

HOW SHOULD THE GOVERNMENT ALLOCATE ITS TAX REVENUES BETWEEN PRODUCTIVITY- ENHANCING AND UTILITY- ENHANCING PUBLIC GOODS?

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We present a fairly standard general equilibrium model of endogenous growth with productive and nonproductive public goods and services. The former enhance private productivity and the latter private utility. We study Ramsey second-best optimal policy, where the latter is summarized by the paths of the income tax rate and the allocation of collected tax revenues between productivity-enhancing and utility-enhancing public expenditures. We show that the properties and macroeconomic implications of the second-best optimal policy (a) are different from the benchmark case of the social planner's first-best allocation and (b) depend crucially on whether public goods and services are subject to congestion.

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1. INTRODUCTION

Public expenditures on goods and services are traditionally classified as productive or nonproductive. The former, known also as productivity-enhancing, include

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expenditures on infrastructure, the law, education and training, etc. The latter, known also as utility-enhancing, include expenditures on national defense, national parks, social programs, etc. Although in practice what is productive or nonproductive is unclear, this classification has been very common in the theoretical growth literature [see, e.g., Turnovsky (2000)].¹

A natural question to ask is “what is the optimal allocation of government scarce resources (i.e., tax revenues) between the above two categories of public expenditures?” This is the main goal of this paper. It studies Ramsey second-best optimal policy (where policy is summarized by the paths of the income tax rate and the allocation of collected tax revenues between productivity-enhancing and utility-enhancing public goods) in a fairly standard general equilibrium model of endogenous growth. By Ramsey second-best optimal policy, we mean that the paths of (non-lump sum) policy instruments are chosen by a benevolent government that takes into account the competitive decentralized equilibrium, where the latter includes the optimal reactions of private agents to policy instruments.

We show that solving for second-best optimal policy produces very different normative results from the benchmark case of the social planner’s first-best allocation, which is usually studied in the related literature. We also show that the properties and macroeconomic implications of second-best optimal policy depend crucially on whether productivity-enhancing public goods and services are subject to congestion.

Congestion is a form of rivalry. It means that, for a given quantity of aggregate public goods and services, the quantity available to an individual declines as other individuals use the facilities. Examples of productivity-enhancing public goods and services with congestion include highways, police and fire services, courts, public schools, etc. Actually, much of the literature on the role of public investment/capital in endogenous growth has assumed that productivity-enhancing public goods and services are subject to congestion.²

We use a growth model that is a straightforward extension of the tractable model introduced by Barro (1990). This is a commonly used model in the literature. We distinguish between productivity-enhancing public goods and services (denoted as PE and included as an externality in the private production function) and utility-enhancing public goods and services (denoted as UE and included as an external argument in private utility function). We allow public goods to be subject to both absolute and relative congestion, where our modeling of congestion is as in, e.g., Fisher and Turnovsky (1998) and Eicher and Turnovsky (2000). Within this model, government expenditures on PE and UE are financed by income tax revenues. Subject to the competitive decentralized equilibrium, the government chooses its tax policy (the path of the income tax rate) and revenue allocation policy (the path of allocation of collected tax revenues between PE and UE).

Our results are as follows. We focus on the long run, where the economy grows at a constant balanced growth rate. We also find it interesting to focus on how this long-run equilibrium is affected by a preference parameter that measures how much the citizen values public consumption (UE) relative to private consumption.

In all cases, the more the citizen values UE, the higher the required tax rate and the higher the provision of UE as a share of private capital. But, beyond that unsurprising result, the provision of PE as a share of private capital should also increase. Thus, UE and PE should move in the same direction, meaning that higher public consumption cannot be sustained without higher public infrastructure. All these effects are monotonic and independent of congestion.

What is more interesting is the way the government allocates its tax revenues and the resulting balanced growth rate. This crucially depends on the degree of congestion (irrespective of whether this takes the form of absolute or relative congestion). Consider first the case in which PE is congested on a one-to-one basis by private activity, in the sense that if a change in total private capital is accompanied by an equal change in total public capital, public services remain constant at the individual level. This case, which we call full or one-to-one congestion, is popular in the growth literature [see, e.g., Barro and Sala-i-Martin (2004, p. 223)]. Our numerical solutions show that the effect of the preference parameter on the allocation decision and the resulting balanced growth rate is not monotonic: above a critical value of the preference parameter (which coincides with the empirically plausible region), the more the citizen values UE, the more tax revenues the government should allocate to PE vis-à-vis UE, and the higher the balanced growth rate is. This is different from the conventional recipe. Only below the critical value of the preference parameter (which, however, looks empirically implausible) do we get the conventional recipe; namely, the more the citizen values UE, the more tax revenues the government should allocate to them vis-à-vis PE, and the lower the balanced growth rate is. When PE is partially congested, or not congested at all, by private activity, the results are monotonic and again obey the conventional recipe.

In other words, the degree of congestion of productive public capital is crucial to second-best policy and the resulting balanced growth rate. In all cases with nonzero congestion, private agents' decisions to expand their own capital reduce the amount of public productive services (PE) available to each individual firm. Because this is not internalized, the Ramsey government intervenes to restore PE at its optimal ratio. In the presence of full or one-to-one congestion, such an intervention requires that public capital grow at the same rate as private capital. The higher the growth rate, the more tax revenue the Ramsey government should allocate to productive public capital. All this turns out to be good for tax bases. Large tax bases allow, in turn, the financing of all public services, including nonproductive ones (UE). In contrast, when congestion is partial or zero, the priority of the financing of PE is not necessary. Now, even if public capital grows by less than private capital, PE can be at its optimal ratio. Hence, the Ramsey government finds it optimal to follow the traditional recipe.

The above refers to second-best optimal policy. In sharp contrast, in the social planner's solution, the planner finds it optimal first to hit a relatively high growth rate independent of preferences over various nonproductive uses, and in turn to make the allocation choices among the latter. The degree of allocation among

nonproductive uses depends simply on how much the society values one vis-à-vis the others. For instance, if the citizen's valuation of public consumption relative to private consumption rises, the planner increases the resources allocated to the former and decreases the resources allocated to the latter. Note that, in our model economy, this first-best allocation cannot be implemented by the government of the decentralized economy. Thus, as is generally recognized, the first-best optimum may be unattainable³ and can thus serve as a reference case only.

Therefore, the properties of optimal fiscal (tax-spending) policy depend crucially on whether (i) we are in a static or growing economy; (ii) a first-best allocation is not attainable, so that the government needs to design a second-best policy problem; and (iii) productivity-enhancing public goods are congested and by how much.

Although the next section reviews the growth literature, here we wish to point out that our main result (namely, that congestion should matter to policy) resembles the result of Jones (1995) and Young (1998), who show that greater and greater quantities of resources need to be devoted to innovative activities to sustain a given growth rate. Our result is similar in the sense that, when there are congestion problems, so that the share of productive public goods falls as the private economy expands, the Ramsey government needs to devote more tax revenues to finance those public goods.

The rest of the paper is as follows. Section 2 reviews the literature. Section 3 presents a model with congestion and solves for Ramsey second-best optimal policy. Section 4 solves the associated social planner's problem. Section 5 studies extensions and transitional dynamics. Section 6 closes the paper.

2. RELATIONSHIP TO THE LITERATURE

Our work is related to several branches of the theoretical literature on growth and fiscal policy. Eicher and Turnovsky (2000), Fisher and Turnovsky (1998) and Turnovsky (1996; 2000, chap. 13) study growth models with a congested public good (either productivity-enhancing or utility-enhancing). Chatterjee and Ghosh (2009) also allow for congestion, where their single public capital provides both productive and utility services. These authors use a rich production structure that allows different degrees and types of congestion (e.g., absolute and relative congestion), as well as different degrees of substitutability between private and public capital. They study the effects of different ways of financing public expenditure (e.g., income taxes, consumption taxes, lump-sum taxes) and how these effects depend on the degree and type of congestion. But, when they analyze policy choices, the above authors solve for policies that can replicate the associated first-best optimum. Here, by contrast, we solve for Ramsey second-best policy. Also, they do not study the policy allocation problem, namely how a government allocates collected tax revenue to different types of public goods, and how this problem is affected by the degree of congestion, which is the focus of our work. Devarajan et al. (1996) include two types of public goods and study the effect of

composition of public expenditure on economic growth, but policy is exogenous and there is no congestion.

Barro and Sala-i-Martin (1992) and Glomm and Ravikumar (1994) have developed well-known growth models with congested public productive services treated both as a flow and as a stock variable. But, because there is a single public good, these papers do not study the policy allocation problem. Park and Philippopoulos (2003, 2004) use two types of public goods and study second-best allocation policies but, because they do not allow for congestion, they get the conventional policy recipe only. Baier and Glomm (2001) present a rich model with various types of public expenditures, but they do not have congestion or solve for growth-maximizing policies and do not choose all categories of government expenditure optimally. Futagami et al. (1993) extend Barro's (1990) model by treating PE public goods as a stock variable without congestion. Ghosh and Gregoriou (2008) study a government's optimal composition problem in a model with two types of public goods but, because their focus is on the empirical side, they solve a simplified government problem in which the government takes private decisions as given; also, they do not allow for congestion. Our present paper is close in spirit to Economides and Philippopoulos (2008); however, that paper is restricted to a specific form of congestion effects that is found in environmental resources only.

To sum up, our work differs from the literature in that at the same time (a) we study Ramsey second-best optimal policy in a general equilibrium model of growth augmented with the two main categories of public goods and services and (b) we allow for congestion effects and show their key role in the design of second-best policy.

Finally, it is worth adding that Ott and Turnovsky (2006) study the role of excludability of public goods, which means that individuals can have access to them only if they pay user fees. Recall that rivalry and excludability are the key features of impure public goods.

3. A GROWTH MODEL WITH PUBLIC GOODS AND SECOND-BEST OPTIMAL POLICY

3.1. Individuals

There are a constant number of identical individuals indexed by the superscript $i = 1, 2, \dots, N$. Each i maximizes intertemporal utility,

$$\int_0^{\infty} u(C^i, K_c) e^{-\rho t} dt, \quad (1)$$

where C^i is i 's private consumption, K_c is the total stock of utility-enhancing public capital (UE), and $\rho > 0$ is the rate of time preference. The utility function $u(\cdot)$ is increasing and concave. For simplicity, we use an additively separable

function,

$$u(C^i, K_c) = \nu \log C^i + (1 - \nu) \log K_c, \tag{2}$$

where the parameter $0 < \nu < 1$ measures how much the agent values private consumption relative to UE.

The flow budget constraint of each i is

$$C^i + I^i = (1 - \tau)Y^i, \tag{3}$$

where I^i is i 's private investment, Y^i is i 's output, and $0 \leq \tau < 1$ is a distorting tax rate. The motion of private capital is

$$\dot{K}^i = -\delta^k K^i + I^i, \tag{4}$$

where K^i is i 's stock of private capital and $\delta^k \geq 0$ is the depreciation rate. The initial stock $K^i(0)$ is given. A dot over a variable denotes its time derivative.

Production is modeled as in, e.g., Fisher and Turnovsky (1998) and Eicher and Turnovsky (2000). Thus, i 's production function is

$$Y^i = A(K^i)^\alpha (K_g^i)^\beta, \tag{5}$$

where K_g^i is the services derived by each individual i from productivity-enhancing public capital. These services are defined as

$$K_g^i \equiv \frac{K_g}{(K)^\theta} \left(\frac{K^i}{K} \right)^\xi, \tag{6}$$

where K_g is the total stock of productivity-enhancing public capital (PE), $K \equiv NK^i$ is the total stock of private capital, $A > 0$, $0 < \alpha, \beta < 1$ are productivity parameters, and $\theta, \xi \geq 0$ are parameters measuring the degree of absolute and relative congestion, respectively.⁴

Using (6), the individual production function (5) can be reexpressed as $Y^i = A(K^i)^{\alpha+\beta\xi-\beta} (K^i (K_g / (K)^{\theta+\xi}))^\beta$. Thus, if the elasticities of absolute and relative congestion sum to unity, $\theta + \xi = 1$, total public capital, K_g , and total private capital, K , must increase at the same rate for public services to remain constant at the individual firm level. This is what we call full, or one-to-one, congestion. The case $0 \leq \theta + \xi < 1$ describes what we call partial and zero congestion. The case $\theta + \xi > 1$ implies that public capital must grow more than the private economy for public services to remain constant at the firm level. We distinguish between these cases because they are important to what follows.

Summing (4) over the N identical individuals gives the aggregate production function

$$Y = N^{1-\alpha-\beta\xi} A(K)^{\alpha-\beta\theta} (K_g)^\beta,$$

where $Y \equiv NY^i$. Thus, for constant population, the production function in (5) and (6) yields a well-defined balanced growth rate if and only if it exhibits CRS in

K and K_g , i.e., $\alpha + \beta(1 - \theta) = 1$. We assume that this condition holds [see also Eicher and Turnovsky (2000, p. 329)].

Using the restriction $\alpha + \beta(1 - \theta) = 1$ in (5) and (6), we can get at least two popular production functions in the literature. First, in the case without any type of congestion, $\theta = \xi = 0$, we get $Y^i = A(K^i)^{1-\beta}(K_g)^\beta$ at the firm level. This is the production function in, e.g., Barro and Sala-i-Martin (1992, p. 649). Second, in the case of proportional congestion, $\theta + \xi = 1$, we get $Y^i = AK^i(K_g/K)^\beta$ at firm level. This is as in, e.g., Barro and Sala-i-Martin (1992, p. 650; 2004, p. 223).

3.2. Individual Optimization

Each agent i acts competitively by choosing the paths of C^i and K^i , while taking policy and economywide variables as given. The first-order conditions include (3), (4), and the Euler equation:

$$\dot{C}^i = C^i \left[(1 - \tau)A(\alpha + \beta\xi)(K^i)^{\alpha-1+\beta\xi} \left(\frac{K_g}{K^{\theta+\xi}} \right)^\beta - \delta^k - \rho \right]. \tag{7}$$

3.3. Public Goods

The stock of productivity-enhancing public capital (PE) evolves according to

$$\dot{K}_g = -\delta^g K_g + G_g, \tag{8a}$$

where the parameter $\delta^g \geq 0$ is the depreciation rate and G_g is public investment spending. The initial stock $K_g(0)$ is given. If $\delta^g = 1$, PE is a flow variable as in, e.g., Barro (1990).

Similarly, the stock of utility-enhancing public capital (UE) evolves according to

$$\dot{K}_c = -\delta^c K_c + G_c, \tag{8b}$$

where the parameter $\delta^c \geq 0$ is the depreciation rate and G_c is public consumption spending. The initial stock $K_c(0)$ is given. If $\delta^c = 1$, we get the popular case in which UE is a flow variable.

3.4. Government Budget Constraint

On the revenue side, the government taxes individuals' income at a rate $0 \leq \tau < 1$. On the expenditure side, it spends G_g and G_c . Using a balanced budget within each period

$$G_g + G_c = \tau Y, \tag{9a}$$

where, at each instant, only two out of the three instruments (τ, G_g, G_c) can be set independently.⁵

Equivalently, it is convenient for what follows to rewrite (9a) as

$$G_g = b\tau Y, \tag{9b}$$

$$G_c = (1 - b)\tau Y, \tag{9c}$$

where $0 \leq b \leq 1$ is the fraction of tax revenues used to finance PE and $0 \leq 1 - b \leq 1$ is the fraction that finances UE. Thus, at each instant, fiscal policy can be summarized by τ and b [see also, e.g., Devarajan et al. (1996)].

3.5. Decentralized Competitive Equilibrium

In a decentralized competitive equilibrium (DCE), (i) each individual maximizes utility, (ii) all constraints are satisfied, and (iii) all markets clear. This holds for any feasible policy, which is summarized by the paths of the two independent policy instruments, $0 \leq \tau < 1$ and $0 \leq b \leq 1$. We solve for a symmetric equilibrium (from now on we omit the superscript i) and for simplicity we set $N = 1$.

Combining (1)–(9), it is straightforward to show that such a DCE is given by

$$\dot{C} = C \left[(1 - \tau)A[1 - \beta(1 - \theta - \xi)] \left(\frac{K_g}{K}\right)^\beta - \delta^k - \rho \right], \tag{10a}$$

$$\frac{\dot{K}}{K} = (1 - \tau)A \left(\frac{K_g}{K}\right)^\beta - \delta^k - \frac{C}{K}, \tag{10b}$$

$$\frac{\dot{K}_c}{K_c} = -\delta^c + (1 - b)\tau A \left(\frac{K_g}{K}\right)^\beta \frac{K}{K_c}, \tag{10c}$$

$$\frac{\dot{K}_g}{K_g} = -\delta^g + b\tau A \left(\frac{K_g}{K}\right)^\beta \frac{K}{K_g}, \tag{10d}$$

which is a four-equation system in the paths of C, K, K_g, K_c . This gives the paths of τ and b .

3.6. Second-Best Optimal Policy

We now endogenize policy as summarized by the paths of the income tax rate, $0 \leq \tau < 1$, and the allocation of tax revenues between the two types of public goods, $0 \leq b \leq 1$. The government chooses the paths of τ and b to maximize the household’s utility subject to the DCE in (10a)–(10d). In doing so, the government will try to control for congestion externalities and raise funds optimally to finance its public good activities. Solving for a commitment (Ramsey) equilibrium, the

current-value Hamiltonian, H , of this second-best problem is

$$\begin{aligned}
 H = & \nu \log C + (1 - \nu) \log K_c \\
 & + \lambda_c C \left[[1 - \beta(1 - \theta - \xi)](1 - \tau) A \left(\frac{K_g}{K} \right)^\beta - \delta^k - \rho \right] \\
 & + \lambda_k \left[(1 - \tau) A \left(\frac{K_g}{K} \right)^\beta K - \delta^k K - C \right] \\
 & + \lambda_{kg} \left[-\delta^g K_g + b\tau A \left(\frac{K_g}{K} \right)^\beta K \right] + \lambda_{kc} \left[-\delta^c K_c + (1 - b)\tau A \left(\frac{K_g}{K} \right)^\beta K \right],
 \end{aligned}
 \tag{11}$$

where $\lambda_c, \lambda_k, \lambda_{kc}$, and λ_{kg} are dynamic multipliers associated with (10a), (10b), (10c), and (10d), respectively.

The first-order conditions include the constraints (10a)–(10d) and the optimality conditions with respect to τ, b, C, K, K_c , and K_g , which are, respectively,⁶

$$[1 - \beta(1 - \theta - \xi)]\lambda_c C + \lambda_k K = \lambda_{kc} K, \tag{12a}$$

$$\lambda_{kc} = \lambda_{kg}, \tag{12b}$$

$$\dot{\lambda}_c = -\frac{\nu}{C} - \lambda_c \left[[1 - \beta(1 - \theta - \xi)](1 - \tau) A \left(\frac{K_g}{K} \right)^\beta - \delta^k - \rho \right] + \lambda_k + \rho \lambda_c, \tag{12c}$$

$$\begin{aligned}
 \dot{\lambda}_k = & \beta[1 - \beta(1 - \theta - \xi)](1 - \tau) A \left(\frac{K_g}{K} \right)^\beta K^{-1} \lambda_c C \\
 & - (1 - \beta)(1 - \tau) A \left(\frac{K_g}{K} \right)^\beta \lambda_k + \delta^k \lambda_k - (1 - \beta)\tau A \left(\frac{K_g}{K} \right)^\beta \lambda_{kc} + \rho \lambda_k,
 \end{aligned}
 \tag{12d}$$

$$\dot{\lambda}_{kc} = -\frac{(1 - \nu)}{K_c} + \delta^c \lambda_{kc} + \rho \lambda_{kc}, \tag{12e}$$

$$\begin{aligned}
 \dot{\lambda}_{kg} = & -\beta[1 - \beta(1 - \theta - \xi)](1 - \tau) A \left(\frac{K_g}{K} \right)^\beta K_g^{-1} \lambda_c C \\
 & - \beta(1 - \tau) A \left(\frac{K_g}{K} \right)^{\beta-1} \lambda_k - \beta\tau A \left(\frac{K_g}{K} \right)^{\beta-1} \lambda_{kg} + \delta^g \lambda_{kg} + \rho \lambda_{kg}.
 \end{aligned}
 \tag{12f}$$

Thus, (12a)–(12f), jointly with the constraints (10a)–(10d), constitute a ten-equation system in the paths of $\tau, b, C, K, K_c, K_g, \lambda_c, \lambda_k, \lambda_{kc}, \lambda_{kg}$. This is a general equilibrium with second-best optimal policy.

3.7. Stationary Second-Best General Equilibrium

Because the model allows long-term growth, we transform variables to make them stationary. We define the auxiliary variables $c \equiv C/K, k_c \equiv K_c/K, k_g \equiv K_g/K, \Lambda_c \equiv \lambda_c C, \Lambda_k \equiv \lambda_k K, \Lambda_{kc} \equiv \lambda_{kc} K_c,$ and $\Lambda_{kg} \equiv \lambda_{kg} K_g$. Thus, $c, k_c,$ and k_g are the ratios of private consumption to private capital, nonproductive public capital to private capital, and productive public capital to private capital, respectively, where $\Lambda_c, \Lambda_k, \Lambda_{kc},$ and Λ_{kg} measure respectively the social value of private consumption, nonproductive public capital, and productive public capital. It is then straightforward to show that the dynamics of (10a)–(10d) and (12a)–(12f) are equivalent to the dynamics of

$$\dot{c} = c^2 - \rho c - \beta(1 - \theta - \xi)(1 - \tau)Ak_g^\beta c, \tag{13a}$$

$$\dot{k}_c = -\delta^c k_c + (1 - b)\tau Ak_g^\beta - (1 - \tau)Ak_g^\beta k_c + ck_c + \delta^k k_c, \tag{13b}$$

$$\dot{k}_g = -\delta^g k_g + b\tau Ak_g^\beta - (1 - \tau)Ak_g^{1+\beta} + ck_g + \delta^k k_g, \tag{13c}$$

$$\dot{\Lambda}_c = -\nu + \rho\Lambda_c + \Lambda_{kc}, \tag{13d}$$

$$\begin{aligned} \dot{\Lambda}_k &= \beta[1 - \beta(1 - \theta - \xi)](1 - \tau)Ak_g^\beta \Lambda_c \\ &+ \beta(1 - \tau)Ak_g^\beta \Lambda_k - (1 - \beta)\tau Ak_g^\beta \frac{\Lambda_{kc}}{k_c} + (\rho - c)\Lambda_k, \end{aligned} \tag{13e}$$

$$\dot{\Lambda}_{kc} = -(1 - \nu) + \rho\Lambda_{kc} + (1 - b)\tau Ak_g^\beta \frac{\Lambda_{kc}}{k_c}, \tag{13f}$$

$$\begin{aligned} \dot{\Lambda}_{kg} &= -\beta[1 - \beta(1 - \theta - \xi)](1 - \tau)Ak_g^\beta \Lambda_c - \beta(1 - \tau)Ak_g^\beta \Lambda_k \\ &- \beta\tau Ak_g^{\beta-1} \Lambda_{kg} + b\tau Ak_g^{\beta-1} \Lambda_{kg} + \rho\Lambda_{kg}, \end{aligned} \tag{13g}$$

$$[1 - \beta(1 - \theta - \xi)]\Lambda_c + \Lambda_k = \frac{\Lambda_{kc}}{k_c}, \tag{13h}$$

$$\Lambda_{kc}k_g = \Lambda_{kg}k_c, \tag{13i}$$

which constitute a nine-equation system in the paths of $\tau, b, c, k_c, k_g, \Lambda_c, \Lambda_k, \Lambda_{kc},$ and Λ_{kg} . This is a stationary general equilibrium with second-best optimal policy. We next study the long run of this economy.

3.8. Long-Run Second-Best General Equilibrium

In the long run, variables do not change in (13a)–(13i). We denote the resulting long-run values as $\tilde{\tau}$, \tilde{b} , \tilde{c} , \tilde{k}_c , \tilde{k}_g , $\tilde{\Lambda}_c$, $\tilde{\Lambda}_k$, $\tilde{\Lambda}_{kg}$, and $\tilde{\Lambda}_{kg}$. In this long run, all components of national income grow at the same nonnegative balanced growth rate, denoted as $\tilde{\gamma}$, and policy instruments do not change. The long-run solution is given by the system

$$\tilde{c} = \rho + \beta(1 - \theta - \xi)(1 - \tilde{\tau})A\tilde{k}_g^\beta, \tag{14a}$$

$$\tilde{k}_c = \frac{(1 - \tilde{b})\tilde{\tau}A\tilde{k}_g^\beta}{(1 - \tilde{\tau})A\tilde{k}_g^\beta + \delta^c - \delta^k - \tilde{c}}, \tag{14b}$$

$$(1 - \tilde{\tau})A\tilde{k}_g^\beta + \delta^g = \tilde{b}\tilde{\tau}A\tilde{k}_g^{\beta-1} + \delta^k + \tilde{c}, \tag{14c}$$

$$\tilde{\Lambda}_k\tilde{c} + \rho\tilde{\Lambda}_c = \nu, \tag{14d}$$

$$(1 - \tilde{b})\tilde{\tau}A\tilde{k}_g^{\beta-1}\tilde{\Lambda}_{kg} - \rho\tilde{\Lambda}_{kg} + (\tilde{c} - \rho)\tilde{\Lambda}_k = 0, \tag{14e}$$

$$\begin{aligned} &\beta(1 - \tilde{\tau})[1 - \beta(1 - \theta - \xi)]A\tilde{k}_g^\beta\tilde{\Lambda}_c + \beta(1 - \tilde{\tau})A\tilde{k}_g^\beta\tilde{\Lambda}_k \\ &- (1 - \beta)\tilde{\tau}A\tilde{k}_g^\beta\frac{\tilde{\Lambda}_{kc}}{\tilde{k}_c} - (\tilde{c} - \rho)\tilde{\Lambda}_k = 0, \end{aligned} \tag{14f}$$

$$\tilde{\Lambda}_{kc} = \frac{1 - \nu}{(1 - \tilde{\tau})A\tilde{k}_g^\beta + \delta^c - \delta^k - \tilde{c} + \rho}, \tag{14g}$$

$$[1 - \beta(1 - \theta - \xi)]\tilde{\Lambda}_c + \tilde{\Lambda}_k = \frac{\tilde{\Lambda}_{kc}}{\tilde{k}_c}, \tag{14h}$$

$$\tilde{\Lambda}_{kc}\tilde{k}_g = \tilde{\Lambda}_{kg}\tilde{k}_c. \tag{14i}$$

The above nonlinear system is solved numerically. We use the following baseline parameter values:⁷ $\beta = 0.15$ (where $\beta \geq 0$ is the productivity of public capital in the production function), $A = 0.25$ (where $A > 0$ is total factor productivity in the production function), $\delta^k = \delta^c = \delta^g = 0.06$ (which are the depreciation rates of private capital, nonproductive public capital, and productive public capital, respectively), and $\rho = 0.04$ (where $\rho > 0$ is the rate of time preference). Regarding the values of θ (the degree of absolute congestion) and ξ (the degree of relative congestion) in (5), we choose to report results for three cases: (a) The case of full, or one-to-one, congestion, $\theta + \xi = 1$. Note that this also captures the subcases in which either $\theta = 1$ and $\xi = 0$ (i.e., absolute full congestion), or $\theta = 0$ and $\xi = 1$ (i.e., relative full congestion). (b) The case of positive but partial congestion, $0 < \theta + \xi < 1$. (c) The case without any type of congestion, $\theta = \xi = 0$.⁸

Tables 1, 2, and 3 report the long-run solution for varying values of the parameter ν in a wide range, $0.1 \leq \nu \leq 0.95$, in all three cases. We focus on the values of ν because it is an important parameter in our setup; it measures how much the household values its own private consumption vis-à-vis public consumption (see

TABLE 1. Effect of ν on long-run second-best equilibrium when $\theta + \xi = 1$

ν	$\tilde{\tau}$	\tilde{b}	\tilde{c}	\tilde{k}_c	\tilde{k}_g	$\tilde{\Lambda}_c + \tilde{\Lambda}_k$	$\tilde{\Lambda}_{kc}$	$\tilde{\Lambda}_{kg}$	\tilde{G}_g/\tilde{Y}	\tilde{G}_c/\tilde{Y}	$\tilde{\gamma}$
0.10											
0.20											
0.30	0.594	0.627	0.04	0.868	1.464	7.50	6.516	10.983	0.372	0.221	0.007
0.40	0.504	0.678	0.04	0.482	1.017	10.00	4.828	10.171	0.341	0.162	0.024
0.50	0.415	0.711	0.04	0.288	0.711	12.50	3.602	8.897	0.295	0.119	0.038
0.60	0.328	0.734	0.04	0.176	0.489	15.00	2.653	7.346	0.241	0.087	0.050
0.70	0.243	0.749	0.04	0.107	0.321	17.50	1.879	5.620	0.182	0.060	0.059
0.80	0.158	0.755	0.04	0.061	0.188	20.00	1.221	3.778	0.120	0.038	0.063
0.85	0.117	0.754	0.04	0.043	0.133	21.25	0.919	2.830	0.088	0.028	0.063
0.90	0.076	0.748	0.04	0.027	0.083	22.50	0.629	1.870	0.057	0.019	0.058
0.95	0.036	0.729	0.04	0.014	0.038	23.75	0.338	0.911	0.026	0.010	0.047

Notes: $A = 0.25$, $\beta = 0.15$, $\delta^k = 0.06$, $\delta^s = 0.06$, $\delta^c = 0.06$, and $\rho = 0.04$. Entries without numbers imply ill-defined solutions, e.g., negative long-run growth rates.

equation (2)). The tables will also report the resulting equilibrium values of the balanced growth rate, $\tilde{\gamma}$, as well as the government resources earmarked for PE and UE as shares of output, denoted as \tilde{G}_g/\tilde{Y} and \tilde{G}_c/\tilde{Y} , respectively.

Case A: Full or one-to-one congestion, $\theta + \xi = 1$. Inspection of results in Table 1 implies the following: (a) The solution is well defined.⁹ For instance, $0 < \tilde{\tau} < 1$, $0 < \tilde{b} \leq 1$, $\tilde{c} > 0$, $\tilde{k}_c > 0$, and $\tilde{k}_g > 0$. Also, for $\nu > 0.2$, the balanced growth rate—along which all national income quantities grow at the same constant rate—is positive, $\tilde{\gamma} > 0$ (for $\nu \leq 0.2$, the economy is shrinking—hence there are no entries in the corresponding rows). (b) As ν falls (i.e., as we care more about UE relative to private consumption), it is optimal to tax more ($\tilde{\tau}$ rises monotonically). (c) The relationship between ν and the fraction of tax revenues allocated to PE relative to UE (\tilde{b}) is not monotonic. Specifically, in the region $0.8 \leq \nu < 1.0$, as ν falls, \tilde{b} rises; in the region $0.2 < \nu < 0.8$, as ν falls, \tilde{b} falls. (d) The balanced growth rate, $\tilde{\gamma}$, behaves like \tilde{b} ; i.e., its behavior is nonmonotonic. (e) Both \tilde{G}_c/\tilde{Y} and \tilde{G}_g/\tilde{Y} , and their associated stocks \tilde{k}_c and \tilde{k}_g , all increase monotonically as ν falls.

In other words, there is a critical value of ν , denoted as ν^* , above which the more the citizen values UE, the more tax revenues the government should allocate to PE relative to UE; i.e., for $\nu \geq \nu^*$, $\partial\tilde{b}/\partial\nu < 0$. This policy allocation effect, in combination with the monotonic increase in \tilde{k}_g as ν falls, more than offsets the adverse effect from higher tax rates ($\partial\tilde{\tau}/\partial\nu < 0$), so that the balanced growth rate rises in this region (i.e., for $\nu \geq \nu^*$, $\partial\tilde{\gamma}/\partial\nu < 0$).

By contrast, in the region $\nu < \nu^*$, $\partial\tilde{b}/\partial\nu > 0$. That is, in this region, we get the conventional policy recipe: the more the citizen values UE, the more tax revenues the government should allocate to them relative to PE. Now the allocation effect works in the same direction as the adverse effect from higher tax rates ($\partial\tilde{\tau}/\partial\nu < 0$), so that the balanced growth rate falls; i.e., for $\nu < \nu^*$, $\partial\tilde{\gamma}/\partial\nu > 0$.

TABLE 2. Effect of ν on long-run second-best equilibrium when $\theta + \xi < 1$

ν	$\bar{\tau}$	\bar{b}	\bar{c}	\bar{k}_c	\bar{k}_g	$\bar{\Lambda}_c$	$\bar{\Lambda}_k$	$\bar{\Lambda}_{kc}$	$\bar{\Lambda}_{kg}$	\bar{G}_g/\bar{Y}	\bar{G}_c/\bar{Y}	$\bar{\gamma}$
0.10	0.395	0.247	0.049	0.817	0.268	66.531	-51.948	7.842	2.573	0.097	0.297	0.014
0.20	0.337	0.301	0.050	0.566	0.244	58.585	-42.814	6.445	2.783	0.101	0.235	0.024
0.30	0.292	0.355	0.050	0.415	0.229	51.374	-34.662	5.339	2.949	0.104	0.188	0.031
0.40	0.256	0.413	0.051	0.309	0.218	44.648	-27.126	4.386	3.092	0.106	0.150	0.036
0.50	0.225	0.477	0.051	0.229	0.209	38.274	-20.025	3.530	3.220	0.107	0.117	0.041
0.60	0.198	0.549	0.052	0.166	0.202	32.173	-13.254	2.742	3.338	0.108	0.089	0.045
0.70	0.173	0.632	0.052	0.114	0.196	26.290	-6.744	2.004	3.449	0.109	0.063	0.049
0.80	0.151	0.731	0.052	0.070	0.191	20.588	-0.449	1.306	3.554	0.110	0.040	0.053
0.85	0.141	0.787	0.052	0.050	0.188	17.796	2.629	0.969	3.604	0.111	0.029	0.054
0.90	0.131	0.850	0.052	0.032	0.186	15.039	5.666	0.640	3.653	0.111	0.019	0.056
0.95	0.121	0.921	0.052	0.015	0.184	12.314	8.665	0.317	3.702	0.111	0.001	0.057

Notes: $A = 0.25$, $\beta = 0.15$, $\delta^k = 0.06$, $\delta^g = 0.06$, $\delta^c = 0.06$, $\rho = 0.04$, $\theta = 0.3$, and $\xi = 0.2$.

TABLE 3. Effect of ν on long-run second-best equilibrium when $\theta = \xi = 0$

ν	$\tilde{\tau}$	\tilde{b}	\tilde{c}	\tilde{k}_c	\tilde{k}_g	$\tilde{\Lambda}_c$	$\tilde{\Lambda}_k$	$\tilde{\Lambda}_{kc}$	$\tilde{\Lambda}_{kg}$	\tilde{G}_g/\tilde{Y}	\tilde{G}_c/\tilde{Y}	$\tilde{\gamma}$
0.10	0.420	0.218	0.058	1.089	0.304	28.739	-24.911	8.732	2.440	0.091	0.328	0.003
0.20	0.351	0.276	0.060	0.713	0.272	35.749	-20.498	7.057	2.691	0.097	0.254	0.013
0.30	0.302	0.331	0.061	0.509	0.252	32.885	-16.568	5.802	2.879	0.100	0.202	0.020
0.40	0.262	0.390	0.062	0.373	0.239	30.153	-12.937	4.745	3.038	0.102	0.160	0.026
0.50	0.229	0.454	0.063	0.274	0.228	27.530	-9.517	3.808	3.178	0.104	0.124	0.031
0.60	0.200	0.528	0.064	0.196	0.220	24.998	-6.257	2.951	3.307	0.105	0.094	0.035
0.70	0.174	0.613	0.064	0.134	0.213	22.541	-3.123	2.154	3.426	0.106	0.067	0.039
0.80	0.150	0.716	0.065	0.082	0.207	20.149	-0.091	1.402	3.539	0.107	0.042	0.042
0.85	0.139	0.775	0.065	0.059	0.205	18.974	1.390	1.040	3.593	0.108	0.031	0.044
0.90	0.129	0.841	0.065	0.038	0.202	17.812	2.853	0.686	3.647	0.108	0.020	0.045
0.95	0.119	0.915	0.065	0.018	0.200	16.662	4.298	0.340	3.698	0.109	0.010	0.047

Notes: $A = 0.25$, $\beta = 0.15$, $\delta^k = 0.06$, $\delta^g = 0.06$, $\delta^c = 0.06$, and $\rho = 0.04$.

Notice that most applied studies work in the region $\nu > 0.7$ [see, e.g., Malley et al. (2007, pp. 1067–1068), who also provide references]. Thus, the conventional policy recipe can hold for values of ν that are too low relative to those commonly used in the applied literature. This means that if we focus on the commonly used parameter region, it is the striking new policy recipe that holds rather than the conventional one.¹⁰

Let us discuss these results. The more the citizen values public consumption, the more resources are eventually allocated to it (i.e., as ν falls, both \tilde{G}_c/\tilde{Y} and \tilde{k}_c rise). This is natural. But, at the same time and in the whole range of parameter values, the stronger the preference over public consumption, the higher should also be the public investment-to-output ratio and the public capital-to-private capital ratio (i.e., as ν falls, \tilde{G}_g/\tilde{Y} and \tilde{k}_g rise monotonically). This implies that higher provision of public consumption should go hand in hand with higher provision of public infrastructure, and this is achieved by the right mix of tax and spending decisions on the part of the Ramsey government. Note that the property that \tilde{G}_g/\tilde{Y} should increase as ν falls is due to the effort of the government to correct for externalities by keeping \tilde{k}_g at its desired ratio (as we shall see below, \tilde{G}_g/\tilde{Y} is independent of ν in the absence of congestion problems and thus externalities to be internalized by the government).

What is more striking is the optimal tax revenue allocation decision and the resulting balanced growth rate. In the empirically plausible region, the more the citizen values UE, the more tax revenues the government should allocate to PE vis-à-vis UE, and the higher is the balanced growth rate. Only below the critical value of the preference parameter (which, however, looks empirically implausible), we get the traditional recipe, namely, the more the citizen values UE, the more tax revenues the government should allocate to it vis-à-vis PE, and the lower is the balanced growth rate. As we show below, the nonmonotonicity result for b and γ , as well as the striking policy recipe in the empirically plausible region, depend on the presence and degree of congestion, and so their interpretation is deferred to the next section.

Thus, summarizing the above:

RESULT 1. *Along the second-best (Ramsey) balanced growth path, when there is full or one-to-one congestion, $\theta + \xi = 1$, there is a critical value of ν , denoted as ν^* , where (i) for $\nu \geq \nu^*$, $\partial \tilde{b}/\partial \nu < 0$ and $\partial \tilde{\gamma}/\partial \nu < 0$; (ii) for $\nu < \nu^*$, $\partial \tilde{b}/\partial \nu > 0$ and $\partial \tilde{\gamma}/\partial \nu > 0$.*

Case B: partial congestion, $0 < \theta + \xi < 1$. Inspection of results in Table 2 implies the following: (a) The solution is again well defined. (b) As ν falls, $\tilde{\tau}$ rises monotonically. This is as in case A above. (c) As ν falls, \tilde{b} falls monotonically. This differs from case A. (d) As ν falls, $\tilde{\gamma}$ falls. Again this differs from case A. (e) The stocks of both public goods, \tilde{k}_c and \tilde{k}_g , increase monotonically as ν falls. This is as in case A.

Thus, when congestion is partial, there is no reason for the government to allocate more resources to PE relative to UE when the citizen cares more about UE.

In other words, we now get the traditional policy recipe and in turn the traditional effect on the balanced growth rate over the whole range of ν ($0 < \nu < 1$).

Therefore, the degree of congestion (absolute or relative) of productive public capital is crucial to the nonmonotonic behavior of second-best tax revenue allocation policy and the resulting balanced growth rate. In all cases with nonzero congestion, private agents' decision to expand their own capital reduces the amount of public productive services (PE) available to each individual firm.¹¹ Because this is not internalized, the Ramsey government intervenes to restore PE at its optimal ratio. In the presence of full or one-to-one congestion, $\theta + \xi = 1$, such an intervention requires that public capital grow at the same rate as private capital. The higher the growth rate, the more tax revenue the Ramsey government should allocate to productive public capital. All this turns out to be good for tax bases. Large tax bases allow, in turn, the financing of all public services including nonproductive ones (UE). This was the case in Table 1 in the empirically plausible region of ν . In contrast, when congestion is only partial, $0 < \theta + \xi < 1$, the priority over the financing of PE is not necessary. Now, public capital can grow by less than private capital in order for PE to be at its optimal ratio. Hence, the Ramsey government finds it optimal to follow the traditional recipe.

Case C: no congestion, $\theta = \xi = 0$. Finally, we check the case of pure public goods without any congestion, absolute or relative. The results, reported in Table 3, are qualitatively the same as those in Table 2. This is how it should be, because the case without congestion belongs to the case with partial congestion.

Thus, summarizing cases B and C above, we have

RESULT 2. *Along the second-best (Ramsey) balanced growth path, when there is partial or zero congestion, $0 \leq \theta + \xi < 1$, we get the conventional policy recipe $\partial \bar{b} / \partial \nu > 0$ and thus $\partial \bar{\gamma} / \partial \nu > 0$.*

Finally, we have

RESULT 3. *Results 1 and 2 also hold in the popular special cases in which $\delta^c = 1$ and/or $\delta^g = 1$, i.e., when public services are flow, rather than stock, variables.*

4. SOCIAL PLANNER'S SOLUTION

This section solves for the benchmark case of a first-best allocation (FBA). This serves as a reference. Now a social planner internalizes externalities and chooses directly the paths of C , K , K_g , K_c , G_g , and G_c (respectively, private consumption, private capital, productive public capital, nonproductive public capital, resources assigned to infrastructure, and resources assigned to public consumption) to maximize households' utility subject to the resource constraints

$$\dot{K} = AK_g^\beta K^{1-\beta} - \delta^k K - C - G_g - G_c, \tag{15a}$$

$$\dot{K}_g = -\delta^g K_g + G_g, \tag{15b}$$

$$\dot{K}_c = -\delta^c K_c + G_c, \tag{15c}$$

where (15a) is the economy’s resource constraint and (15b) and (15c) are the motions of productive and nonproductive public capital, respectively.

4.1. Solution of Social Planner’s Problem

The current-value Hamiltonian, H , of this first-best problem is

$$H \equiv v \log C + (1 - v) \log K_c + \lambda_k [AK_g^\beta K^{1-\beta} - \delta^k K - C - G_g - G_c] + \lambda_{kg} [-\delta^g K_g + G_g] + \lambda_{kc} [-\delta^c K_c + G_c], \tag{16}$$

where λ_k , λ_{kg} , and λ_{kc} are new dynamic multipliers associated with (15a)–(15c).

Deriving the first-order conditions with respect to C , G_g , G_c , λ_k , K , λ_{kg} , K_g , λ_{kc} , and K_c and using the new stationary auxiliary variables $c \equiv C/K$, $k_g \equiv K_g/K$, $k_c \equiv K_c/K$, $g_g \equiv G_g/K$, and $g_c \equiv G_c/K$, we have

$$\frac{\dot{c}}{c} = -\beta Ak_g^\beta - \rho + c + g_g + g_c, \tag{17a}$$

$$\frac{\dot{k}_c}{k_c} = -\delta^c + g_c k_c^{-1} - Ak_g^\beta + \delta^k + c + g_g + g_c, \tag{17b}$$

$$\frac{\dot{k}_g}{k_g} = -\delta^g + g_g k_g^{-1} - Ak_g^\beta + \delta^k + c + g_g + g_c, \tag{17c}$$

$$\beta Ak_g^{\beta-1} - (1 - \beta) Ak_g^\beta = \delta^g - \delta^k, \tag{17d}$$

$$\frac{c}{k_c} = \frac{v[(1 - \beta) Ak_g^\beta + \delta^c - \delta^k]}{(1 - v)}, \tag{17e}$$

where (17a)–(17e) constitute a system of five equations in the paths of c , k_c , k_g , g_g , and g_c . In turn, the consumption growth rate can follow from

$$\frac{\dot{C}}{C} = (1 - \beta) Ak_g^\beta - \delta^k - \rho. \tag{17f}$$

This is a stationary first-best allocation (FBA). We next study this economy in the long run.

TABLE 4. Effect of ν on long-run first-best equilibrium

ν	\bar{c}	\bar{k}_c	\bar{k}_g	\bar{g}_g	\bar{g}_c	\bar{G}_g/\bar{Y}	\bar{G}_c/\bar{Y}	$\bar{\gamma}$
0.10	0.006	0.331	0.176	0.021	0.041	0.038	0.072	0.063
0.20	0.011	0.285	0.176	0.021	0.035	0.038	0.062	0.063
0.30	0.017	0.242	0.176	0.021	0.030	0.038	0.053	0.063
0.40	0.022	0.201	0.176	0.021	0.025	0.038	0.044	0.063
0.50	0.026	0.163	0.176	0.021	0.020	0.038	0.035	0.063
0.60	0.031	0.127	0.176	0.021	0.015	0.038	0.027	0.063
0.70	0.035	0.092	0.176	0.021	0.011	0.038	0.020	0.063
0.80	0.039	0.060	0.176	0.021	0.007	0.038	0.013	0.063
0.90	0.043	0.029	0.176	0.021	0.003	0.038	0.006	0.063

Notes: $A = 0.25$, $\beta = 0.15$, $\delta^k = 0.06$, $\delta^g = 0.06$, $\delta^c = 0.06$, and $\rho = 0.04$.

4.2. Long-Run First-Best Allocation

In the long run, variables do not change in (17a)–(17e). We denote the resulting long-run values as \bar{c} , \bar{k}_c , \bar{k}_g , \bar{g}_g , and \bar{g}_c . Thus, variables with upper bars denote long-run values in a first-best allocation. To make our results directly comparable to those in the preceding section, we present numerical results in Table 4.¹² The parameter values used are the same as above. We also report the resulting solutions of \bar{G}_g/\bar{Y} , \bar{G}_c/\bar{Y} , and $\bar{\gamma}$, as we did in Section 3.

Inspection of numerical results, again for varying values of ν , reveals the following: (a) The solution is well defined. For instance, $\bar{c} > 0$, $\bar{k}_c > 0$, $\bar{k}_g > 0$, $\bar{g}_g > 0$, $\bar{g}_c > 0$, $\bar{G}_g/\bar{Y} > 0$, $\bar{G}_c/\bar{Y} > 0$, and $\bar{\gamma} \geq 0$. (b) The positive balanced growth rate ($\bar{\gamma} > 0$) is independent of ν . This differs from the second-best equilibrium above, where the balanced growth rate did depend on ν (compare Tables 1–3 with Table 4). (c) Not only the growth rate, but also all variables associated with the production side of the economy, are now independent of ν (see the flat values of \bar{k}_g , \bar{g}_g , and \bar{G}_g/\bar{Y}). This differs from the second-best equilibrium, where the same variables decreased with ν . (d) The social resources earmarked for nonproductive uses do depend on ν . Specifically, \bar{c} increases, whereas \bar{k}_c , \bar{g}_c , and \bar{G}_c/\bar{Y} decrease, with ν . (e) The balanced growth rate is never smaller than that in a second-best equilibrium (compare the long-run values of γ in Tables 1–3 with those in Table 4).

Therefore, in our model economy, the social planner finds it optimal to first hit a relatively high growth rate independent of preferences over alternative nonproductive goods and services, and in turn to make the allocation choice among the latter. Thus, we get a form of dichotomy. Having achieved an efficient use of productive factors, the planner allocates social resources to various consumption uses by following the conventional recipe: the more the citizen values public consumption relative to private consumption, the more social resources the planner allocates to the former relative to the latter.

We finally report that all the above results also hold in the popular special cases in which $\delta^c = 1$ and/or $\delta^g = 1$, i.e., when public services are flow variables.

5. ROBUSTNESS AND DYNAMICS

The aim of this section is to enrich the basic model and thus check robustness of results (this is in Section 5.2), as well as to study transitional dynamics (this is in Section 5.3). Throughout the section, we focus on the interesting case of full or one-to-one congestion, $\theta + \xi = 1$. But, before we do so, we examine whether the first-best solution in Section 4 is implementable (this is in Section 5.1).

5.1. Implementation of First Best: Is It Possible?

We check whether the government in the decentralized economy can choose its policy instruments to implement the first-best allocation—in other words, whether it is possible to choose τ, b so that the long-run DCE solution from equations (14a)–(14c)¹³ coincides with the long-run solution of the social planner. It is straightforward to see that, in our model economy, this is not possible. This happens because (in addition to the classic Tinbergen target-policy problem, which states that the number of independent policy instruments should not be less than the number of policy targets), the second-best equilibrium has the property $\tilde{c} = \rho$ (see (14a) above if we set $\theta + \xi = 1$), where ρ is the exogenous rate of time preference. This means that the government does not have any freedom to affect the consumption-to-capital ratio. Of course, this is a model-specific result; in other economies, the first-best solution is attainable depending upon availability of appropriate policy instruments [see, e.g., Turnovsky (2000, chap. 13)].

5.2. Adding Congested Utility-Enhancing Public Goods

We now present the more general case in which both categories of public goods, productivity-enhancing and utility-enhancing ones, are subject to congestion. In particular, we assume that the instantaneous utility function of each individual i changes from equation (2) to

$$u(C^i, K_c) = \nu \log C^i + (1 - \nu) \log K_c^i, \tag{18}$$

where K_c^i is the services derived by each individual from UE. Using the same modeling as in equation (6), these services are defined as

$$K_c^i \equiv \frac{K_c}{(K)^\pi} \left(\frac{K^i}{K} \right)^\eta, \tag{19}$$

where the parameters $\pi, \eta \geq 0$ measure the degrees of absolute and relative congestion, respectively.

Working as in Section 3,¹⁴ it can be shown that the qualitative results of Result 1 do not change. This is for any value of the new parameters $\pi, \eta \geq 0$. The new numerical solution is reported in Table 5 and delivers the same message as Table 1.

TABLE 5. Effect of ν on long-run second-best equilibrium when $\theta + \xi = 1$ and there is congestion in UEPG

ν	$\tilde{\tau}$	\tilde{b}	\tilde{c}	\tilde{k}_c	\tilde{k}_g	$\tilde{\Lambda}_c + \tilde{\Lambda}_k$	$\tilde{\Lambda}_{kc}$	$\tilde{\Lambda}_{kg}$	\tilde{G}_g/\tilde{Y}	\tilde{G}_c/\tilde{Y}	$\tilde{\gamma}$
0.10	0.619	0.436	0.014	1.086	0.841	7	7.607	5.892	0.270	0.349	0.018
0.20	0.517	0.491	0.022	0.678	0.654	9	6.105	5.894	0.254	0.263	0.031
0.30	0.429	0.528	0.027	0.449	0.504	11	4.947	5.552	0.227	0.202	0.041
0.40	0.350	0.554	0.030	0.308	0.384	13	4.005	4.994	0.194	0.156	0.049
0.50	0.279	0.572	0.033	0.213	0.286	15	3.206	4.293	0.160	0.119	0.055
0.60	0.214	0.582	0.035	0.147	0.205	17	2.505	3.494	0.124	0.089	0.059
0.70	0.153	0.584	0.037	0.078	0.138	19	1.568	2.181	0.072	0.052	0.059
0.80	0.097	0.575	0.038	0.060	0.082	21	1.273	1.726	0.055	0.041	0.057
0.85	0.070	0.563	0.038	0.044	0.057	22	0.981	1.268	0.039	0.030	0.052
0.90	0.045	0.541	0.039	0.029	0.035	23	0.687	0.812	0.024	0.020	0.045
0.95	0.021	0.491	0.039	0.015	0.015	24	0.381	0.368	0.010	0.010	0.031

Notes: $A = 0.25$, $\beta = 0.15$, $\delta^k = 0.06$, $\delta^g = 0.06$, $\delta^c = 0.06$, $\rho = 0.04$, $\pi = 0.2$, and $\eta = 0.2$.

5.3. Dynamics

We next study dynamic stability. For algebraic simplicity, we consider the popular case in which only PE are a stock variable [as in (8a)], whereas UE are a flow variable. Thus, with G_c denoting public consumption goods and services, the instantaneous utility function of each individual is

$$u(C^i, G_c) = \nu \log C^i + (1 - \nu) \log G_c, \quad (20)$$

so that the model is as in Section 3 except that equation (2) changes to (20) and we omit equation (8b). This is a special case of the model solved in Section 3.

If we linearly approximate the equilibrium equations around the long-run solution, we end up with a three-equation linear differential system. Studying the properties of the associated Jacobian evaluated in the long run, we can show that there are two positive roots and one negative root. Because there are two jump variables and one predetermined, we have saddle-path stability.¹⁵

6. CONCLUSIONS

We have set up a dynamic general equilibrium model of endogenous growth in which productive and nonproductive public goods are financed by distorting taxes, and policy decisions are made by a Ramsey-type government that solves a second-best optimal policy problem. We focused on the optimal allocation of collected tax revenues between productive and nonproductive public goods and provided some new results in the case of congested public goods.

Because specific results are in the Introduction, we close with a general policy lesson. In a growing economy, the government realizes that it needs large tax bases to finance the provision of public consumption goods and services. It thus makes its allocation decision to boost economic growth and enlarge the tax base. Having achieved this, it can afford to raise the tax rates to finance nonproductive public spending. In other words, nongrowing societies cannot afford the provision of public consumption goods and services. Actually, our results show that, when PE are congested, the more “socialist” a society is, in the sense that it values more public consumption goods and services, the more growth-promoting policies it should choose, here in the form of giving priority to public investment. Only when there are “unrealistically” strong preferences over public consumption is it optimal to follow the conventional policy recipe; namely, not only to tax more but also to allocate more tax revenues to public consumption.

NOTES

1. See also Glomm and Ravikumar (1994), Turnovsky and Fisher (1995), Turnovsky (1996), Fisher and Turnovsky (1998), Eicher and Turnovsky (2000), Baier and Glomm (2001), Park and Philippopoulos (2003, 2004), Ghosh and Gregoriou (2008), Chatterjee and Ghosh (2009), and many others. On the other hand, see, e.g., Devarajan et al. (1996) and Kneller et al. (1999) for a practical classification of government expenditures.

2. See, e.g., Barro and Sala-i-Martin (1992, 2004), Glomm and Ravikumar (1994), Turnovsky (1996, 2000, chap. 13), Fisher and Turnovsky (1998), Eicher and Turnovsky (2000), Ott and Turnovsky (2006), and Chatterjee and Ghosh (2009).

3. See, e.g., Atkinson and Stiglitz (1980, p. 14) and Turnovsky (2000, p. 430).

4. When $\xi > 0$ and $\theta = 0$, there is pure relative congestion, in the sense that the agent can maintain a fixed level of government services, if and only if the usage of his own individual capital/activity increases in proportion to the usage of the aggregate capital/activity. When $\theta > 0$ and $\xi = 0$, there is pure absolute congestion, in the sense that congestion is proportional to the aggregate level of private capital/activity in the economy. Examples of public goods for each case can be found in Eicher and Turnovsky (2000, pp. 327–328).

5. Here we use a single tax. As discussed in Section 2 above, several related papers have used more than one tax (e.g., income, consumption, and lump-sum taxes). But these papers focus on how different taxes can replicate the first-best optimum [see Turnovsky (2000, chap. 13) for a detailed analysis]. This is not the focus of our paper.

6. The transversality condition that guarantees utility is bounded is also satisfied. With a log-linear utility function, this requires that $\rho > 0$.

7. Qualitative results are robust to the parameter values chosen, except as otherwise stated.

8. The case $\theta + \xi > 1$ is omitted because it describes an extreme situation [see, e.g., Eicher and Turnovsky (2000, p. 328)]. Results for this case are available upon request.

9. Throughout the paper, except as otherwise stated, for the parameter values used, there is only one well-defined long-run solution, whose properties we discuss. For instance, solutions that imply a shrinking economy in the long run (i.e., negative balanced growth rate) are not reported. Also, we do not report solutions that become problematic in some popular special cases, e.g., when there are only utility-enhancing public goods or only productivity-enhancing public goods (details are in Appendix A).

10. It is reported that for relatively low values of β (in our numerical solutions, $0 \leq \beta \leq 0.1$), the conventional recipe holds over the whole range of ν . This makes sense because the role of public capital must be high enough to affect policy choices.

11. See, e.g., Turnovsky (2000, p. 408) for a more detailed discussion of how congestion leads to inefficiencies in a decentralized equilibrium and how this provides arguments for (tax) policy intervention.

12. An analytical solution for the first-best is in Appendix B.

13. We compare the long-run solution of these three equations to the long-run first-best solution reported in Appendix B. Then, it is not possible to find values of τ, b that make these two solutions equal.

14. The algebra is in Appendix C.

15. The algebra is in Appendix D.

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APPENDIX A: LONG-RUN SECOND-BEST SOLUTION

(i) When $\theta + \xi = 1$, equations (14a)–(14i) are simplified to

$$\tilde{c} = \rho, \tag{A.1a}$$

$$\tilde{k}_c = \frac{(1 - \tilde{b})\tilde{\tau}A\tilde{k}_g^\beta}{(1 - \tilde{\tau})A\tilde{k}_g^\beta + \delta^c - \delta^k - \tilde{c}}, \tag{A.1b}$$

$$(1 - \tilde{\tau})A\tilde{k}_g^\beta + \delta^g = \tilde{b}\tilde{\tau}A\tilde{k}_g^{\beta-1} + \delta^k + \tilde{c}, \tag{A.1c}$$

$$(\tilde{\Lambda}_k + \tilde{\Lambda}_c)\rho = \nu, \tag{A.1d}$$

$$(1 - \tilde{b})\tilde{\tau}A\tilde{k}_g^{\beta-1} - \rho = 0, \tag{A.1e}$$

$$\beta(1 - \tilde{\tau})A\tilde{k}_g^\beta - (1 - \beta)\tilde{\tau}A\tilde{k}_g^\beta = 0, \tag{A.1f}$$

$$\tilde{\Lambda}_{kc} = \frac{1 - \nu}{(1 - \tilde{\tau})A\tilde{k}_g^\beta + \delta^c - \delta^k - \tilde{c} + \rho}, \tag{A.1g}$$

$$(\tilde{\Lambda}_c + \tilde{\Lambda}_k) = \frac{\tilde{\Lambda}_{kc}}{\tilde{k}_c}, \tag{A.1h}$$

$$\tilde{\Lambda}_{kc}\tilde{k}_g = \tilde{\Lambda}_{kg}\tilde{k}_c, \tag{A.1i}$$

Inspection of the above equations reveals that $\tilde{\Lambda}_k$ and $\tilde{\Lambda}_c$ do not enter separately but as a sum, $\tilde{\Lambda}_k + \tilde{\Lambda}_c$. This happens because $c = \rho$ (or $C = \rho K$) along the whole optimal path, including the long run, so that the first two dynamic constraints on the government's optimization problem coincide; namely, $\dot{C}/C = \dot{K}/K = (1 - \tau)A(K_g/K)^\beta - \delta^k - \rho$. Thus, the above nine equations are solved for eight variables, $\tilde{\tau}$, \tilde{b} , \tilde{c} , \tilde{k}_c , \tilde{k}_g , $(\tilde{\Lambda}_c + \tilde{\Lambda}_k)$, $\tilde{\Lambda}_{kc}$, and $\tilde{\Lambda}_{kg}$. Thus there are two long-run solutions. One solution is the one studied in the main text. The other solution implies that $\tilde{\tau} = \beta$, etc.; but this is not well defined because the model breaks down when there are only UE (in this case, $\tilde{\tau} = \beta = 0$, which implies zero tax rates and zero provision of public consumption). We report that in the popular special case in which public consumption is a flow variable, there is only one solution whose properties are those summarized in Result 1 and Table 1 (see also Appendix D for the solution of this special case).

(ii) When $0 \leq \theta + 1$, there is a single solution, as reported in the main text.

APPENDIX B: FIRST-BEST ALLOCATION

Inspection of (17d) reveals that this is an atemporal equation in k_g only. Hence, k_g is constant along the optimal path. In turn, (17a), (17b), and (17c) in the long run give

$$\tilde{c} + \tilde{g}_g + \tilde{g}_c = \beta A\tilde{k}_g^\beta + \rho, \tag{B.1a}$$

$$\tilde{c} + \tilde{g}_g + \tilde{g}_c = A\tilde{k}_g^\beta - \tilde{g}_c\tilde{k}_c^{-1} + \delta^c - \delta^k, \tag{B.1b}$$

$$\tilde{c} + \tilde{g}_g + \tilde{g}_c = A\tilde{k}_g^\beta - \tilde{g}_g\tilde{k}_g^{-1} + \delta^g - \delta^k. \tag{B.1c}$$

Equations (B.1a) and (B.1b) together imply that

$$\tilde{g}_g = \tilde{k}_g [(1 - \beta)A\tilde{k}_g^\beta + \delta^g - \delta^k - \rho]. \tag{B.1d}$$

Given the solution for \tilde{k}_g from (17d), (B.1d) is an equation in \tilde{g}_g only. Combining equations (17e), (B.1a), and (B.1b), we get

$$\tilde{k}_c = \frac{\beta A\tilde{k}_g^\beta + \rho - \tilde{g}_g}{\frac{\nu}{(1 - \nu)} [(1 - \beta)A\tilde{k}_g^\beta + \delta^c - \delta^k] + (1 - \beta)A\tilde{k}_g^\beta + \delta^c - \delta^k - \rho}, \tag{B.1e}$$

$$\tilde{g}_c = [(1 - \beta)A\tilde{k}_g^\beta + \delta^c - \delta^k - \rho]\tilde{k}_c. \tag{B.1f}$$

Given the solutions for \tilde{k}_g and \tilde{g}_g from (17d) and (B.1d), (B.1e) is an equation in \tilde{k}_c only. Once we solve for \tilde{k}_c , (B.1f) gives \tilde{g}_c . Given \tilde{k}_g , \tilde{g}_g , and \tilde{g}_c from (17d), (B.1d), and (B.1f), (B.1a) gives \tilde{c} . As said in the text, we solve (17d), (B.1d), (B.1e), (B.1f), and (B.1a) numerically for \tilde{k}_g , \tilde{g}_g , \tilde{k}_c , \tilde{g}_c , and \tilde{c} respectively. Finally, notice from (17d) that \tilde{k}_g is independent of ν . This implies that the first-best growth rate and \tilde{g}_g , given by (17f) and (B.1d), respectively, are also independent of ν . (B.1e) implies that $\partial\tilde{k}_c/\partial\nu < 0$. Given that $\partial\tilde{k}_c/\partial\nu < 0$, (B.1f) and (B.1a) imply that $\partial\tilde{g}_c/\partial\nu < 0$ and $\partial\tilde{c}/\partial\nu > 0$, respectively.

APPENDIX C: ADDING CONGESTED UTILITY-ENHANCING PUBLIC GOODS

We now consider the more general case in which both types of public goods, PE and UE, are subject to congestion. We focus on the interesting case in which $\theta + \xi = 1$. The instantaneous utility function of each individual i changes from (2) to

$$u(C^i, K_c) = \nu \log C^i + (1 - \nu) \log K_c^i, \tag{C.1}$$

where

$$K_c^i \equiv \frac{K_c}{(K)^\pi} \left(\frac{K^i}{K} \right)^\eta, \tag{C.2}$$

where the parameters $\pi, \eta \geq 0$ measure the degrees of absolute and relative congestion, respectively. It is straightforward to show that a DCE is given by

$$\dot{C} = C \left[(1 - \tau)A \left(\frac{K_g}{K} \right)^\beta - \delta^k - \rho + \frac{\eta(1 - \nu) C}{\nu} \frac{C}{K} \right], \tag{C.3a}$$

$$\frac{\dot{K}}{K} = (1 - \tau)A \left(\frac{K_g}{K} \right)^\beta - \delta^k - \frac{C}{K}, \tag{C.3b}$$

$$\frac{\dot{K}_c}{K_c} = -\delta^c + (1 - b)\tau A \left(\frac{K_g}{K} \right)^\beta \frac{K}{K_c}, \tag{C.3c}$$

$$\frac{\dot{K}_g}{K_g} = -\delta^g + b\tau A \left(\frac{K_g}{K} \right)^\beta \frac{K}{K_g}. \tag{C.3d}$$

Working as in Section 3.6, and defining the same auxiliary variables as in Section 3.7, a stationary general equilibrium with second-best policy is given by

$$\dot{c} = \left[\frac{\nu + \eta(1 - \nu)}{\nu} \right] c^2 - \rho c, \tag{C.4a}$$

$$\dot{k}_c = -\delta^c k_c + (1 - b)\tau A k_g^\beta - (1 - \tau)A k_g^\beta k_c + c k_c + \delta^k k_c, \tag{C.4b}$$

$$\dot{k}_g = -\delta^g k_g + b\tau A k_g^\beta - (1 - \tau)A k_g^{1+\beta} + c k_g + \delta^k k_g, \tag{C.4c}$$

$$\dot{\Lambda}_c = -\nu - \left[\frac{\eta(1 - \nu)}{\nu} \right] c \Lambda_c + \rho \Lambda_c + \Lambda_k c, \tag{C.4d}$$

$$\begin{aligned} \dot{\Lambda}_k &= (1 - \nu)\pi + \left[\frac{\eta(1 - \nu)}{\nu} \right] c \Lambda_c + \beta(1 - \tau)A k_g^\beta \Lambda_c + \beta(1 - \tau)A k_g^\beta \Lambda_k \\ &\quad - (1 - \beta)\tau A k_g^\beta \frac{\Lambda_{kc}}{k_c} + (\rho - c)\Lambda_k, \end{aligned} \tag{C.4e}$$

$$\dot{\Lambda}_{kc} = -(1 - \nu) + \rho \Lambda_{kc} + (1 - b)\tau A k_g^\beta \frac{\Lambda_{kc}}{k_c}, \tag{C.4f}$$

$$\begin{aligned} \dot{\Lambda}_{kg} &= -\beta(1 - \tau)Ak_g^\beta \Lambda_c - \beta(1 - \tau)Ak_g^\beta \Lambda_k \\ &\quad - \beta\tau Ak_g^{\beta-1} \Lambda_{kg} + b\tau Ak_g^{\beta-1} \Lambda_{kg} + \rho \Lambda_{kg}, \end{aligned} \tag{C.4g}$$

$$\Lambda_c + \Lambda_k = \frac{\Lambda_{kc}}{k_c}, \tag{C.4h}$$

$$\Lambda_{kc} k_g = \Lambda_{kg} k_c. \tag{C.4i}$$

With dotted variables set to zero, equations (C.4a)–(C.4i) give in the long run

$$\tilde{c} = \frac{\rho v}{v + \eta(1 - v)}, \tag{C.5a}$$

$$\tilde{k}_c = \frac{(1 - \tilde{b})\tilde{\tau} A \tilde{k}_g^\beta}{(1 - \tilde{\tau})A \tilde{k}_g^\beta + \delta^c - \delta^k - \tilde{c}}, \tag{C.5b}$$

$$(1 - \tilde{\tau})A \tilde{k}_g^\beta + \delta^s = \tilde{b}\tilde{\tau} A \tilde{k}_g^{\beta-1} + \delta^k + \tilde{c}, \tag{C.5c}$$

$$\tilde{\Lambda}_k = \tilde{\Lambda}_c = \frac{v + \eta(1 - v)}{2\rho}, \tag{C.5d}$$

$$\tilde{\Lambda}_{kc} = \frac{1 - v}{(1 - \tilde{\tau})A \tilde{k}_g^\beta + \delta^c - \delta^k - \tilde{c} + \rho}, \tag{C.5e}$$

$$\tilde{\Lambda}_{kg} = \frac{(1 - v)(\pi + \eta)}{(1 - \tilde{b})\tilde{\tau} A \tilde{k}_g^{\beta-1} - \rho}, \tag{C.5f}$$

$$\begin{aligned} &[v + \eta(1 - v)](1 - \tilde{b})\tilde{\tau} A \tilde{k}_g^\beta \left[(1 - \tilde{\tau})A \tilde{k}_g^\beta + \delta^c - \delta^k + \frac{\rho\eta(1 - v)}{v + \eta(1 - v)} \right] \\ &= \rho(1 - v) \left[(1 - \tilde{\tau})A \tilde{k}_g^\beta + \delta^c - \delta^k - \frac{\rho v}{v + \eta(1 - v)} \right], \end{aligned} \tag{C.5g}$$

$$\rho(1 - v)(\pi + \eta) = [v + \eta(1 - v)][(1 - \tilde{b})\tilde{\tau} A \tilde{k}_g^{\beta-1} - \rho]\tilde{k}_g. \tag{C.5h}$$

This system is solved numerically by using the same parameter values as in Section 3.8. Regarding π and η , we set $\pi = 0.2$ and $\eta = 0.2$. Results are presented in Table 5 and are qualitatively similar to those in Table 1.

APPENDIX D: TRANSITIONAL DYNAMICS

We focus on the popular case in which public consumption services are a flow variable. We also set $\delta^c = \delta^s \equiv \delta$ for expositional simplicity. The instantaneous utility function is

$$u(C^i, G_c) = v \log C^i + (1 - v) \log G_c, \tag{D.1}$$

where G_c is public consumption. Then the DCE is simplified from (10a)–(10d) to

$$\dot{C} = C \left[(1 - \tau)A \left(\frac{K_g}{K} \right)^\beta - \delta - \rho \right], \tag{D.2a}$$

$$\frac{\dot{K}}{K} = (1 - \tau)A \left(\frac{K_g}{K} \right)^\beta - \delta - \frac{C}{K}, \tag{D.2b}$$

$$\frac{\dot{K}_g}{K_g} = -\delta + b\tau A \left(\frac{K_g}{K} \right)^\beta \frac{K}{K_g}. \tag{D.2c}$$

In turn, the stationary general equilibrium is simplified from (13a)–(13i) to

$$\dot{c} = c^2 - \rho c, \tag{D.3a}$$

$$\dot{k}_g = b\tau Ak_g^\beta - (1 - \tau)Ak_g^{1+\beta} + ck_g, \tag{D.3b}$$

$$\dot{\Lambda}_c = -\nu + \rho\Lambda_c + \Lambda_k c, \tag{D.3c}$$

$$\dot{\Lambda}_k = -(1-\nu)(1-\beta) + \beta(1-\tau)Ak_g^\beta(\Lambda_c + \Lambda_k) - (1-\beta)b\tau Ak_g^\beta \frac{\Lambda_{kg}}{k_g} + (\rho - c)\Lambda_k, \tag{D.3d}$$

$$\dot{\Lambda}_{kg} = -\beta(1 - \nu) - \beta(1 - \tau)Ak_g^\beta(\Lambda_c + \Lambda_k) + [(1 - \beta)b\tau Ak_g^{\beta-1} + \rho]\Lambda_{kg}, \tag{D.3e}$$

$$\Lambda_c + \Lambda_k = \frac{(1 - \nu)}{\tau Ak_g^\beta} + b \frac{\Lambda_{kg}}{k_g}, \tag{D.3f}$$

$$\Lambda_{kg} = \frac{(1 - \nu)}{(1 - b)\tau Ak_g^{\beta-1}}. \tag{D.3g}$$

Equations (D.3a)–(D.3g) in the long run give

$$\tilde{c} = \rho, \tag{D.4a}$$

$$\tilde{k}_g = \frac{\tilde{\tau}}{1 - \tilde{\tau}}, \tag{D.4b}$$

$$\tilde{\Lambda}_k + \tilde{\Lambda}_c = \frac{\nu}{\rho}, \tag{D.4c}$$

$$\tilde{\Lambda}_{kg} = \frac{(1 - \nu)}{\rho} \tag{D.4d}$$

$$\frac{(1 - \nu)}{\tilde{\tau}} + \frac{\nu Ak_g^\beta}{\rho} + \frac{(1 - \nu)bAk_g^{\beta-1}}{\rho} = 0, \tag{D.4e}$$

$$\rho = (1 - \tilde{b})\tilde{\tau}A\tilde{k}_g^{\beta-1}, \tag{D.4f}$$

which is a simplified version of (A.1a)–(A.1i). This is solved numerically by using the same parameter values as in Section 3.8 (of course, the solution carries the same qualitative properties as the solution presented in the first part of Section 3.8).

We now study the transitional dynamics of (D.3a)–(D.3g) around (D.4a)–(D.4f). Notice three things: (i) $c = \rho$ along the optimal path. (ii) The last two equations, (D.3f) and (D.3g),

are atemporal. These equations are the first-order conditions for τ and b . (iii) We define $\Lambda \equiv \Lambda_c + \Lambda_k$.

We first work with the atemporal equations. Combining (D.3f) and (D.3g), we get $\Lambda_{kg} = \Lambda k_g$. If we differentiate this expression with respect to time and use (D.3b), (D.3c), (D.3d), and (D.3e), we get

$$\frac{\partial \tau}{\partial k_g} = - \frac{(1 - \nu)(1 - \beta) + \nu\tau(1 - b) - \frac{\rho(1 - \nu)(1 - \beta)}{Ak_g^\beta}}{\nu(1 - b)k_g} \tag{D.5a}$$

$$\frac{\partial b}{\partial k_g} = \frac{(1 - \nu)(1 - \beta) + \nu\tau(1 - b) - \frac{\rho(1 - \nu)(1 - \beta)}{Ak_g^\beta}}{\nu\tau k_g}, \tag{D.5b}$$

where derivatives are evaluated in the long run.

Thus, the dynamics of (D.3a)–(D.3g) are equivalent to the dynamics of the following system of equations:

$$\dot{k}_g = b\tau Ak_g^\beta - (1 - \tau)Ak_g^{1+\beta} + ck_g, \tag{D.6a}$$

$$\dot{\Lambda} = -\nu + \rho\Lambda - (1 - \nu)(1 - \beta) + \beta(1 - \tau)Ak_g^\beta\Lambda - (1 - \beta)b\tau Ak_g^\beta \frac{\Lambda_{kg}}{k_g}, \tag{D.6b}$$

$$\dot{\Lambda}_{kg} = -\beta(1 - \nu) - \beta(1 - \tau)Ak_g^\beta\Lambda + [(1 - \beta)b\tau Ak_g^{\beta-1} + \rho]\Lambda_{kg}. \tag{D.6c}$$

The Jacobian of (D.6a)–(D.6c) is

$$J = \begin{bmatrix} \alpha_{11} = \left. \frac{\partial \dot{k}_g}{\partial k_g} \right|_{BGP} & \alpha_{12} = \left. \frac{\partial \dot{k}_g}{\partial \Lambda} \right|_{BGP} & \alpha_{13} = \left. \frac{\partial \dot{k}_g}{\partial \Lambda_{kg}} \right|_{BGP} \\ \alpha_{21} = \left. \frac{\partial \dot{\Lambda}}{\partial k_g} \right|_{BGP} & \alpha_{22} = \left. \frac{\partial \dot{\Lambda}}{\partial \Lambda} \right|_{BGP} & \alpha_{23} = \left. \frac{\partial \dot{\Lambda}}{\partial \Lambda_{kg}} \right|_{BGP} \\ \alpha_{31} = \left. \frac{\partial \dot{\Lambda}_{kg}}{\partial k_g} \right|_{BGP} & \alpha_{32} = \left. \frac{\partial \dot{\Lambda}_{kg}}{\partial \Lambda} \right|_{BGP} & \alpha_{33} = \left. \frac{\partial \dot{\Lambda}_{kg}}{\partial \Lambda_{kg}} \right|_{BGP} \end{bmatrix}, \tag{D.7}$$

where the elements of Jacobian evaluated at the BGP are

$$\alpha_{11} = \frac{(1 - \nu)(1 - \beta) + \nu\bar{\tau}(1 - \bar{b}) - \frac{\rho(1 - \nu)(1 - \beta)}{A\bar{k}_g^\beta}}{\nu\bar{k}_g} A\bar{k}_g^\beta \left(\frac{1 - 2\bar{b} - \bar{k}_g}{1 - \bar{b}} \right) + \beta\bar{b}\bar{\tau}A\bar{k}_g^{\beta-1} + (1 + \beta)(1 - \bar{\tau})A\bar{k}_g^\beta + \rho, \quad \alpha_{12} = 0, \alpha_{13} = 0,$$

$$\alpha_{21} = \frac{(1 - \nu)(1 - \beta) + \nu\bar{\tau}(1 - \bar{b}) - \frac{\rho(1 - \nu)(1 - \beta)}{A\bar{k}_g^\beta}}{\nu(1 - \bar{b})\bar{k}_g} \beta A\bar{k}_g^\beta \bar{\Lambda} + (1 - \beta)^2 \bar{b}\bar{\tau}A\bar{k}_g^{\beta-2} \bar{\Lambda}_{kg} + \beta^2(1 - \bar{\tau})A\bar{k}_g^{\beta-1} \bar{\Lambda}$$

$$\begin{aligned}
 & + (1 - \beta)A\tilde{k}_g^{\beta-1}\tilde{\Lambda}_{kg} \frac{(1 - \nu)(1 - \beta) + \nu\tilde{\tau}(1 - \tilde{b}) - \frac{\rho(1 - \nu)(1 - \beta)}{A\tilde{k}_g^\beta}}{\nu\tilde{k}_g} \left(\frac{2\tilde{b} - 1}{1 - \tilde{b}} \right), \\
 & \alpha_{22} = \rho + \beta(1 - \tilde{\tau})A\tilde{k}_g^\beta > 0, \alpha_{23} = -(1 - \beta)\tilde{b}\tilde{\tau}A\tilde{k}_g^{\beta-1} > 0, \\
 & \alpha_{31} = - \frac{(1 - \nu)(1 - \beta) + \nu\tilde{\tau}(1 - \tilde{b}) - \frac{\rho(1 - \nu)(1 - \beta)}{A\tilde{k}_g^\beta}}{\nu(1 - \tilde{b})\tilde{k}_g} \beta A\tilde{k}_g^\beta \tilde{\Lambda} \\
 & \quad - \beta^2(1 - \tilde{\tau})A\tilde{k}_g^{\beta-1}\tilde{\Lambda} - (1 - \beta)^2\tilde{b}\tilde{\tau}A\tilde{k}_g^{\beta-2}\tilde{\Lambda}_{kg} \\
 & \quad + (1 - \beta)A\tilde{k}_g^{\beta-1}\tilde{\Lambda}_{kg} \frac{(1 - \nu)(1 - \beta) + \nu\tilde{\tau}(1 - \tilde{b}) - \frac{\rho(1 - \nu)(1 - \beta)}{A\tilde{k}_g^\beta}}{\nu\tilde{k}_g} \left(\frac{1}{1 - \tilde{b}} \right), \\
 & \alpha_{32} = -\beta(1 - \tilde{\tau})A\tilde{k}_g^\beta < 0, \alpha_{33} = (1 - \beta)\tilde{b}\tilde{\tau}A\tilde{k}_g^{\beta-1} + \rho > 0.
 \end{aligned}$$

The characteristic third-order polynomial of (D.7) is given by

$$\lambda^3 - (\alpha_{11} + \alpha_{22} + \alpha_{33})\lambda^2 + (\alpha_{11}\alpha_{22} + \alpha_{11}\alpha_{33} + \alpha_{22}\alpha_{33} - \alpha_{32}\alpha_{33})\lambda - \alpha_{11}\alpha_{22}\alpha_{33} = 0, \tag{D.8}$$

where λ is an eigenvalue of the characteristic polynomial. The determinant of the Jacobian is given by

$$\det J = \alpha_{11}\alpha_{22}\alpha_{33} = \lambda_1\lambda_2\lambda_3. \tag{D.9}$$

Recall that there are two jump variables (Λ and Λ_{kg}) and one predetermined variable (k_g). If the determinant is positive, there are either two negative roots and one positive root (i.e., indeterminacy) or three positive roots (i.e., instability). If it is negative, there are either two positive roots and one negative root (i.e., saddle-path stability) or three negative roots (i.e., indeterminacy). Because α_{22} and α_{33} are unambiguously positive, the sign of the determinant depends only on the sign of α_{11} . Although we cannot specify analytically the sign of α_{11} , we can calculate it numerically using the same parameter values as in Section 3.8. Thus, it follows that α_{11} is negative. Hence, the determinant of the Jacobian is negative, which implies either saddle-path stability or indeterminacy. Inspection of the characteristic polynomial in (D.8) reveals that because the constant term is positive, the most we can have is two sign alterations. Hence, using Descartes' theorem, which states that the number of positive roots cannot be greater than the number of sign alterations, we can have at most two positive roots. Next, we define $\lambda' = -\lambda$. In this case, and after calculating numerically the coefficients on λ' , we find that the most we can have is one sign alteration. Thus, the most we can have is one negative root. Combining results, there are two positive roots and one negative root. With two jump variables (Λ and Λ_{kg}) and one predetermined variable (k_g), we have saddle-path stability.