilde{\mathrm{Tr}} DD\right] + \frac{1}{4}\frac{m^2}{C^2}\left[(\partial^\mu C)(\partial_\mu C) + \frac{1}{6} H^{\mu\nu\rho} H_{\mu\nu\rho}\right]. \end{aligned} \tag{6.3}$$

[38] We assume here that μ is sufficiently far from particle thresholds, to avoid a detailed treatment of these thresholds.

[39] And Jones for super-Yang–Mills theory [30].

[40] Omitting a derivative.

With the identification $C = m^2 g^2$ of the tree-level gauge coupling field and

$$C = m^2\, g^2(M) \tag{6.4}$$

in general, the quantity m does not play any role in the gauge Lagrangian. It appears in the kinetic Lagrangian of the linear superfield where it naturally keeps track of the violation of scale invariance unavoidable with the gauge coupling field. Actually,

$$\Delta_{(0)} = 2\hat{L}\frac{\partial}{\partial \hat{L}}\mathcal{H}_{(0)} - 2\mathcal{H}_{(0)} = -2\mathcal{H}_{(0)} + 2m^2 = -m\frac{d}{dm}\mathcal{H}_{(0)} \tag{6.5}$$

indicates that the logarithmic choice (6.2) appropriate for the tree-level Yang–Mills Lagrangian in expression (6.3) is a function $\mathcal{H}_{(0)}$ with scale dimension zero.[41] The last equality indicates that since m is used to obtain the appropriate mass dimensions, a scale transformation of C can be compensated by a rescaling of m[42]: for any $\mathcal{H} = m^2\mathcal{F}(\hat{L}/m^2)$,

$$2\hat{L}\mathcal{H}_{\hat{L}} - 2\mathcal{H} = -m\frac{d}{dm}\mathcal{H}. \tag{6.6}$$

We wish to write a loop-corrected Wilson Lagrangian

$$\mathcal{L}_{W,\mu} = \frac{1}{g^2_{W,\mu}}\mathcal{L}_{SYM} + \ldots = -\frac{1}{4g^2_{W,\mu}}\widetilde{\mathrm{Tr}}F_{\mu\nu}F^{\mu\nu} + \ldots, \tag{6.7}$$

where the Wilson gauge coupling $g^2_{W,\mu}$ is expressed as a function of C/m^2 identified with the ordinary observable gauge coupling constant $g^2(M)$ at an arbitrary normalization scale M,[43] as in eq. (6.4).[44] Without matter superfield, we certainly have

$$\mathcal{L}_{W,\mu} = \int d^2\theta d^2\bar{\theta}\, \mathcal{H}(\hat{L}/m^2, \mu/M) \tag{6.8}$$

and the Lagrangian has necessarily (classical) $\widetilde{R}$ symmetry rotating Grassmann coordinates and fermions. This implies that the corresponding natural supercurrent structure (4.19) including the Belinfante energy-momentum tensor has vanishing chiral source superfield X:

$$\begin{aligned}
\overline{D}^{\dot{\alpha}} J_{(W)\alpha\dot{\alpha}} &= \chi_{(W)\alpha}, \\
J_{(W)\alpha\dot{\alpha}} &= 2\,\mathcal{H}_{LL}(\overline{D}_{\dot{\alpha}}\hat{L})(D_\alpha\hat{L}) - 4\,\mathcal{H}_L\widetilde{\mathrm{Tr}}(\mathcal{W}_\alpha e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot{\alpha}}e^{\mathcal{A}}), \\
\chi_{(W)\alpha} &= \overline{DD}D_\alpha(\mathcal{H} - \hat{L}\mathcal{H}_L).
\end{aligned} \tag{6.9}$$

The lowest component of $\frac{3}{8}J_{(W)\alpha\dot{\alpha}}$ is the current of $\widetilde{R}$ symmetry:

$$j^{\widetilde{R}}_\mu = \frac{3}{2}\mathcal{H}_C\,\widetilde{\mathrm{Tr}}\,\lambda\sigma^\mu\bar{\lambda} + \frac{3}{4}\mathcal{H}_{CC}\,\chi\sigma_\mu\bar{\chi} = \frac{1}{g^2_{W,\mu}}q_\lambda\,\widetilde{\mathrm{Tr}}\,\lambda\sigma^\mu\bar{\lambda} - \frac{\mathcal{H}_{CC}}{2}q_\chi\,\chi\sigma_\mu\bar{\chi} \tag{6.10}$$

---

41 The constant terms $2m^2$ in $\Delta_{(0)}$ is irrelevant.

42 Which is not a scale transformation.

43 Strictly speaking, we always work at a finite nonzero value of g^2 and we are not concerned with the definition of or the relation with a perturbative renormalization scheme. This question is discussed for instance in refs. [31] (relation with the DR scheme) or in refs. [32] (higher-derivative regularization).

44 And $\mathcal{L}_{SYM} = \frac{1}{4}\int d^2\theta\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \text{h.c.}$ is defined in eqs. (2.1).

with $\widetilde{R}$ charges $q_\lambda = 3/2$ and $q_\chi = -3/2$ as in eq. (4.20). Quantum corrections to the effective Lagrangian appear in the metric factors $\mathcal{H}_C$ and $-\mathcal{H}_{CC}/2$. But the one-loop chiral anomaly of the R-symmetry current generated by massless gauginos leads formally to[45] a chiral source superfield

$$X_{(W),anomaly} = -\frac{C(G)}{8\pi^2}\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}, \tag{6.11}$$

as in the anomaly-corrected supercurrent structure (5.17).

Two different renormalization-group equations follow. Firstly, since the perturbative dependence on μ is restricted to one-loop [6],

$$\mu\frac{d}{d\mu}\mathcal{L}_{W,\mu} = \frac{b_0}{32\pi^2}\int d^2\theta\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \text{h.c.}, \qquad b_0 = 3C(G), \tag{6.12}$$

we infer that[46]

$$\begin{aligned} \mathcal{L}_{W,\mu} &= \int^2 \theta d^2\bar\theta\, \hat{\mathcal{H}}(\hat{L}) + \frac{b_0}{32\pi^2}\ln\frac{\mu}{M}\int^2 \theta\, \widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \text{h.c.}, \\ \mathcal{H}(\hat{L},\mu/M) &= \hat{\mathcal{H}}(\hat{L}) + \frac{b_0}{8\pi^2}\ln\left(\frac{\mu}{M}\right)\hat{L}. \end{aligned} \tag{6.13}$$

The one-loop correction is scale invariant: it will not appear in the divergence of the dilatation current: $\Delta_{(1\text{-}loop)} = \Delta_{(0)}$. But it is not invariant under the rescalings of the parameters M or μ. Since the Wilson coupling $g_{W,\mu}$, which is not a physically significant quantity, is

$$\frac{1}{g^2_{W,\mu}} = \hat{\mathcal{H}}_C(C) + \frac{b_0}{8\pi^2}\ln\frac{\mu}{M}, \tag{6.14}$$

a rescaling of μ in $\mathcal{L}_{W,\mu}$ is controlled by

$$\beta_W(g^2_{W,\mu}) \equiv \mu\frac{d}{d\mu}g^2_{W,\mu} = -\frac{b_0}{8\pi^2}\,g^4_{W,\mu} \tag{6.15}$$

which is exhausted at one-loop.

Secondly, since M is arbitrary, the RG implies that

$$0 = M\frac{d}{dM}\left[\hat{\mathcal{H}}_C(C) + \frac{b_0}{8\pi^2}\ln\frac{\mu}{M}\right] \tag{6.16}$$

and, with[47]

$$M\frac{d}{dM}C = \beta(C) = m^2\,\beta(g^2) \tag{6.17}$$

since we identify C/m^2 with the physical gauge coupling $g^2(M)$,

$$\beta(C) = \frac{1}{8\pi^2}\frac{b_0}{\hat{\mathcal{H}}_{CC}}. \tag{6.18}$$

45 Eq. (5.10).

46 Since $\int d^2\theta d^2\bar\theta\,\hat{L} = \frac{1}{4}\int d^2\theta\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \text{h.c.} + \text{derivative}$.

47 We always define the β function as $\beta \equiv M\frac{d}{dM}g^2$.

The β function is then proportional to the inverse of the linear gauge coupling superfield kinetic metric $-\frac{1}{2}\mathcal{H}_{CC}$, which is positive. With identifications (6.4) and (6.17),

$$\beta(g^2(M)) = \frac{1}{8\pi^2 m^2}\,\frac{b_0}{\hat{\mathcal{H}}_{CC}} = -\frac{m^2}{C^2\hat{\mathcal{H}}_{CC}}\beta_{1\text{-}loop}. \tag{6.19}$$

The tree-level $\mathcal{H}_{(0)}$, eq. (6.2), leads to $\beta(g^2) = \beta_{1\text{-}loop}$, corrections to $\mathcal{H}_{(0)}$ generate higher order contributions.[48]

Under a rescaling $\mu \to e^{\lambda}\mu$ of the Wilson scale,

$$\delta\mathcal{L}_{W,\mu} = \lambda\frac{b_0}{32\pi^2}\int d^2\theta\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \text{c.c.} = -\lambda\frac{b_0}{32\pi^2}\widetilde{\mathrm{Tr}}F_{\mu\nu}F^{\mu\nu} + \ldots \tag{6.20}$$

This variation is the supersymmetry partner of the anomalous variation induced by the $\widetilde{R}$ symmetry rotating the gaugino: under $\lambda_\beta \to e^{\frac{3}{2}i\alpha}\lambda_\beta$,

$$\delta\mathcal{L}_{W,\mu} = -i\alpha\frac{b_0}{32\pi^2}\int d^2\theta\,\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \text{c.c.} = -\alpha\frac{b_0}{32\pi^2}\widetilde{\mathrm{Tr}}F_{\mu\nu}\widetilde{F}^{\mu\nu} + \ldots, \tag{6.21}$$

a variation which can be deduced from the anomaly-matching term (5.24). In this sense, the one-loop term in the Wilson Lagrangian can be understood as a matching term for the anomaly of the $\widetilde{R}$-symmetry.

Following section 5.2, we should then cancel the residual scaling anomaly (5.16) with coefficient $d(w) = C(G)$ by adding to the tree-level Lagrangian function $\mathcal{H}_{(0)}$ the contribution (5.19) with $\epsilon = -1$. This counterterm removes all dependence on the physical scale M and defines the β function. The resulting function $\mathcal{H}$ is

$$\mathcal{H}(\hat{L}) = m^2 \ln\frac{\hat{L}}{m^2} + \frac{C(G)}{8\pi^2}\left[\hat{L}\ln\frac{\hat{L}}{m^2} - \hat{L}\right] + \frac{b_0}{8\pi^2}\ln\frac{\mu}{M}\hat{L}, \tag{6.22}$$

which in turn leads to the Wilson gauge coupling

$$\begin{aligned}\frac{1}{g^2_{W,\mu}} &= \mathcal{H}_C = \frac{m^2}{C} + \frac{C(G)}{8\pi^2}\ln\frac{C}{m^2} + \frac{b_0}{8\pi^2}\ln\frac{\mu}{M} \\ &= \frac{1}{g^2(M)} + \frac{C(G)}{8\pi^2}\ln g^2(M) + \frac{b_0}{8\pi^2}\ln\frac{\mu}{M}.\end{aligned} \tag{6.23}$$

Arbitrariness of M in this expression, or directly formula (6.19), leads to the beta function

$$\beta(g^2) = -\frac{g^4}{8\pi^2}\,\frac{3\,C(G)}{1 - \frac{C(G)}{8\pi^2}g^2} \tag{6.24}$$

which is the all-order NSVZ beta function [12,28,30].

In the function $\mathcal{H}(\hat{L})$, the first term is the classical, tree-level contribution, the second term encodes all perturbative contributions beyond one-loop and the third, μ-dependent term, is the one-loop correction. Hence, the NSVZ beta function can be derived from algebraic and anomaly arguments only, including its denominator, in the formalism with the gauge coupling field which

48 At one-loop only, the equality of the β functions implies $\mathcal{H} = m^2 \ln\hat{L} + b\hat{L}$, with an arbitrary constant b which can be eliminated by a redefinition of the scale μ.

introduces a second, real, anomaly-matching (or canceling) superfield $\hat{L}$.[49] Choosing $M = \mu$ leads to the relation [6]

$$\mathcal{H}_C = \frac{1}{g^2_{W,\mu}} = \frac{m^2}{C} + \frac{C(G)}{8\pi^2} \ln\left(\frac{C}{m^2}\right) = \frac{1}{g^2(\mu)} + \frac{C(G)}{8\pi^2} \ln g^2(\mu) \tag{6.25}$$

and in the chiral version $g^{-2}_{W,\mu} = \mathrm{Re}\, s$. We note however that with the higher-order terms, the Legendre transformation (3.3) leading to the chiral version of the theory cannot be solved analytically: the Wilson gauge coupling field $\mathrm{Re}\, s$ is well-defined (and physically meaningless) but its Kähler potential $\mathcal{K}(S + \overline{S})$ cannot be obtained in a closed form. In this sense, the linear theory (6.22) contains more information than the dual chiral version and its symmetry or anomaly behavior is explicit.

The loop-corrected Wilson Lagrangian for pure super-Yang–Mills is then

$$\begin{aligned} \mathcal{L}_{W,\mu} &= \int d^2\theta d^2\overline{\theta} \left(m^2 \ln \frac{\hat{L}}{m^2} + \frac{C(G)}{8\pi^2} \left[\hat{L} \ln \frac{\hat{L}}{m^2} - \hat{L} \right] \right) \\ &\quad + \frac{b_0}{32\pi^2} \ln \frac{\mu}{M} \int d^2\theta\, \widetilde{\mathrm{Tr}} \mathcal{W}\mathcal{W} + \mathrm{h.c.} \end{aligned} \tag{6.26}$$

Notice that this Lagrangian does not have a potential since the auxiliary D vanishes and the linear superfield does not have an auxiliary field. The value of the coupling constant remains arbitrary in $\mathcal{L}_{W,\mu}$.

In the supercurrent structure (6.9), the source superfield $\chi_{(W)\alpha}$ dictates the behavior of the Wilson Lagrangian under scale transformations and includes then the anomaly contribution (5.16)[50]:

$$\begin{aligned} \chi_{(W)\alpha} &= -\frac{1}{2} \overline{DD} D_\alpha \Delta_{W,\mu} \\ \Delta_{W,\mu} &= 2[\hat{L}\mathcal{H}_{\hat{L}} - \mathcal{H}] = 2m^2 \left[1 - \ln \frac{\hat{L}}{m^2} \right] + \frac{C(G)}{4\pi^2} \hat{L} = \Delta_{(0)} + \frac{C(G)}{4\pi^2} \hat{L}. \end{aligned} \tag{6.27}$$

Again, the first term is due to the classical scale breaking with the gauge coupling field, as induced by the scale dimension $w = 2$ of C, while the second term is due to the anomaly-canceling counterterm which encodes the corrections beyond one-loop. The source superfield $\chi_{(W)\alpha}$ generates the trace of the Belinfante energy-momentum tensor using the on-shell equality $T^\mu{}_\mu = D/4$. Off-shell, omitting fermions,

$$\begin{aligned} \frac{D}{4} &= -\frac{m^2}{C} \left[1 - \frac{C(G)}{8\pi^2} \frac{C}{m^2} \right] (\Box C + 2\mathcal{L}_{SYM}) + \frac{m^2}{2C^2} \left[(\partial_\mu C)(\partial^\mu C) - \frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] \\ &= \delta - \partial^\mu \mathcal{V}_\mu, \end{aligned} \tag{6.28}$$

where δ is the scale variation of the bosonic Lagrangian and $\mathcal{V}_\mu$ is the virial current,

$$\mathcal{V}_\mu = \frac{m^2}{C} \left[1 - \frac{C(G)}{8\pi^2} \frac{C}{m^2} \right] \partial_\mu C = \partial_\mu \left[m^2 \ln \frac{C}{m^2} - \frac{C(G)}{8\pi^2} C \right], \tag{6.29}$$

---

[49] This Lagrangian has been obtained long ago, using similar arguments and somewhat obscure conformal supergravity methods, in ref. [13].

[50] It is an off-shell expression.

according to expressions (4.26) and (4.34).[51] Concentrating on the super-Yang–Mills part $\mathcal{H}_C \mathcal{L}_{SYM}$ of the Wilson Lagrangian (6.26), or equivalently working in a constant background $C = m^2 g^2(M)$, we have firstly

$$\mu \frac{d}{d\mu} \mathcal{H}_C \mathcal{L}_{SYM} = M \frac{dC}{dM} \mathcal{H}_{CC} \mathcal{L}_{SYM} = \frac{b_0}{8\pi^2} \mathcal{L}_{SYM} \tag{6.30}$$

since M is arbitrary. This expresses the one-loop dependence of the Wilson coupling on the Wilson scale μ. Secondly

$$\delta = \frac{D}{4} = -\frac{2}{g^2(M)} \left[1 - \frac{C(G)}{8\pi^2} g^2(M)\right] \mathcal{L}_{SYM} = 2C\mathcal{H}_{CC}\mathcal{L}_{SYM}, \tag{6.31}$$

so that

$$M \frac{d}{dM} \delta = M \frac{dC}{dM} \left[\frac{2}{C^2} \mathcal{L}_{SYM}\right] = 2 \frac{\beta(g^2)}{g^4} \mathcal{L}_{SYM}. \tag{6.32}$$

Since $\delta = T^\mu{}_\mu$ on-shell, this result is a version of the trace anomaly formula [33].

The Wilson Lagrangian defined by the function (6.22) also describes the dynamics of the three-index tensor $H_{\mu\nu\rho} = 3\, \partial_{[\mu} B_{\nu\rho]} - \omega_{\mu\nu\rho}$ with a simple quadratic Lagrangian:

$$\begin{aligned} \mathcal{L}_B &= -\frac{1}{24} \mathcal{H}_{CC}\, H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{24} \epsilon^{\mu\nu\rho\sigma} H_{\mu\nu\rho}\, \mathcal{J}_\sigma \\ &= \frac{m^2}{C^2} \left[1 - \frac{C(G)}{8\pi^2} \frac{C}{m^2}\right] \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{24} \epsilon^{\mu\nu\rho\sigma} H_{\mu\nu\rho}\, \mathcal{J}_\sigma \\ &= \frac{1}{g^4 m^2} \left[1 - \frac{C(G)}{8\pi^2} g^2\right] \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{24} \epsilon^{\mu\nu\rho\sigma} H_{\mu\nu\rho}\, \mathcal{J}_\sigma, \end{aligned} \tag{6.33}$$

where the current J_σ is

$$J_\sigma = \mathcal{H}_{CCC}\, \hat{\chi} \sigma^\mu \bar{\hat{\chi}} = \frac{2m^2}{C^3} \left[1 - \frac{C(G)}{16\pi^2} \frac{C}{m^2}\right] \hat{\chi} \sigma^\mu \bar{\hat{\chi}}, \tag{6.34}$$

in terms of the gauge invariant spinor $\hat{\chi} = \chi - \frac{1}{2} \sigma^\mu \widetilde{\mathrm{Tr}}\, \bar{\lambda} a_\mu$. The antisymmetric tensor with gauge invariance is equivalent to a pseudoscalar with shift symmetry. The duality transformation is performed by first considering $H_{\mu\nu\rho}$ as an unconstrained three-form field with Bianchi identity

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu H_{\nu\rho\sigma} = -3\, \widetilde{\mathrm{Tr}} \Big[F_{\mu\nu} \widetilde{F}^{\mu\nu} - 2\, \partial_\mu (\lambda \sigma^\mu \bar{\lambda})\Big] \tag{6.35}$$

imposed by a Lagrange multiplier scalar a. Eliminating $H_{\mu\nu\rho}$ with

$$H_{\mu\nu\rho} = -2\, \mathcal{H}_{CC}^{-1}\, \epsilon_{\mu\nu\rho\sigma} [\partial^\sigma a + \frac{1}{4} \mathcal{J}^\sigma], \tag{6.36}$$

the dual theory is

$$\mathcal{L}_{axion} = -\frac{2}{\mathcal{H}_{CC}} \frac{1}{2} \left[\partial^\mu a + \frac{1}{4} \mathcal{J}^\mu\right] \left[\partial_\mu a + \frac{1}{4} \mathcal{J}_\mu\right] - \frac{a}{2}\, \widetilde{\mathrm{Tr}} \Big[F_{\mu\nu} \widetilde{F}^{\mu\nu} - 2\, \partial_\mu (\lambda \sigma^\mu \bar{\lambda})\Big] \tag{6.37}$$

[51] Notice that $D = 2D_\Delta$, as defined in eq. (4.32).

and a is an axion field with standard coupling to $\widetilde{\mathrm{Tr}} F_{\mu\nu}\widetilde{F}^{\mu\nu}$ and a kinetic metric inverse of the gauge coupling field metric. Since

$$-\frac{2}{\mathcal{H}_{CC}} = 2m^2 g^4 \left[1 - \frac{C(G)}{8\pi^2}\right]^{-1} = -m^2 \frac{16\pi^2}{3C(G)}\beta, \tag{6.38}$$

the canonically normalized axion field ma couples with scale m^{-1} to $\widetilde{\mathrm{Tr}} F_{\mu\nu}\widetilde{F}^{\mu\nu}$.

Notice that it is legitimate to use a instead of $h_{\mu\nu\rho}$ as supersymmetry partner of the gauge coupling field C. Simply, while C and $h_{\mu\nu\rho}$ belong to an off-shell linear representation of supersymmetry, a and C have nonlinear supersymmetry variations depending on the Lagrangian function $\mathcal{H}$, as prescribed by the duality transformation.

6.1.2. $\mathcal{N} = 2$ *super-Yang–Mills*

The $\mathcal{N} = 2$ super-Yang–Mills theory adds a chiral X in the adjoint representation to the gauge superfield $\mathcal{W}_\alpha$. The chiral superfield

$$\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} - \frac{1}{2}\overline{DD}(\overline{X}e^{\mathcal{A}}X)$$

transforms with a derivative under the second supersymmetry and a superpotential for X is not permitted.[52] The classical Lagrangian can be written in various equivalent forms[53]:

$$\begin{aligned}
\mathcal{L}_{cl.} &= \frac{1}{g^2}\int d^2\theta \left[\frac{1}{4}\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} - \frac{1}{8}\overline{DD}(\overline{X}e^{\mathcal{A}}X)\right] + \text{h.c.} \\
&= \frac{1}{g^2}\int d^2\theta d^2\bar\theta\, \overline{X}e^{\mathcal{A}}X + \frac{1}{4g^2}\int d^2\theta\, \widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \text{h.c.} + \text{derivative} \\
&= \frac{1}{g^2}\int d^2\theta d^2\bar\theta \left[\hat{L} + \overline{X}e^{\mathcal{A}}X\right] + \text{derivative.}
\end{aligned} \tag{6.39}$$

The supersymmetry variations are of course independent from the gauge coupling constant g. In the last expression, the linear superfield L would be non-dynamical: its contribution to the Lagrangian is a derivative.

The traditional introduction of renormalized quantities in a theory with chiral matter superfields amounts to writing

$$\mathcal{L} = \frac{1}{4g^2}\int d^2\theta\, \widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \text{h.c.} + \sum_i \int d^2\theta d^2\bar\theta\, Z_i \overline{\Phi}_{r_i} e^{\mathcal{A}_{r_i}} \Phi_{r_i}, \tag{6.40}$$

where the sum is over irreducible components r_i of the matter representation r. Then, in $\mathcal{N} = 2$ super-Yang–Mills theory,

$$Z_X = \frac{1}{g^2} \tag{6.41}$$

and the corresponding anomalous dimension is

$$\gamma_X = -M\frac{d}{dM}\ln Z_X = \frac{1}{g^2}\beta(g^2) \tag{6.42}$$

---

[52] A Fayet–Iliopoulos term linear in X would be allowed in a $U(1)$ theory.

[53] Even though written in $\mathcal{N} = 1$ superspace, this Lagrangian has $\mathcal{N} = 2$ off-shell supersymmetry. This is not the case for theories with hypermultiplets.

in terms of the renormalization scale M used to normalize quantities. In $\mathcal{N}=2$, the beta function is purely one-loop and, with $b_0 = 2C(G)$ for super-Yang–Mills theory,

$$\beta(g^2) = -\frac{g^4}{4\pi^2}C(G), \qquad \gamma_X(g^2) = -\frac{g^2}{4\pi^2}C(G). \tag{6.43}$$

The last result provides a derivation of the gauge contribution to the scheme-independent one-loop anomalous dimension of a chiral superfield in irreducible representation r: since the anomalous dimension follows from the two-point function of this superfield, the relevant group quantity is

$$\sum_{a,j}(T_r^a)^i{}_j(T_r^a)^j{}_k \equiv C(r)\,\delta^i_k \tag{6.44}$$

instead of $T(r)$ in β functions. But $C(AdjG) = T(AdjG) = C(G)$ and then

$$\gamma_{r,gauge} = -\frac{g^2}{4\pi^2}C(r). \tag{6.45}$$

Inserting the values of γ_X and b_0 in the NSVZ formula [12]

$$\beta_{\rm NSVZ}(g^2) = -\frac{g^4}{8\pi^2}\frac{b_0 + \sum_i \gamma_{r_i}T(r_i)}{1-\frac{C(G)}{8\pi^2}g^2}, \qquad \gamma_{r_i} = -M\frac{d}{dM}\ln Z_i, \tag{6.46}$$

the denominator simplifies and the one-loop β function (6.43) is obtained.

An alternative formulation is to redefine the renormalization constant Z_X as

$$Z_X = \frac{1}{g^2}\widehat{Z}_X \tag{6.47}$$

and to reexpress the $\mathcal{N}=2$ super-Yang–Mills $\beta_{\rm NSVZ}$ as

$$\beta_{\rm NSVZ}(g^2) = -\frac{g^4}{8\pi^2}\frac{b_0 + \widehat{\gamma}_X T(r_X)}{1-\frac{g^2}{8\pi^2}[C(G)-T(r_X)]}, \qquad \widehat{\gamma}_X = -M\frac{d}{dM}\ln\widehat{Z}_X. \tag{6.48}$$

Since $T(r_X) = C(G)$ the denominator disappears and $\widehat{\gamma}_X = 0$, see eq. (6.41).

With the gauge coupling field C, the natural $\mathcal{N}=2$ extension of the super-Yang–Mills Wilson Lagrangian (6.26) is clearly

$$\begin{aligned}\mathcal{L}_{W,\mu} &= m^2\int d^2\theta d^2\overline{\theta}\,\ln\Big[\hat{L} + \overline{X}e^{\mathcal{A}}X\Big] \\ &\qquad + \frac{b_0}{32\pi^2}\ln\frac{\mu}{M}\int d^2\theta\left[\widetilde{\rm Tr}\mathcal{W}\mathcal{W} - \frac{1}{2}\overline{DD}(\overline{X}e^{\mathcal{A}}X)\right] + \text{h.c.}\end{aligned} \tag{6.49}$$

Since we now have scalar fields in X, we will use this Lagrangian for zero background value of $X|_{\theta=0}$, *i.e.* in the phase with unbroken gauge symmetry. Expand:

$$\begin{aligned}\mathcal{L}_{W,\mu} &= \int d^2\theta d^2\overline{\theta}\left[m^2\ln\hat{L} + \left(\frac{m^2}{\hat{L}} + \frac{b_0}{8\pi^2}\ln\frac{\mu}{M}\right)\overline{X}e^{\mathcal{A}}X\right] + \dots \\ &\qquad + \frac{b_0}{32\pi^2}\ln\frac{\mu}{M}\int d^2\theta\,\widetilde{\rm Tr}\mathcal{W}\mathcal{W} + \text{h.c.} \\ &= \frac{1}{g^2_{W,\mu}}\left[\frac{1}{4}\int d^2\theta\,\widetilde{\rm Tr}\mathcal{W}\mathcal{W} + \text{h.c.} + \int d^2\theta d^2\overline{\theta}\,\overline{X}e^{\mathcal{A}}X\right] + \dots\end{aligned} \tag{6.50}$$

with $b_0 = 2C(G)$ and

$$\frac{1}{g^2_{W,\mu}} = \frac{m^2}{C} + \frac{b_0}{8\pi^2}\ln\frac{\mu}{M}. \tag{6.51}$$

The Wilson wave-function renormalization constant for X is then

$$Z_{W,X} = \frac{1}{g^2_{W,\mu}} = \frac{m^2}{C} + \frac{b_0}{8\pi^2}\ln\frac{\mu}{M} = \frac{1}{g^2(M)} + \frac{b_0}{8\pi^2}\ln\frac{\mu}{M} \tag{6.52}$$

with anomalous dimension

$$\gamma_{W,X} = -\mu\frac{d}{d\mu}\ln Z_{W,X} = -\frac{C(G)}{4\pi^2}g^2_{W,\mu}, \tag{6.53}$$

as expected. Notice that, as earlier, the scale M is arbitrary in the Wilson Lagrangian which only changes if the Wilson scale μ is varied.

We should maybe remark here that the anomalous dimension of the superfield X, as defined in eqs. (6.40) and (6.46), does not vanish[54]: its purely one-loop (and anyway scheme-independent) value is needed to cancel the higher-order terms in the NSVZ β function (6.46). The point is that the second supersymmetry correlates Z_X and γ_X with the inverse gauge coupling and the β function. After rescaling to canonical gauge kinetic terms, all fields in the super-Yang–Mills multiplet have the canonical scale dimension required by gauge invariance. As observed in ref. [7], the rescaling is not anomalous: in our formulation, this is the absence of the contribution (5.19).

The introduction of $\mathcal{N} = 2$ hypermultiplets implies the presence of a superpotential. It should be g-independent to be compatible with the real coupling field. Using chiral superfields $\mathcal{H}_i$ and $\widetilde{\mathcal{H}}^i$ in representations $r_{\mathcal{H}}$ and $\bar{r}_{\mathcal{H}}$ to describe the hypermultiplets, the appropriate $\mathcal{N} = 2$ Lagrangian reads

$$\begin{aligned}\mathcal{L} = {}& \frac{1}{g^2}\int d^2\theta\left[\frac{1}{4}\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} - \frac{1}{8}\overline{DD}(\overline{X}e^{\mathcal{A}}X)\right] + \text{h.c.} \\ & + \int d^2\theta d^2\bar\theta\left[\overline{\widetilde{\mathcal{H}}}e^{-\mathcal{A}_{\mathcal{H}}}\widetilde{\mathcal{H}} + \overline{\mathcal{H}}e^{\mathcal{A}_{\mathcal{H}}}\mathcal{H}\right] + \frac{i}{\sqrt{2}}\int d^2\theta\,\widetilde{\mathcal{H}}X_{\mathcal{H}}\mathcal{H} + \text{h.c.}\end{aligned} \tag{6.54}$$

where $\mathcal{A}_{\mathcal{H}}$ and $X_{\mathcal{H}}$ are matrix-valued in the representation $r_{\mathcal{H}}$ of the hypermultiplets. The wave-function renormalization constants are then

$$Z_X = \frac{1}{g^2}\widehat{Z}_X, \qquad \widehat{Z}_X = Z_{\mathcal{H}} = Z_{\widetilde{\mathcal{H}}} = 1, \tag{6.55}$$

where the last equalities are due to the non-renormalization theorem of $\mathcal{N} = 2$ theories. With these choices, the NSVZ β function becomes

$$\beta_{\text{NSVZ}}(g^2) = -\frac{g^4}{8\pi^2}\frac{b_0 + \widehat{\gamma}_X T(r_X) + 2\gamma_{\mathcal{H}}T(r_{\mathcal{H}})}{1 - \frac{g^2}{8\pi^2}[C(G) - T(r_X)]} = -\frac{g^4}{4\pi^2}[C(G) - T(r_{\mathcal{H}})]. \tag{6.56}$$

In this expression,

$$\widehat{\gamma}_X = -M\frac{d}{dM}\ln\widehat{Z}_X = 0, \qquad \gamma_{\mathcal{H}} = -M\frac{d}{dM}\ln Z_{\mathcal{H}} = 0, \qquad T(r_X) = C(G). \tag{6.57}$$

If the hypermultiplet is in the adjoint representation, $\beta = 0$ and the theory has $\mathcal{N} = 4$ supersymmetry.

[54] As occasionally stated, see for instance ref. [34].

6.2. Gaugino condensates, nonperturbative superpotentials

The effective action describing gaugino condensates $\langle\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}\rangle = -\langle\widetilde{\mathrm{Tr}}\lambda\lambda\rangle$ is formally derived by coupling the operator $\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}$ to a classical source field J in the path integral and taking the Legendre transformation exchanging J with the condensate classical field U. In the supersymmetric context, J and U are expected to be chiral superfields since $\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}$ is chiral, but U should also keep track of the relation $\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} = \overline{DD}\Omega$. Consider again the microscopic theory (4.5):

$$\mathcal{L} = \int d^2\theta d^2\bar\theta\, \mathcal{H}(\hat{L}, Y) + \int d^2\theta\, W(\Phi) + \int d^2\bar\theta\, \overline{W}(\bar\Phi).$$

As explained in Section 3, this is equivalent to

$$\begin{aligned}\mathcal{L} &= \int d^2\theta d^2\bar\theta \left[\mathcal{H}(V,Y) - \frac{1}{2}(S+\bar S)(V+2\Omega)\right] + \int d^2\theta\, W(\Phi) + \int d^2\bar\theta\, \overline{W}(\bar\Phi) \\ &= \int d^2\theta d^2\bar\theta \left[\mathcal{H}(V,Y) - \frac{1}{2}(S+\bar S)V\right] + \text{derivative} \\ &\quad + \int d^2\theta \left[W(\Phi) + \frac{1}{4} S\, \widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}\right] + \int d^2\bar\theta \left[\overline{W}(\bar\Phi) + \frac{1}{4}\bar S\, \widetilde{\mathrm{Tr}}\overline{\mathcal{W}\mathcal{W}}\right].\end{aligned} \tag{6.58}$$

The real, unconstrained gauge-invariant superfield V has the same canonical dimension two as $\hat{L}$. We assume that S has natural scale dimension and $U(1)_R$ charge $w = q = 0$. The S-dependent terms in the Lagrangian are then scale invariant and do not modify the scale-breaking superfield Δ.

The last equality (6.58) firstly shows that S is actually the source superfield J.[55] Secondly, the integration over the gauge superfield is now confined in a universal ($\mathcal{H}$-independent) term and in the matter dependence of $\mathcal{H}$. Finally, the Lagrangian has an axionic shift symmetry $\delta S = ic$ which in the last line exists because $\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} = \overline{DD}\Omega$.

Consider first pure super-Yang–Mills theory:

$$\mathcal{L}_{SYM} = \int d^2\theta d^2\bar\theta \left[\mathcal{H}(V) - \frac{1}{2}(S+\bar S)V\right] + \frac{1}{4}\int d^2\theta\, S\, \widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W} + \frac{1}{4}\int d^2\bar\theta\, \bar S\, \widetilde{\mathrm{Tr}}\overline{\mathcal{W}\mathcal{W}}. \tag{6.59}$$

Using anomaly-matching, the effective Lagrangian is then of the form

$$\begin{aligned}\mathcal{L}_{SYM,eff.} &= \int d^2\theta d^2\bar\theta \left[\mathcal{H}(V) + \frac{d(w)}{8\pi^2} V\left(\ln\frac{V}{m^2} - 1\right) + \mathcal{K}_{(U)}\right] \\ &\quad + \frac{1}{4}\int d^2\theta \left[S\left(U + \frac{1}{2}\overline{DD}V\right) + \frac{b(w)}{24\pi^2} U\left(\ln\frac{U}{M^3} - 1\right)\right] \\ &\quad + \frac{1}{4}\int d^2\bar\theta \left[\bar S\left(\bar U + \frac{1}{2}DDV\right) + \frac{b(w)}{24\pi^2}\bar U\left(\ln\frac{\bar U}{M^3} - 1\right)\right],\end{aligned} \tag{6.60}$$

with $b(w) = 3C(G) = 3d(w)$ in the absence of chiral superfields. In the first line, m is the irrelevant mass parameter already present, for instance, in the Wilson Lagrangian (6.22). However,

[55] Introducing the source is equivalent to replace S by J. The condensate superfield U is then the Legendre dual of J calculated at $J = S$ and the effective Lagrangian depends then on U and S.

the chiral, U-dependent contributions match the one-loop anomaly induced by a rescaling of the physical scale M, as in the Wilson Lagrangian (6.13), and corresponding to the identification (6.4). The field equation for S is

$$U = -\frac{1}{2}\overline{DD}\,V \qquad \Longleftrightarrow \qquad \langle \widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}\rangle = -\frac{1}{2}\overline{DD}\,\langle \hat{L}\rangle \tag{6.61}$$

as required and the effective Lagrangian is then a function of V, DDV and $\overline{DD}V$,

$$\begin{aligned}\mathcal{L}_{SYM,eff.} = & \int d^2\theta d^2\bar{\theta}\left[\mathcal{H}(V) + \frac{d(w)}{8\pi^2}\left(V\ln\frac{V}{m^2} - V\right) + \mathcal{K}_{(U)}\right]_{U=-\frac{1}{2}\overline{DD}\,V} \\ & + \frac{b(w)}{96\pi^2}\int d^2\theta\left[U\ln\frac{U}{M^3} - U\right]_{U=-\frac{1}{2}\overline{DD}\,V} + \text{h.c.}\end{aligned} \tag{6.62}$$

The real function $\mathcal{K}_{(U)}$ of U is the induced Kähler potential which generates kinetic terms for the components of the condensate superfield U.[56]

To derive the gaugino condensate, we need the bosonic component expansion of V:

$$\begin{aligned} V &= C - \theta\theta\,\overline{F} - \overline{\theta\theta}\,F + \theta\sigma^\mu\bar{\theta}v_\mu + \theta\theta\overline{\theta\theta}\left(\tfrac{1}{2}D + \tfrac{1}{4}\Box C\right), \\ U &= u - \theta\theta f_u, \\ u &= -2F, \qquad \mathrm{Re}\, f_u = -D, \qquad \mathrm{Im}\, f_u = -\partial^\mu v_\mu. \end{aligned} \tag{6.63}$$

The gaugino condensate is then the value of the (classical) superfield U at the minimum of the effective potential included in the effective Lagrangian: $\langle\widetilde{\mathrm{Tr}}\lambda\lambda\rangle = -\langle u\rangle = 2\langle F\rangle$. The scalar potential is the sum of three squares induced by the field equations of the three real auxiliary fields D, $\mathrm{Re}\, F$ and $\mathrm{Im}\, F$ included in V. As usual with a coupling field, gaugino condensation alone does not lead to a stabilized ground state: there is a runaway behavior and further contributions would be needed to determine the ground state value of C, *i.e.* to dynamically determine the value of the gauge coupling. But the gaugino condensate is determined as a function of C by the cancellation of the terms linear in D, which in $\mathcal{L}_{SYM,eff.}$ are

$$D\Big[\frac{1}{2}\mathcal{H}_C + \frac{d(w)}{16\pi^2}\ln\frac{C}{m^2} + \frac{b(w)}{48\pi^2}\ln\frac{|u|}{M^3}\Big] \tag{6.64}$$

and a quadratic term is generated by $K_{(U)}$. The linear terms cancel at the supersymmetric ground state:

$$|u| = M^3\left[\frac{C}{m^2}\right]^{-3d(w)/b(w)}\exp\left(-\frac{24\pi^2}{b(w)}\mathcal{H}_C\right) \tag{6.65}$$

or, with the identification $C = m^2 g^2(M)$,

$$|u| = M^3[g^2(M)]^{-3d(w)/b(w)}\exp\left(-\frac{24\pi^2}{b(w)}\mathcal{H}_C\right). \tag{6.66}$$

This formula holds for super-Yang–Mills theory in which $\mathcal{H} = m^2\ln(\hat{L}/m^2)$, and the gaugino condensate is then

[56] While $\mathcal{H}$ generates the kinetic terms of the gauge coupling field and supersymmetry partners as in the microscopic Lagrangian.

$$|u| = \frac{M^3}{g^2(M)} \exp\left(-\frac{8\pi^2}{C(G)g^2(M)}\right). \tag{6.67}$$

The gaugino condensate is a physical quantity which is then invariant under the renormalization group. The condition $M\frac{d}{dM}|u| = 0$ applied on formula (6.65) provides then a derivation of the beta function

$$\beta(g^2) = -\frac{g^4}{4\pi^2}\frac{b(w)}{1-\frac{g^2}{8\pi^2}d(w)},$$

and it also provides a definition for the RG-invariant scale characterizing the strength of the gauge interaction,

$$|u| = \Lambda^3, \tag{6.68}$$

which is a derived quantity.

The effective Lagrangian (6.60) has a continuous $\widetilde{R}$ symmetry with transformations

$$U \longrightarrow e^{3i\alpha}U, \qquad S \longrightarrow S - i\frac{b(w)}{8\pi^2}\alpha \tag{6.69}$$

and $\theta \to e^{\frac{3}{2}i\alpha}\theta$. The rotation of the Grassmann coordinates induces the rotation of U once S imposes $U = -\frac{1}{2}\overline{DD}V$ and the shift of S is induced by the chiral anomaly. In version (6.62) of the theory, the $\widetilde{R}$ symmetry is manifest since $U = -\frac{1}{2}\overline{DD}V$ and $\mathrm{Re}\int d^2\theta\, U \ln U$ transforms then with a derivative. The $\widetilde{R}$ symmetry is spontaneously broken by the gaugino condensate and, since the D contribution to the scalar potential only specifies the modulus $|\langle \widetilde{\mathrm{Tr}}\lambda\lambda\rangle|$, the condensate phase provides the expected ground state degeneracy.

The effective Lagrangian (6.62) has been derived from perturbative anomaly arguments. It is expected that non-perturbative contributions discretize the $\widetilde{R}$ transformations. Concentrating now on $SU(N)$ super-Yang–Mills theory, $b(w) = 3C(G) = 3N$, discretization to Z_{2N} implies that the parameter α in transformations (6.69) has values

$$\frac{3}{2}\alpha = \frac{\pi k}{N}, \qquad k \text{ integer}. \tag{6.70}$$

This follows from the shift in S (6.69) which effectively corresponds to an anomalous $\widetilde{R}$ transformation

$$\delta\mathcal{S}_{SYM} = -3Nq\,\alpha, \qquad q = \frac{1}{32\pi^2}\int d^4x\, \widetilde{\mathrm{Tr}}F_{\mu\nu}\widetilde{F}^{\mu\nu} \tag{6.71}$$

of the super-Yang–Mills action. Since for non-trivial gauge field configurations q is an integer, condition (6.70) follows.

Since U^{kN} is now invariant, we may add non-perturbative contributions to the superpotential term of the effective Lagrangian (6.60):

$$\begin{aligned} W(S,U) &= W_{pert.}(S,U) + W_{np}(U) + \frac{1}{8}S\overline{DD}V, \\ W_{pert.}(S,U) &= \frac{1}{4}U\left[S + \frac{N}{8\pi^2}\left(\ln\frac{U}{M^3} - 1\right)\right], \qquad W_{np}(U) = \frac{1}{4}U\sum_{n\geq 1}\frac{1}{kN}c_k\,U^{kN}, \end{aligned} \tag{6.72}$$

with complex coefficients c_k.

To verify that the non-perturbative contribution sums k-instanton terms, a standard approach is to neglect the Kähler potential $\mathcal{K}_{(U)}$, omit the last term in W and eliminate U as a function of S. Both S and U are then unconstrained chiral superfields and the superpotential $W(S,U)$ generates the Legendre transformation

$$0 = \frac{\partial}{\partial U}[W_{pert.}(S,U) + W_{np}(U)] \tag{6.73}$$

which expresses U as a function of S. Perturbatively,

$$0 = \frac{\partial}{\partial U} W_{pert.}(S,U) \qquad \Longrightarrow \qquad U = M^3 e^{-8\pi^2 S/N}. \tag{6.74}$$

Replacing in $W_{pert.}(S,U) + W_{np}(U)$ leads to

$$W_{(S)} = \frac{1}{4} M^3 e^{-8\pi^2 S/N} \left[-\frac{N}{8\pi^2} + \sum_{k\geq 1} \frac{1}{kN} c_k \left(M^3 e^{-8\pi^2 S} \right)^k \right] \tag{6.75}$$

as expected from k-instanton contributions expressed in terms of the Wilson holomorphic coupling field S. The complete Legendre transformation (6.73) is of course much more complicated. In any case, this procedure is a crude approximation of the effective Lagrangian (6.60) which in particular turns background equations into overconstrained superfield equations. For instance, a term quadratic in D is generated by $\mathcal{K}_{(U)}$. Without this term, the field equation for D is the constraint $\frac{\partial}{\partial C}\mathcal{L}_{SYM,eff.} = 0$.

7. Discussion

In this work we have studied effective actions obtained by replacing the Yang–Mills coupling constant of $\mathcal{N}=1$ SYM by a real field embedded in a linear superfield. As a consequence, holomorphicity is not relevant. This choice introduces a second real gauge-invariant super-Yang–Mills operator $\hat{L}$ to be used in D-terms of effective Lagrangians. We have then shown how this approach allows to correctly treat the quantum anomalies of $U(1)_R$ and dilatation transformations. In particular, we have shown that the one-loop running of the Wilsonian action and anomaly matching in this effective approach are sufficient to derive the NSVZ β function, provided a specific D-term anomaly counterterm constructed with $\hat{L}$ is used to account for the discrepancy in the anomalous behaviors of R and dilatation transformations. This counterterm is at the origin of the denominator of β. We have also shown that a similar approach leads to an effective Lagrangian for super-Yang–Mills condensates, in terms of a real superfield V (for $\langle\hat{L}\rangle$) and a chiral $U = -\frac{1}{2}\overline{DD}V$ (for $\langle\widetilde{\mathrm{Tr}}\mathcal{W}\mathcal{W}\rangle$), with two outcomes: a scalar potential predicting the value of the modulus of the gaugino condensate, as function of the physical gauge coupling with correct all-order behavior and another derivation of the NSVZ β function. Since the real counterterm in terms of $\hat{L}$ cannot be (analytically) transformed into a F-term by chiral-linear duality, we conclude then that embedding the coupling field in a linear superfield is not only useful but actually necessary for a correct description of super-Yang–Mills theory.

That R and dilatation transformations must be considered in a super-Poincaré theory has a simple origin. These transformations are symmetries of the $\mathcal{N}=1$ superconformal algebra and the multiplets of Poincaré supersymmetry carry a representation of the full superconformal algebra (with same scale dimension and R charge for chiral superfields, as unique restriction). Hence, the R and dilatation currents are well-defined but not conserved in a generic Poincaré theory. We have illustrated this point in our construction of the supercurrent structures of $\mathcal{N}=1$ gauge theories coupled to a linear superfield, with the occurrence of the superfields Δ and $\widetilde{\Delta}$ in the source superfields X and χ_α. The use of the $16_B + 16_F$ operators of the $\mathcal{S}$ supercurrent structure [8] is extremely useful in this respect, in contrast with the Ferrara–Zumino $12_B + 12_F$ structure with X only. This construction is also the main tool in our treatment of R and dilatation anomalies, and then in the construction of effective Lagrangians.

Since we have considered anomalies for a generic simple gauge group with an arbitrary matter content, allowing also anomalous dimensions for the chiral superfields, it is tempting to generalize the NSVZ β function (6.19), (6.24) derived for pure super-Yang–Mills theory to

$$\beta(C) = -\frac{C^2}{8\pi^2}\frac{b(w)}{1-\frac{1}{8\pi^2}d(w)C} \qquad\qquad \beta(g^2) = -\frac{g^4}{8\pi^2}\frac{b(w)}{1-\frac{g^2}{8\pi^2}d(w)}, \tag{7.1}$$

where $b(w)$ and $d(w)$ are given by expressions (5.12) and (5.15). This equation would hold for background matter superfields and anomalous dimensions. However, anomalous dimensions are related to wave-function renormalization, $\gamma = -M\frac{d}{dM}\ln Z$, and the significance of eq. (7.1) can only be established in relation with a dynamical Lagrangian for the matter superfields, a point illustrated in our discussion of $\mathcal{N}=2$ theories. In $\mathcal{N}=1$ theory with chiral matter superfields, a treatment with the field coupling C also requires an appropriate formulation of Konishi anomalies, and the outcome should be an expression of the anomalous dimension as a function of C, in analogy with the case of β. This problem goes far beyond the present paper and our present knowledge, which some of us try to improve.

Acknowledgements

The work of J.H. is partially supported by a Marina Solvay fellowship and by the advanced ERC grant ‘Symmetries and Dualities in Gravity and M-theory’ (SyDuGraM) of Marc Henneaux. The work of N.A., D.A. and J.-P.D. is supported by the Albert Einstein Center for Fundamental Physics (AEC) at the University of Bern. J.-P.D. has benefited from extended stays at Ecole Normale Supérieure, Paris and Ecole Polytechnique, Palaiseau. He wishes to thank his colleagues for hospitality and intelligence. J.H. wishes to thank the University of Bern for its hospitality and financial support. Early stages of this research have been supported by the Swiss National Science Foundation.

Appendix A. The supercurrent superfield equation

This appendix presents the superfield and component formulas for the supercurrent structures used in the main text. One needs to solve the supercurrent superfield equations

$$\overline{D}^{\dot\alpha} J_{\alpha\dot\alpha} = D_\alpha X + \chi_\alpha, \qquad \overline{D}_{\dot\alpha} X = 0, \qquad \chi_\alpha = -\frac{1}{4}\overline{DD}D_\alpha U, \qquad U = U^\dagger \tag{A.1}$$

for the components of the supercurrent superfield $J_{\alpha\dot\alpha}$, as a function of the anomaly sources X and U. We use the following expansion of the chiral superfields X and χ_α:

$$\begin{aligned} X &= x + \sqrt{2}\theta\psi_X - \theta\theta\, f_X - i\theta\sigma^\mu\bar\theta\partial_\mu x - \tfrac{i}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi_X - \tfrac{1}{4}\theta\theta\overline{\theta\theta}\Box x, \\ \chi_\alpha &= -i\lambda_\alpha + \theta_\alpha D + \tfrac{i}{2}(\theta\sigma^\mu\overline{\sigma}^\nu)_\alpha F_{\mu\nu} - \theta\sigma^\mu\bar\theta\partial_\mu\lambda_\alpha - \theta\theta(\sigma^\mu\partial_\mu\bar\lambda)_\alpha \\ &\quad - \tfrac{1}{2}\theta\theta(\sigma^\mu\bar\theta)_\alpha(\partial_\nu F^\nu{}_\mu - i\partial_\mu D) + \tfrac{i}{4}\theta\theta\overline{\theta\theta}\Box\lambda_\alpha, \end{aligned} \tag{A.2}$$

where

$$U = \theta\sigma^\mu\bar{\theta}\, U_\mu + i\,\theta\theta\overline{\theta\lambda} - i\,\overline{\theta\theta}\theta\lambda + \frac{1}{2}\theta\theta\overline{\theta\theta}\, D + \dots, \tag{A.3}$$

$F_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu$ and the dots denote contributions in U which do not appear in χ_α. The real field D is defined by

$$\frac{1}{2}D = U|_{\theta\theta\overline{\theta\theta}} + \frac{1}{4}\Box U|_{\theta=0}. \tag{A.4}$$

With this definition, the lowest component $U|_{\theta=0}$ of U does not appear in the expansion (A.2) of χ_α. Eq. (A.4) is used in theories with a linear superfield, and in the next Appendix.

Then, the supercurrent equation is solved by the component expansion[57]

$$\begin{aligned} J_\mu(x,\theta,\bar{\theta}) &= (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} J_{\alpha\dot{\alpha}} \\ &= \frac{8}{3}\, j_\mu + \theta(S_\mu + 2\sqrt{2}\,\sigma_\mu\overline{\psi}_X) + \bar{\theta}(\bar{S}_\mu - 2\sqrt{2}\,\bar{\sigma}_\mu\psi_X) \\ &\quad - 2i\,\theta\theta\,\partial_\mu\bar{x} + 2i\,\overline{\theta\theta}\,\partial_\mu x \\ &\quad + \theta\sigma^\nu\bar{\theta}\left[8\,T_{\mu\nu} - 4\,\eta_{\mu\nu}\,\mathrm{Re}\, f_X - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left(\frac{8}{3}\,\partial^\rho j^\sigma - F^{\rho\sigma}\right)\right] \\ &\quad - \frac{i}{2}\theta\theta\bar{\theta}(\partial_\nu S_\mu\sigma^\nu + 2\sqrt{2}\,\bar{\sigma}_\mu\sigma^\nu\partial_\nu\overline{\psi}_X) \\ &\quad + \frac{i}{2}\overline{\theta\theta}\theta(\sigma^\nu\partial_\nu\bar{S}_\mu + 2\sqrt{2}\,\sigma_\mu\bar{\sigma}^\nu\partial_\nu\psi_X) \\ &\quad - \frac{2}{3}\theta\theta\overline{\theta\theta}\left(2\,\partial_\mu\partial^\nu j_\nu - \Box j_\mu\right), \end{aligned} \tag{A.5}$$

with $T_{\mu\nu} = T_{\nu\mu}$. Eq. (A.1) implies that $T_{\mu\nu}$ and S_μ are conserved. They are identified with the energy-momentum tensor and the supercurrent of the super-Poincaré theory with supercurrent superfield $J_{\alpha\dot{\alpha}}$. Eq. (A.1) also imposes the relations

$$\begin{aligned} &4\,T^\mu{}_\mu = D + 6\,\mathrm{Re}\, f_X, \qquad \partial^\mu j_\mu = -\tfrac{3}{2}\,\mathrm{Im}\, f_X, \\ &(\sigma^\mu\bar{S}_\mu)_\alpha = 6\sqrt{2}\,\psi_{X\,\alpha} + 2i\,\lambda_\alpha \end{aligned} \tag{A.6}$$

between components of $J_{\alpha\dot{\alpha}}$ and the anomaly superfields X and χ_α. The first equation is useful when discussing the behavior of the theory under dilatations: the trace of the energy-momentum tensor is related, but in general not equal, to the divergence of the dilatation current. The second equation controls the behavior of the theory under a $U(1)_R$ transformation with current j_μ. Note that these relations can be used to modify the component expansion (A.5), which is then not unique. Our expansion is as in ref. [8].

The canonical scale dimension w and chiral R charge q of the supercurrent superfield $J_{\alpha\dot{\alpha}}$ are $w = 3$ and $q = 0$. In the superconformal case where $\overline{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = 0$, these dimensions are as required for a conserved dimension three R-current and a conserved dimension four symmetric tensor. The natural weights (w, q) of the source superfields X, χ_α and U are then respectively $(3, 3)$, $(7/2, 3/2)$ and $(2, 0)$.

[57] We do not choose a particularly suitable normalization for the supercurrent S_μ, which we never use here.

The supercurrent superfield equation (A.1) is sufficient for all theories considered in this article. The superfields X and χ_α are usually called *chiral* and *linear* sources or anomalies. Their existence has been known for a long time [16,35] but an unfortunate claim that their simultaneous presence in the supercurrent equation is not compatible with a conserved energy-momentum tensor [36] soon propagated in the literature. It is the merit of Komargodski and Seiberg [8] to have eliminated this mistake.[58] Relations (A.6) indicate that the linear source leads to a conserved R current while the chiral source correlates $T^\mu{}_\mu$ and the divergence of the R current. Hence, different order parameters for $\partial^\mu j_\mu$ and $T^\mu{}_\mu$ require both sources. Notice also that the dilatation current j^D_μ is not present in the supercurrent structure. It is defined (up to the addition of identically conserved currents) as the current for which the variation δ of the Lagrangian under scale transformations equals $\partial^\mu j^D_\mu$ on-shell.

Improvement transformations of the energy-momentum tensor and the supercurrent can be induced by observing that the superfield identity

$$2\overline{D}^{\dot\alpha}[D_\alpha, \overline{D}_{\dot\alpha}]\mathcal{G} = D_\alpha \overline{DD}\mathcal{G} + 3\overline{DD}D_\alpha\mathcal{G}, \tag{A.7}$$

which holds for any superfield $\mathcal{G}$, is a solution of the supercurrent equation (A.1). It can thus be used to transform the supercurrent structure as

$$\begin{aligned} J_{\alpha\dot\alpha} \quad &\longrightarrow \quad \widetilde{J}_{\alpha\dot\alpha} = J_{\alpha\dot\alpha} + 2\,[D_\alpha, \overline{D}_{\dot\alpha}]\,\mathcal{G}, \\ X \quad &\longrightarrow \quad \widetilde{X} = X + \overline{DD}\,\mathcal{G}, \\ \chi_\alpha \quad &\longrightarrow \quad \widetilde{\chi}_\alpha = \chi_\alpha + 3\,\overline{DD}\,D_\alpha\,\mathcal{G}, \end{aligned} \tag{A.8}$$

with any real $\mathcal{G}$. If $\mathcal{G}$ has the expansion

$$\begin{aligned} \mathcal{G} \;=\; & C_g + i\theta\chi_g - i\bar\theta\bar\chi_g + \theta\sigma^\mu\bar\theta v_{g\mu} + \tfrac{i}{2}\theta\theta(M_g + iN_g) - \tfrac{i}{2}\overline{\theta\theta}(M_g - iN_g) \\ & + i\theta\theta\bar\theta(\bar\lambda_g + \tfrac{i}{2}\partial_\mu\chi_g\sigma^\mu) - i\overline{\theta\theta}\theta(\lambda_g - \tfrac{i}{2}\sigma^\mu\partial_\mu\bar\chi_g) + \tfrac{1}{2}\theta\theta\overline{\theta\theta}(D_g - \tfrac{1}{2}\Box C_g), \end{aligned} \tag{A.9}$$

then the components of the transformed superfields $\widetilde{J}_\mu$, $\widetilde{X}$ and $\widetilde{\chi}_\alpha$ read

$$\begin{aligned} \widetilde{j}_\mu \;&=\; j_\mu - 3v_{g\mu}, \\ \widetilde{S}_\mu \;&=\; S_\mu + 8\sigma_{[\mu}\bar\sigma_{\nu]}\partial^\nu\chi_g, \\ \widetilde{\psi}_X \;&=\; \psi_X + 2\sqrt{2}i\lambda_g + 2\sqrt{2}\sigma^\mu\partial_\mu\bar\chi_g, \\ \widetilde{x} \;&=\; x + 2i(M_g - iN_g), \\ \widetilde{T}_{\mu\nu} \;&=\; T_{\mu\nu} + (\partial_\mu\partial_\nu C_g - \eta_{\mu\nu}\Box C_g), \\ \widetilde{f}_X \;&=\; f_X + 2D_g - 2\Box C_g + 2i\partial_\mu v_g^\mu, \\ \widetilde{F}_{\mu\nu} \;&=\; F_{\mu\nu} - 24\partial_{[\mu}v_{g\nu]}, \\ \widetilde{\lambda} \;&=\; \lambda - 12\lambda_g, \\ \widetilde{D} \;&=\; D - 12D_g. \end{aligned} \tag{A.10}$$

58 See also for instance ref. [24].

Hence, the scalar quantity C_g, the lowest component of $\mathcal{G}$ which defines the whole superfield, induces an improvement of the energy-momentum tensor. It also modifies Re f_X to verify the first equation (A.6). Similarly, the fermionic quantity χ_g improves the supercurrent S_μ and changes ψ_X to maintain the validity of the third eq. (A.6). The vector field v_g^μ modifies the nature of the $U(1)$ current j_μ, which becomes in general the current of another $U(1)$ transformation. The other components of $\mathcal{G}$ only exchange quantities in the anomaly superfields X and χ_α.

In practice, we use the superfield transformation (A.8) to improve the energy-momentum tensor and then to modify the relation between its trace and the divergence of the dilatation current, which does not appear in the supercurrent structure. This is useful to have a firm control of scale invariance anomalies. We are, of course, particularly interested in the improvement in which the trace of the energy-momentum tensor *equals* the divergence of the dilatation current, if it exists. The object to consider is the virial current $\mathcal{V}_\mu$ which, under the improvement (A.10) of the energy-momentum tensor, transforms according to

$$\widetilde{\mathcal{V}}_\mu = \mathcal{V}_\mu + 3\,\partial_\mu C_g, \tag{A.11}$$

where we used that the dilatation current satisfies $j_\mu^D = \mathcal{V}_\mu + x^\nu T_{\mu\nu} = \widetilde{\mathcal{V}}_\mu + x^\nu \widetilde{T}_{\mu\nu}$ up to identically conserved currents. This is the subject of Subsection 4.3 and Appendix C. It would also be interesting to see what the inclusion of the virial current superfield introduced in ref. [37] would bring to this analysis.

Appendix B. On the superfield $\Delta(L, \Phi, \overline{\Phi})$ and omitted derivatives

This appendix applies to any real function of L, Φ and $\overline{\Phi}$, like the Lagrangian function $\mathcal{H}(L, \Phi, \overline{\Phi})$ (omitting gauge superfields) but more specifically to Δ and to the discussion in subsection 4.3.

A chiral superfield is usually expanded as

$$\Phi = z + \sqrt{2}\theta\psi - \theta\theta f - i\theta\sigma^\mu\overline{\theta}\,\partial_\mu z + \frac{i}{\sqrt{2}}\theta\theta\,\partial_\mu\psi\sigma^\mu\overline{\theta} - \frac{1}{4}\theta\theta\overline{\theta\theta}\,\Box z$$

to solve $\overline{D}_{\dot\alpha}\Phi = 0$. To solve $DDL = 0$, a real linear superfield writes

$$L = C + i\theta\chi - i\overline{\theta\chi} + \theta\sigma^\mu\overline{\theta}\,v_\mu + \frac{1}{2}\theta\theta\,\partial_\mu\chi\sigma^\mu\overline{\theta} + \frac{1}{2}\overline{\theta\theta}\,\theta\sigma^\mu\partial_\mu\overline{\chi} + \frac{1}{4}\theta\theta\overline{\theta\theta}\,\Box C,$$

with $v_\mu = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}h^{\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu B^{\rho\sigma}$. The opposite sign in the highest component introduces a subtle novelty in the highest component of a function $\Delta(L, \Phi, \overline{\Phi})$ of both chiral and linear superfields:

$$\begin{aligned}\Delta(L, \Phi, \overline{\Phi})|_{\theta\theta\overline{\theta\theta}} &= \mathcal{L}_\Delta - \tfrac{1}{4}\Box\Delta(C, z, \overline{z}) + \tfrac{1}{2}\partial^\mu[\Delta_C\partial_\mu C],\\ \mathcal{L}_\Delta &= -\tfrac{1}{4}\Delta_{CC}[(\partial^\mu C)(\partial_\mu C) + \tfrac{1}{6}h^{\mu\nu\rho}h_{\mu\nu\rho}] + \Delta_{z\overline{z}}[(\partial^\mu\overline{z})(\partial_\mu z) + \overline{f}f]\\ &\quad - \tfrac{i}{2}v^\mu[\Delta_{Cz}\partial_\mu z - \Delta_{C\overline{z}}\partial_\mu\overline{z}] + \text{fermion terms.}\end{aligned} \tag{B.1}$$

Since total derivatives are irrelevant in a Lagrangian, we for instance use in Section 4

$$\int d^4x \int d^2\theta d^2\overline{\theta}\,\mathcal{H}(L, \Phi, \overline{\Phi}) = \int d^4x\,\mathcal{L}_{\mathcal{H}}. \tag{B.2}$$

From this Lagrangian, we derive canonical (Noether) energy-momentum tensor and dilatation current which are not gauge invariant. The symmetric gauge-invariant Belinfante tensor[59] is then obtained by improving the terms involving the antisymmetric tensor and the gauge fields, using field equations.

It is however important to realize that $\mathcal{L}_\Delta$ *differs* from the D component of the superfield Δ, as defined in the expansion (A.3), (A.4) of a real superfield. Instead,

$$D = 2\mathcal{L}_\Delta + \partial^\mu[\Delta_C \partial_\mu C] \tag{B.3}$$

and the derivative term due to the linear superfield must be included when using D. This is in particular the case when evaluating $T^\mu{}_\mu$ in any supercurrent structure using the first eq. (A.6).

The derivatives present in the expansion (B.1) and neglected in Lagrangians would however, if retained, contribute to the naive form of Noether currents. Since these derivatives do not break translation symmetry, they would affect the energy-momentum tensor by an irrelevant improvement term. For instance, applying standard Noether methods to $-\frac{1}{4}\Box\Delta$ leads to the contribution $-\frac{1}{4}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\Delta$ to the energy-momentum tensor. But if scale or chiral $U(1)$ transformations are broken by Δ, the corresponding dilatation and $U(1)$ currents could receive new derivative contributions which can always be safely omitted. But the point is that superfield expressions in general include some of these derivatives, as displayed for instance in eq. (B.1).

To illustrate this remark, consider a single chiral superfield Φ with Kähler potential $\mathcal{K}(\Phi, \overline{\Phi})$ and a non-R chiral $U(1)$ variation $\delta\Phi = iq\Phi$, $\delta\overline{\Phi} = -iq\overline{\Phi}$. The canonical Noether current derived from the standard $\mathcal{N} = 1$ sigma-model Lagrangian is

$$V_\mu = iq\,\mathcal{K}_{z\bar{z}}\left(z\partial_\mu\bar{z} - \bar{z}\partial_\mu z\right) + q\left(\mathcal{K}_{z\bar{z}} + \frac{1}{2}z\mathcal{K}_{zz\bar{z}} + \frac{1}{2}\bar{z}\mathcal{K}_{z\bar{z}\bar{z}}\right)\psi\sigma_\mu\overline{\psi}. \tag{B.4}$$

The vector current in the $\theta\sigma_\mu\bar{\theta}$ component of the current superfield $\mathcal{Z} = \frac{q}{2}(\mathcal{K}_\Phi\Phi + \mathcal{K}_{\overline{\Phi}}\overline{\Phi})$ is however different:

$$\begin{aligned} \mathcal{V}_\mu &= \frac{iq}{2}\left(\bar{z}\mathcal{K}_{\bar{z}\bar{z}} + \mathcal{K}_{\bar{z}} + z\mathcal{K}_{z\bar{z}}\right)\partial_\mu\bar{z} - \frac{iq}{2}\left(z\mathcal{K}_{zz} + \mathcal{K}_z + \bar{z}\mathcal{K}_{z\bar{z}}\right)\partial_\mu z \\ &\quad + \frac{q}{2}\left(2\,\mathcal{K}_{z\bar{z}} + z\mathcal{K}_{zz\bar{z}} + \bar{z}\mathcal{K}_{z\bar{z}\bar{z}}\right)\psi\sigma_\mu\overline{\psi}. \end{aligned} \tag{B.5}$$

The difference

$$\mathcal{V}_\mu - V_\mu = -\frac{iq}{2}\,\partial_\mu\left(\bar{z}\mathcal{K}_{\bar{z}} - z\mathcal{K}_z\right) \tag{B.6}$$

is not an improvement term and vanishes if the Kähler potential has $U(1)$ symmetry. Neglecting fermions since these derivatives affect scalar contributions only,

$$\delta_{U(1)}\left[\mathcal{K}_{z\bar{z}}(\partial_\mu z)(\partial^\mu\bar{z})\right] = iq\frac{\partial^2}{\partial z\partial\bar{z}}(z\mathcal{K}_z - \bar{z}\mathcal{K}_{\bar{z}}) = \partial^\mu V_\mu, \tag{B.7}$$

the second equality holding on-shell, while

$$\partial^\mu\mathcal{V}_\mu = iq\left[\frac{\partial^2}{\partial z\partial\bar{z}} + \frac{1}{2}\Box\right](z\mathcal{K}_z - \bar{z}\mathcal{K}_{\bar{z}}). \tag{B.8}$$

[59] Omitting fermions and gauge fields, its expression is given in eq. (4.21).

The new contribution does not use any field equations. What actually matters is that the quantity $z\mathcal{K}_z - \bar{z}\mathcal{K}_{\bar{z}}$ measures the violation of the $U(1)$ symmetry in both cases, and one can safely use either the standard Noether current V_μ or the superfield current $\mathcal{V}_\mu$.

Appendix C. Improving the energy-momentum tensor

This appendix is mostly concerned with scale (non-)invariance and also to its relation to conformal symmetry in the context of classical theories and at the Lagrangian level.

It should be familiar that, in general, for a given field theory, an energy-momentum tensor $T_{\mu\nu}$ verifies

$$j_\mu^{(dilatations)} \neq x^\nu T_{\mu\nu}, \qquad \partial^\mu j_\mu^{(dilatations)} \neq T^\mu{}_\mu, \tag{C.1}$$

where $j_\mu^{(dilatations)}$ is the current for scale transformations. An improvement of $T_{\mu\nu}$ may turn these relations into equalities, modifying the dilatation current while keeping the (on-shell) value of its divergence unchanged. But this improvement transformation does not always exist.

For instance, in the canonical formalism, in a Poincaré-invariant Lagrangian depending on fields [60] φ_i with scaling dimension w_i and their first derivatives, $\mathcal{L}(\varphi_i, \partial_\mu\varphi_i)$, the Noether current for dilatations is

$$j_\mu^{(dilatations)} = \sum_i w_i \frac{\partial\mathcal{L}}{\partial\partial^\mu\varphi_i}\varphi_i + x^\nu T_{\mu\nu}^{(can.)}, \tag{C.2}$$

where

$$T_{\mu\nu}^{(can.)} = \sum_i \frac{\partial\mathcal{L}}{\partial\partial^\mu\varphi_i}\partial_\nu\varphi_i - \eta_{\mu\nu}\mathcal{L} \tag{C.3}$$

is the canonical (Noether) energy-momentum tensor. The first term is induced by the transformation of the fields at fixed x, the second by the transformation of the coordinates. The field

$$\Delta \equiv \delta\mathcal{L} - 4\mathcal{L} = \sum_i w_i \frac{\partial\mathcal{L}}{\partial\varphi_i}\varphi_i + \sum_i (w_i+1)\frac{\partial\mathcal{L}}{\partial\partial^\mu\varphi_i}\partial^\mu\varphi_i - 4\mathcal{L} \tag{C.4}$$

is a measure for the violation of scale invariance.[61] The currents $T_{\mu\nu}^{(can.)}$ and $j_\mu^{(dilatations)}$ and the quantity Δ are in general calculated in terms of off-shell fields, but the conservation equations

$$\partial^\mu T_{\mu\nu}^{(can.)} = 0, \qquad \partial^\mu j_\mu^{(dilatations)} = \Delta \tag{C.5}$$

are verified on shell. From eq. (C.2), the trace of the canonical energy-momentum tensor satisfies on shell

$$T^{(can.)\mu}{}_\mu = \Delta - \partial^\mu \left(\sum_i w_i \frac{\partial\mathcal{L}}{\partial\partial^\mu\varphi_i}\varphi_i \right), \tag{C.6}$$

and it is in particular not traceless in a scale-invariant theory.

[60] Which are not necessarily scalars only.

[61] We assume that the assignment of scale dimensions w_i has some justification even if Δ does not vanish with any assignment, as in a generic theory without scale invariance.

Except in general for the contribution of scalar fields, the canonical energy-momentum tensor is not symmetric (and not gauge invariant). Lorentz invariance of the theory can be used to improve $T_{\mu\nu}^{(can.)}$ to a symmetric Belinfante tensor[62] $\mathcal{T}_{\mu\nu}$, which also turns out to be gauge invariant. The improvement procedure uses a tensor $\mathcal{X}_{\mu\rho\nu}^{(Bel.)} = -\mathcal{X}_{\rho\mu\nu}^{(Bel.)}$, and

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu}^{(can.)} + \partial^\rho \mathcal{X}_{\mu\rho\nu}^{(Bel.)}. \tag{C.7}$$

In view of eq. (C.2), the canonical dilatation current improves to a Belinfante current according to

$$\begin{aligned} j_\mu^{(dilatations)} &= V_\mu + x^\nu T_{\mu\nu}^{(can.)} \longrightarrow \mathcal{J}_\mu^{(dilatations)} = \mathcal{V}_\mu + x^\nu \mathcal{T}_{\mu\nu}, \\ \mathcal{V}_\mu &= V_\mu + \mathcal{X}^{(Bel.)\rho}{}_{\mu\rho}, \end{aligned} \tag{C.8}$$

omitting in $\mathcal{J}_\mu$ the improvement term $-\partial^\rho(\mathcal{X}_{\mu\rho\nu}x^\nu)$. The vector field $\mathcal{V}_\mu$ is called the *virial current*.

The possibility to improve the energy-momentum tensor suggests that there may exist another symmetric energy-momentum tensor $\Theta_{\mu\nu}$ verifying

$$\partial^\mu J_\mu^{(dilatations)} = \Theta^\mu{}_\mu, \qquad \partial^\mu \Theta_{\mu\nu} = 0. \tag{C.9}$$

Its existence is linked to the interplay of scale and conformal transformations in Poincaré theories: with any symmetric energy-momentum tensor $T_{\mu\nu}$, one can define four additional currents

$$\mathcal{K}_{\mu\nu} = x^2 T_{\mu\nu} - 2x_\nu x^\rho T_{\mu\rho} = -(2x_\nu x_\rho - \eta_{\nu\rho}x^2) T_\mu{}^\rho \tag{C.10}$$

verifying

$$\partial^\mu \mathcal{K}_{\mu\nu} = -2x_\nu T^\mu{}_\mu. \tag{C.11}$$

Hence, if $\Theta_{\mu\nu}$ exists, $\partial^\mu \mathcal{K}_{\mu\nu} = -2x_\nu\, \partial^\mu J_\mu^{(dilatations)}$ (on shell) and the four currents $\mathcal{K}_{\mu\nu}$ constructed with $\Theta_{\mu\nu}$ are always conserved in a scale-invariant theory. Since the $\mathcal{K}_{\mu\nu}$ appear to be the currents for conformal transformations (conformal boosts), a scale-invariant theory is then also conformal. The non-existence of such an energy-momentum tensor is then a feature of field theories where scale invariance does not imply conformal invariance. These Lagrangians are not renormalizable and scale invariance, if present, is in general spontaneously broken.

We then wish to construct a symmetric tensor such that $J_\mu^{(dilatations)} = x^\nu \Theta_{\mu\nu}$ (off shell), or equivalently such that the improved virial current vanishes up to an improvement or a conserved current: $\Theta^\mu{}_\mu = \Delta$ on shell. A first method would be to improve $\mathcal{T}_{\mu\nu}$ to

$$\Theta_{\mu\nu} = \mathcal{T}_{\mu\nu} - \frac{1}{3}\Big(\partial_\nu \mathcal{V}_\mu - \eta_{\mu\nu}\partial^\rho \mathcal{V}_\rho\Big). \tag{C.12}$$

In terms of the improved tensor, the dilatation current is

$$\mathcal{J}_\mu^{(dilatations)} = x^\nu \Theta_{\mu\nu} - \frac{1}{3}\partial^\nu (x_\mu \mathcal{V}_\nu - x_\nu \mathcal{V}_\mu), \tag{C.13}$$

the second term is an improvement with zero divergence which can be omitted to obtain

62 For detail, see for instance refs. [10,11,38,39]. Field equations are used. In this sense, the transformation from the canonical to the symmetric tensor is not truly an improvement.

$$J_\mu^{(dilatations)} = x^\nu \Theta_{\mu\nu}, \qquad \partial^\mu J_\mu^{(dilatations)} = \Theta^\mu{}_\mu, \qquad \partial^\mu \Theta_{\mu\nu} = 0. \tag{C.14}$$

However, both energy-momentum tensors are symmetric only if $\partial_{[\mu}\mathcal{V}_{\nu]} = 0$, up maybe to an improvement term. It is clearly solved if $\mathcal{V}_\mu = \partial_\mu \mathcal{G}$, for some function $\mathcal{G}$ of the off-shell fields. In this case,

$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\mathcal{G}, \tag{C.15}$$

but the existence of $\mathcal{G}$ in terms of off-shell fields is a non-trivial conditions on $\mathcal{V}_\mu$ and then on the Lagrangian.[63]

How to improve the energy-momentum tensor and the dilatation current to obtain equalities (C.14) has been discussed in more general terms long ago and in particular by Callan, Coleman and Jackiw (CCJ) [10,11].[64] They first observe that the tensor $\Theta_{\mu\nu}$ differs by an improvement from the Belinfante tensor only for spin (or helicity) zero fields. To summarize the improvement procedure, it is assumed that there exists a tensor $\sigma_{\mu\nu}$ such that (off shell)

$$\mathcal{V}_\mu = \partial^\nu \sigma_{\mu\nu} = \partial^\nu \sigma_{[\mu\nu]} + \partial^\nu \sigma_{(\mu\nu)}, \qquad \sigma_{[\mu\nu]} = -\sigma_{[\nu\mu]}, \qquad \sigma_{(\mu\nu)} = \sigma_{(\nu\mu)}. \tag{C.16}$$

The first term $\partial^\nu \sigma_{[\mu\nu]}$ is an improvement which can be omitted in the dilatation current, and the second term can be written

$$\partial^\nu \sigma_{(\mu\nu)} = \widehat{\mathcal{X}}_{\mu\nu}{}^\nu \tag{C.17}$$

with

$$\widehat{\mathcal{X}}_{\mu\rho\nu} = \frac{1}{2}\Big[\partial_\mu \sigma_{(\rho\nu)} - \partial_\rho \sigma_{(\mu\nu)} - \eta_{\mu\nu}\partial^\lambda \sigma_{(\rho\lambda)} + \eta_{\rho\nu}\partial^\lambda \sigma_{(\mu\lambda)}\Big] + \frac{1}{6}\Big[\eta_{\mu\nu}\partial_\rho \sigma^\lambda{}_\lambda - \eta_{\rho\nu}\partial_\mu \sigma^\lambda{}_\lambda\Big], \tag{C.18}$$

verifying also

$$\widehat{\mathcal{X}}_{\mu\rho\nu} = -\widehat{\mathcal{X}}_{\rho\mu\nu} \qquad \partial^\rho \widehat{\mathcal{X}}_{\mu\rho\nu} = \partial^\rho \widehat{\mathcal{X}}_{\nu\rho\mu}. \tag{C.19}$$

Then, the improvement formula (C.12) can be extended to

$$\Theta_{\mu\nu} = \mathcal{T}_{\mu\nu} - \partial^\rho \widehat{\mathcal{X}}_{\mu\rho\nu} \tag{C.20}$$

which relates two symmetric energy-momentum tensors. The corresponding improvement of the dilatation current is then

$$\mathcal{J}_\mu^{(dilatations)} = \partial^\nu \sigma_{\mu\nu} + x^\nu \mathcal{T}_{\mu\nu} \qquad \Longrightarrow \qquad J_\mu^{(dilatations)} = x^\nu \Theta_{\mu\nu} \tag{C.21}$$

omitting improvement terms.

Hence, if the condition (C.16) on the virial current is verified, there exists an improved energy-momentum tensor and a dilatation current verifying conservation equations (C.14) and then scale invariance implies conformal symmetry.

In a two-derivative theory with scalar fields only, the virial current is linear in the field derivatives

[63] With a single field φ in a two-derivative Lagrangian, $\mathcal{V}_\mu = f(\varphi)\partial_\mu\varphi$, which can be written as $\partial_\mu g(\varphi)$ with $g' = f$. The condition is already nontrivial with two real scalar fields, as in supersymmetric theories with chiral superfields.

[64] See also Coleman [38], Polchinski [40] or Ortin's book [39], section 2.4.

$$\mathcal{V}_\mu = \sum_i \mathcal{F}_i(\varphi_j)\,\partial_\mu\varphi_i \tag{C.22}$$

and the only available tensor is then $\sigma_{\mu\nu} = \eta_{\mu\nu}\mathcal{F}$, leading to

$$\begin{aligned} \widehat{\chi}_{\mu\rho\nu} &= \tfrac{1}{3}(\eta_{\nu\rho}\partial_\mu\mathcal{F} - \eta_{\mu\nu}\partial_\rho\mathcal{F}), \\ \partial^\rho\widehat{\chi}_{\mu\rho\nu} &= \tfrac{1}{3}(\partial_\mu\partial_\nu\mathcal{F} - \eta_{\mu\nu}\Box\mathcal{F}), \end{aligned} \tag{C.23}$$

as in eq. (C.15). If $\mathcal{V}_\mu = \partial_\mu\mathcal{F}$, or

$$\partial_{[\mu}\mathcal{V}_{\nu]} = 0, \tag{C.24}$$

the tensor $\Theta_{\mu\nu}$ exists and scale invariance implies conformal symmetry. There are however many scalar Lagrangians for which this condition is not verified.

Consider for instance the scalar sector of a Wess–Zumino model with a single chiral superfield and Kähler potential $\mathcal{K}(z,\bar{z})$. Since

$$\begin{aligned} \mathcal{V}_\mu &= wz\mathcal{K}_{z\bar{z}}\,\partial_\mu\bar{z} + w\bar{z}\mathcal{K}_{z\bar{z}}\,\partial_\mu z, \\ \partial_{[\mu}\mathcal{V}_{\nu]} &= w(z\mathcal{K}_{zz\bar{z}} - \bar{z}\mathcal{K}_{z\bar{z}\bar{z}})(\partial_{[\mu}z)(\partial_{\nu]}\bar{z}), \end{aligned} \tag{C.25}$$

the condition (C.24) is $z\mathcal{K}_{zz\bar{z}} = \bar{z}\mathcal{K}_{z\bar{z}\bar{z}}$ which integrates[65] into

$$z\mathcal{K}_z = \bar{z}\mathcal{K}_{\bar{z}}. \tag{C.26}$$

Then,

$$\mathcal{V}_\mu = w\,\partial_\mu(z\mathcal{K}_z) = w\,\partial_\mu(\bar{z}\mathcal{K}_{\bar{z}}) \tag{C.27}$$

and the improved energy momentum tensor is

$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)wz\mathcal{K}_z. \tag{C.28}$$

The outcome is that the scale dimension w must be chosen to correspond to a $U(1)$ symmetry of the Kähler potential acting with charge w on z, $\delta z = iwz$. If such a $U(1)$ symmetry does not exist, one can of course assign $w = 0$, in which case the condition $j_\mu^{(dilatations)} = x^\nu T_{\mu\nu}$ is trivially true already for canonical (Noether) currents. But this choice does not lead to scale invariance.

The simple Kähler potential $\mathcal{K} = \frac{1}{2}(z^2\bar{z} + \bar{z}^2 z)$ does not have a $U(1)$ symmetry. The Lagrangian is

$$\mathcal{L} = (z+\bar{z})(\partial^\mu z)(\partial_\mu\bar{z}). \tag{C.29}$$

It is scale-invariant with dimension $w = 2/3$,

$$\Delta = (3w-2)\mathcal{L}, \tag{C.30}$$

but since the point $z = 0$ is excluded, scale invariance is spontaneously broken by $\langle \mathrm{Re}\, z\rangle \neq 0$. Under the conformal transformation

$$\delta_\alpha z = (2x_\alpha x_\rho - \eta_{\alpha\rho}x^2)\partial^\rho z + 2wzx_\alpha, \tag{C.31}$$

[65] Up to an irrelevant Kähler transformation of $\mathcal{K}$.

the variation of the Lagrangian is

$$\delta_\alpha \mathcal{L} = \partial^\mu[(2x_\alpha x_\mu - \eta_{\alpha\mu}x^2)\mathcal{L}] + 2\mathcal{V}_\alpha + 2x_h a\Delta, \tag{C.32}$$

where

$$\mathcal{V}_\alpha = w(z+\bar{z})(z\partial_\alpha \bar{z} + \bar{z}\partial_\alpha z) = j_\alpha^{(dilatations)} - x^\nu T_{\mu\nu}^{(can.)}, \tag{C.33}$$

using the field equation. Scale invariance ($\Delta = 0$) does not imply conformal invariance: $\delta_\alpha\mathcal{L}$ is not a derivative since $\mathcal{V}_\alpha$ is not a derivative.

In this paper, we are mostly interested in theories without scale invariance, but we need to have control of the relation between the divergence of the dilatation current and the trace of the energy-momentum tensor. The introduction of the linear superfield leads to more subtleties, discussed in Subsection 4.3.

Appendix D. The Ferrara–Zumino supercurrent structure

The supercurrent structure originally found by Ferrara and Zumino (FZ) [16] has $\chi_\alpha = 0$. It is obtained by improving the supercurrent structure (4.42) using identity (A.7) with $\mathcal{G} = \Delta_{(w)}/6$ to eliminate χ_α. The resulting $J_{(FZ)\alpha\dot\alpha}$ does not depend on w:

$$\begin{aligned}
\overline{D}^{\dot\alpha} J_{(FZ)\alpha\dot\alpha} &= D_\alpha X_{(FZ)}, \\
J_{(FZ)\alpha\dot\alpha} &= -2\Big[(\overline{\mathcal{D}}_{\dot\alpha}\overline{\Phi})\mathcal{H}_{\Phi\overline{\Phi}}(\mathcal{D}_\alpha\Phi) - \mathcal{H}_{LL}(\overline{D}_{\dot\alpha}\hat{L})(D_\alpha\hat{L}) + 2\,\mathcal{H}_L\widetilde{\mathrm{Tr}}(\mathcal{W}_\alpha e^{-\mathcal{A}}\overline{\mathcal{W}}_{\dot\alpha}e^{\mathcal{A}})\Big] \\
&\quad - \tfrac{2}{3}[D_\alpha, \overline{D}_{\dot\alpha}](\mathcal{H} - \hat{L}\mathcal{H}_L), \\
X_{(FZ)} &= -\tfrac{4}{3}\widetilde{\Delta}_{(w)} + \tfrac{1}{6}\overline{DD}\Delta_{(w)} + \tfrac{1}{6}\overline{DD}(w\mathcal{H}_\Phi\Phi - w\overline{\Phi}\mathcal{H}_{\overline{\Phi}}) \\
&= 4W - \tfrac{1}{3}\overline{DD}(\mathcal{H} - \hat{L}\mathcal{H}_L).
\end{aligned} \tag{D.34}$$

In the second equality for $X_{(FZ)}$, the superfield equation for Φ has been used but the first expression is actually more significant since it depends on the three *off-shell* superfields

$$\widetilde{\Delta}_{(w)}, \qquad \Delta_{(w)}, \qquad w\mathcal{H}_\Phi\Phi - w\overline{\Phi}\mathcal{H}_{\overline{\Phi}}$$

which control the scale and R symmetries of the theory. With chiral and gauge multiplets only, the function $\mathcal{H}$ is replaced by the gauge invariant Kähler potential $\mathcal{K}$. In the scale-invariant case, $\Delta = \widetilde{\Delta} = 0$, $\mathcal{H} - \hat{L}\mathcal{H}_L = \frac{1}{2}(w\Phi\mathcal{H}_\Phi + w\overline{\Phi}\mathcal{H}_{\overline{\Phi}})$, the FZ structure coincides with our improved supercurrent structure (4.42). But both structures significantly differ if scale transformations are not symmetries.[66]

This is of minor importance for the energy-momentum tensor: the structures differ by improvements. In the FZ supercurrent superfield $J_{\alpha\dot\alpha}$,

$$\widetilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)(\mathcal{H} - C\mathcal{H}_C), \tag{D.35}$$

where $T_{\mu\nu}$ is the Belinfante tensor present in the natural structure (4.19), to be compared with expression (4.46) for the improved supercurrent. Accordingly, the dilatation current becomes

[66] Without a linear superfield, this has been observed in ref. [9].

$$j_\mu^D = \widetilde{\mathcal{V}}_\mu + x^\nu \widetilde{T}_{\mu\nu} \tag{D.36}$$

with virial current[67]

$$\widetilde{\mathcal{V}}_\mu = \frac{1}{2}\,\partial_z \Delta_{(w)} D_\mu z + \frac{1}{2}\,\partial_{\bar z} \Delta_{(w)} D_\mu \bar z, \tag{D.37}$$

omitting an improvement term.

More significant is the chiral $U(1)$ current present in the lowest component of the supercurrent superfield $\widetilde{J}_{\alpha\dot\alpha}$. Compared with the $U(1)_{\widetilde{R}}$ current $j_\mu^{\widetilde{R}}$ (4.20) present in the natural, Belinfante structure (4.19), we now find

$$\begin{aligned}\widetilde{j}_\mu - j_\mu^{\widetilde{R}} &= -C\mathcal{H}_{CC}\tfrac{1}{6}\epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma} - i(\mathcal{H} - C\mathcal{H}_C)_z\, D_\mu z + i(\mathcal{H} - C\mathcal{H}_C)_{\bar z}\, D_\mu \bar z \\ &\quad - C\mathcal{H}_{CC}\widetilde{\mathrm{Tr}}\lambda\sigma_\mu\bar\lambda + [\mathcal{H}_{z\bar z} - C\mathcal{H}_{Cz\bar z}]\,\psi\sigma_\mu\overline{\psi} - \tfrac{1}{2}[\mathcal{H}_{CC} + C\mathcal{H}_{CCC}]\,\chi\sigma_\mu\overline{\chi} \\ &\quad - \tfrac{i}{\sqrt{2}}\,C\mathcal{H}_{CC\bar z}\,\chi\sigma_\mu\overline{\psi} + \tfrac{i}{\sqrt{2}}\,C\mathcal{H}_{CCz}\,\psi\sigma_\mu\overline{\chi}.\end{aligned} \tag{D.38}$$

Alternatively, in terms of derivatives of $\Delta = 2(C\mathcal{H}_C - \mathcal{H})$,

$$\begin{aligned}\widetilde{j}_\mu - j_\mu^{\widetilde{R}} &= -\tfrac{1}{12}\Delta_C\epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma} + \tfrac{i}{2}\Delta_z\, D_\mu z - \tfrac{i}{2}\Delta_{\bar z}\, D_\mu\bar z \\ &\quad - \tfrac{1}{2}\Delta_C\widetilde{\mathrm{Tr}}\lambda\sigma_\mu\bar\lambda - \tfrac{1}{2}\Delta_{z\bar z}\,\psi\sigma_\mu\overline{\psi} - \tfrac{1}{2}\,\Delta_{CC}\,\chi\sigma_\mu\overline{\chi} \\ &\quad - \tfrac{i}{2\sqrt{2}}\,\Delta_{C\bar z}\,\chi\sigma_\mu\overline{\psi} + \tfrac{i}{2\sqrt{2}}\,\Delta_{Cz}\,\psi\sigma_\mu\overline{\chi}.\end{aligned} \tag{D.39}$$

With chiral multiplets only, $\mathcal{H}_C = 0$,

$$\widetilde{j}_\mu - j_\mu^{\widetilde{R}} = -i\mathcal{H}_z\,\partial_\mu z + i\mathcal{H}_{\bar z}\,\partial_\mu\bar z + \mathcal{H}_{z\bar z}\,\psi\sigma_\mu\overline{\psi} \tag{D.40}$$

and $\mathcal{H}$ is the Kähler potential. Hence, the Ferrara–Zumino structure includes the current $\widetilde{j}_\mu$ which is actually the Kähler connection derived from Kähler potential $\mathcal{H}$.

The conclusion is that while our natural, Belinfante (4.19) or improved (4.42) supercurrent structures include the currents naturally related to $U(1)_R$ transformations rotating chiral superfields with angle zero or w, the Ferrara–Zumino structure includes a Kähler current which is not the Noether current of $U(1)$ transformations acting on superfields. Of course, if the theory is scale invariant with scale dimensions w, $2(\mathcal{H} - C\mathcal{H}_C) = -w[\mathcal{H}_\Phi\Phi + \overline{\Phi}\mathcal{H}_{\overline{\Phi}}]$ and the Ferrara–Zumino and improved structures coincide.

Appendix E. Legendre identities

This Appendix collects some useful formula induced by the Legendre transformation

$$\mathcal{K}(X,Y) = \mathcal{F}(L,Y) - \frac{1}{2}XL, \tag{E.1}$$

which generates the chiral-linear duality (Section 3). It implies in particular:

[67] See eqs. (4.34) and (A.11).

$$\mathcal{K}_X = -\frac{1}{2}L, \qquad \mathcal{F}_L = \frac{1}{2}X, \qquad \mathcal{K}_Y = \mathcal{F}_Y, \tag{E.2}$$

$$\frac{\partial L}{\partial X} = -2\,\mathcal{K}_{XX}, \qquad \frac{\partial X}{\partial L} = 2\,\mathcal{F}_{LL}, \qquad -4\,\mathcal{K}_{XX}\mathcal{F}_{LL} = 1, \tag{E.3}$$

$$\frac{\partial X}{\partial Y} = 2\mathcal{F}_{LY}, \qquad \frac{\partial L}{\partial Y} = -2\mathcal{K}_{XY}, \tag{E.4}$$

$$\mathcal{K}_{XY} = \frac{\mathcal{F}_{LY}}{2\mathcal{F}_{LL}}, \qquad \mathcal{F}_{LY} = -\frac{\mathcal{K}_{XY}}{2\mathcal{K}_{XX}}, \tag{E.5}$$

$$\mathcal{K}_{YY} = \mathcal{F}_{YY} - \frac{\mathcal{F}_{LY}^2}{\mathcal{F}_{LL}}, \qquad \mathcal{F}_{YY} = \mathcal{K}_{YY} - \frac{\mathcal{K}_{XY}^2}{\mathcal{K}_{XX}}. \tag{E.6}$$

References

[1] N. Seiberg, Phys. Lett. B 318 (1993) 469, arXiv:hep-ph/9309335;
N. Seiberg, Phys. Rev. D 49 (1994) 6857, arXiv:hep-th/9402044.

[2] N. Seiberg, Phys. Lett. B 206 (1988) 75.

[3] N. Seiberg, E. Witten, Nucl. Phys. B 426 (1994) 19, arXiv:hep-th/9407087;
N. Seiberg, E. Witten, Nucl. Phys. B 430 (1994) 485 (Erratum);
N. Seiberg, E. Witten, Nucl. Phys. B 431 (1994) 484, arXiv:hep-th/9408099.

[4] F. Buccella, J.-P. Derendinger, S. Ferrara, C.A. Savoy, Phys. Lett. B 115 (1982) 375;
C. Procesi, G.W. Schwarz, Phys. Lett. B 161 (1985) 117.

[5] M.A. Shifman, A.I. Vainshtein, Nucl. Phys. B 277 (1986) 456.

[6] M.A. Shifman, A.I. Vainshtein, Nucl. Phys. B 359 (1991) 571.

[7] N. Arkani-Hamed, H. Murayama, J. High Energy Phys. 0006 (2000) 030, arXiv:hep-th/9707133.

[8] Z. Komargodski, N. Seiberg, J. High Energy Phys. 1007 (2010) 017, arXiv:1002.2228 [hep-th].

[9] D. Arnold, J.-P. Derendinger, J. Hartong, Nucl. Phys. B 867 (2013) 370, arXiv:1208.1648 [hep-th].

[10] C.G. Callan Jr., S.R. Coleman, R. Jackiw, Ann. Phys. 59 (1970) 42.

[11] S.R. Coleman, R. Jackiw, Ann. Phys. 67 (1971) 552.

[12] V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B 229 (1983) 381;
V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B 229 (1983) 407;
V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Phys. Lett. B 166 (1986) 329.

[13] J.-P. Derendinger, S. Ferrara, C. Kounnas, F. Zwirner, Phys. Lett. B 271 (1991) 307.

[14] C.P. Burgess, J.-P. Derendinger, F. Quevedo, M. Quiros, Phys. Lett. B 348 (1995) 428, arXiv:hep-th/9501065.

[15] W. Siegel, Phys. Lett. B 85 (1979) 333.

[16] S. Ferrara, B. Zumino, Nucl. Phys. B 87 (1975) 207.

[17] S. Cecotti, S. Ferrara, M. Villasante, Int. J. Mod. Phys. A 2 (1987) 1839.

[18] G. Lopes Cardoso, B.A. Ovrut, Nucl. Phys. B 369 (1992) 351;
G. Lopes Cardoso, B.A. Ovrut, Nucl. Phys. B 392 (1993) 315, arXiv:hep-th/9205009.

[19] J.-P. Derendinger, S. Ferrara, C. Kounnas, F. Zwirner, Nucl. Phys. B 372 (1992) 145.

[20] L.J. Dixon, V. Kaplunovsky, J. Louis, Nucl. Phys. B 355 (1991) 649.

[21] S. Ferrara, J. Wess, B. Zumino, Phys. Lett. B 51 (1974) 239.

[22] J.P. Derendinger, F. Quevedo, M. Quiros, Nucl. Phys. B 428 (1994) 282, arXiv:hep-th/9402007.

[23] G. Mack, Commun. Math. Phys. 55 (1977) 1.

[24] M. Magro, I. Sachs, S. Wolf, Ann. Phys. 298 (2002) 123, arXiv:hep-th/0110131.

[25] J.M. Pons, J. Math. Phys. 52 (2011) 012904, arXiv:0902.4871 [hep-th].

[26] I. Antoniadis, M. Buican, Phys. Rev. D 83 (2011) 105011, arXiv:1102.2294 [hep-th].

[27] G. Veneziano, S. Yankielowicz, Phys. Lett. B 113 (1982) 231.

[28] M.A. Shifman, A.I. Vainshtein, Nucl. Phys. B 296 (1988) 445, Sov. Phys. JETP 66 (1987) 1100.

[29] D.J. Gross, Applications of the renormalization group to high-energy physics, in: Methods in Field Theory, Les Houches 1975, North–Holland, Amsterdam, 1976.

[30] D.R.T. Jones, Phys. Lett. B 123 (1983) 45.

[31] I. Jack, D.R.T. Jones, A. Pickering, Phys. Lett. B 435 (1998) 61, http://dx.doi.org/10.1016/S0370-2693(98)00769-2, arXiv:hep-ph/9805482;

I. Jack, D.R.T. Jones, C.G. North, Nucl. Phys. B 486 (1997) 479, http://dx.doi.org/10.1016/S0550-3213(96)00637-2, arXiv:hep-ph/9609325.
[32] A.L. Kataev, K.V. Stepanyantz, Nucl. Phys. B 875 (2013) 459, http://dx.doi.org/10.1016/j.nuclphysb.2013.07.010, arXiv:1305.7094 [hep-th];
A.L. Kataev, K.V. Stepanyantz, Theor. Math. Phys. 181 (2014) 1531, http://dx.doi.org/10.1007/s11232-014-0233-3, arXiv:1405.7598 [hep-th].
[33] S.L. Adler, J.C. Collins, A. Duncan, Phys. Rev. D 15 (1977) 1712;
P. Minkowski, Unpublished Bern report 76-0813, September 1976;
N.K. Nielsen, Nucl. Phys. B 120 (1977) 212;
J.C. Collins, A. Duncan, S.D. Joglekar, Phys. Rev. D 16 (1977) 438.
[34] R.G. Leigh, M.J. Strassler, Nucl. Phys. B 447 (1995) 95, arXiv:hep-th/9503121.
[35] M.F. Sohnius, P.C. West, Phys. Lett. B 105 (1981) 353.
[36] T.E. Clark, O. Piguet, K. Sibold, Nucl. Phys. B 143 (1978) 445.
[37] Y. Nakayama, Phys. Rev. D 87 (8) (2013) 085005, arXiv:1208.4726 [hep-th].
[38] S. Coleman, Aspects of Symmetry, Cambridge University Press, Cambridge, 1988, 402 pp., Chapter 3.
[39] T. Ortin, Gravity and Strings, Cambridge University Press, Cambridge, 2004, 684 pp.
[40] J. Polchinski, Nucl. Phys. B 303 (1988) 226.