Contributions to the theory of density properties for Stein manifolds

Inauguraldissertation

der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Alexandre Ramos-Peon

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Leiter der Arbeit:

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Introduction

Since the days of Felix Klein and his Erlangen Program, it has been clear that the study of the symmetries of an object can give us a great deal of information about the object itself. In complex analysis the study of symmetries becomes particularly interesting in higher dimensions. This was demonstrated spectacularly by Henri Poincaré, who showed that the Riemann Mapping Theorem, one of the deepest results in complex analysis, does not hold in higher dimensions. He did this by computing and comparing groups of symmetries that are natural in this context, namely the groups of holomorphic automorphisms. The most simple complex manifold of higher dimension, complex affine space $\mathbb{C}^n$ with $n \geq 2$, enjoys a property which is relatively rare among complex manifolds: it has a great abundance of automorphisms. This thesis is about complex manifolds that share this feature. Before stating any definitions, let alone the main results, let us put into perspective the ideas surrounding this work.

The structure of the group $\mathrm{Aut}(\mathbb{C}^n)$ is poorly understood. Some very simple automorphisms, called shears, were already studied by J-P. Rosay and W. Rudin in 1988, who were interested in interpolation results for countable sets, as well as in dynamics. Motivated by the questions left open, E. Andersén and L. Lempert proved around 1992 one of the most remarkable results about $\mathrm{Aut}(\mathbb{C}^n)$, that would spark a refreshed interest in affine complex geometry. Namely, they showed that that the group generated by shears is *dense* in $\mathrm{Aut}(\mathbb{C}^n)$, in the compact-open topology. This led F. Forstnerič and J-P. Rosay to the formulation which is now commonly called the Andersén-Lempert theorem: *any local holomorphic flow defined near a holomorphically compact set can be approximated by global holomorphic automorphisms.* In particular, $\mathrm{Aut}(\mathbb{C}^n)$ is exceptionally large, and this method opens the possibility to obtain automorphisms with prescribed local behavior, which compensates to a certain extent the lack of holomorphic partitions of unity. This has deep and interesting consequences, that we will review in Section 1.1 and 1.2.

The *density property* was introduced by D. Varolin in his thesis to allow to generalize these techniques to other complex manifolds. He pointed out that the main feature making the construction of Andersén and Lempert possible is this:

holomorphic vector fields on $\mathbb{C}^n$ that are *complete*, meaning those whose flow generates a one-parameter group of automorphisms, generate a Lie subalgebra that is *dense* in the Lie algebra of all holomorphic vector fields on $\mathbb{C}^n$.

Definition. *Let X be a complex manifold and* $\mathrm{Lie}(X) \subset \mathrm{VF}(X)$ *the Lie algebra generated by all Lie combinations of complete holomorphic vector fields on X. We say X has the* density property *if* $\mathrm{Lie}(X)$ *is dense in* $\mathrm{VF}(X)$ *in the compact-open topology.*

This accurately captures the idea of a manifold having a "large" group of automorphisms. The density property will be discussed in detail in Section 1.4. Most interesting phenomena will take place when the manifold is Stein, which is a fundamental notion in complex analysis that we will review in Section 1.3: briefly said, a complex manifold is Stein if it has "many" holomorphic functions $X \to \mathbb{C}$.

One the main aspects distinguishing *smooth geometry* from *complex geometry* is the rigidity of the latter. For example, any continuous map between smooth manifolds can be approximated by a smooth map, in any reasonable topology, but this is far from true for complex manifolds: there are *no* nonconstant holomorphic maps of $\mathbb{C}$ into any bounded domain in $\mathbb{C}$, or into $\mathbb{C} \setminus \{0, 1\}$, or more generally into any *hyperbolic* manifold. Varolin writes "the basic intention [of the density property] is to isolate those complex manifolds for which the gap between differential topology and holomorphic geometry is considerably narrowed". Indeed, a Stein manifold Y with density property displays a high degree of flexibility: there will be many maps from $\mathbb{C}$ into Y, and even from any Stein manifold X. In particular the density property can be thought as an exact "opposite" of hyperbolicity.

Stein manifolds with the density property also satisfy an approximation property which might be understood as being "dual" to Stein manifolds. This approximation property, called the Oka property and which was formalized by Forstnerič, leads to the concept of *Oka-Forstnerič manifolds*. These are characterized by being "natural targets" of holomorphic maps, in some sense that can be made precise. They also arise as a way to generalize the homotopy principles of Oka, Grauert and Gromov, about continuous sections of fiber bundles being homotopic to holomorphic sections. These are *heuristic Oka principles*: "there are only topological obstructions to solving complex-analytic problems on Stein spaces that can be cohomologically, or even homotopically, formulated." Some notions on this fascinating subject will be introduced in Section 1.5, as it is relevant for the implications of one of the main results of Chapter 2.

Main results

There are two main results in this document. One is about a consequence of the density property for a Stein manifold: a parametric transitivity of the automorphism group, and a corresponding Oka principle. The other is about finding a new class of examples of Stein manifolds with a *volume* density property, and which are not algebraic. Each contribution has been communicated in a preprint available on the arXiv, and is each is discussed in an independent chapter. There is also a preliminary chapter whose main purpose is to place these results in context, as well as to introduce notation and previously known techniques.

Concerning a consequence of the density property

On a connected smooth manifold M of dimension at least 2, the action of the (smooth) diffeomorphism group of M is infinitely transitive. This means that M is "flexible" enough to allow any pair of N-tuples of distinct points of M to be mapped to each other with a diffeomorphism of M. Even more is true: given a manifold W and two smooth parametrizations $W \to M^N$ of such N-tuples, there exists a smooth family of diffeomorphisms mapping for each $w \in W$ the corresponding parametrized tuple to the other, as soon as this is topologically possible. It is natural to consider the analogue of this flexibility in the holomorphic category, where diffeomorphisms are replaced by holomorphic automorphisms, which are *a priori* much more rigid.

The action by holomorphic automorphisms on any Stein manifold with the density property is infinitely transitive (see Section 1.4.1). For the parametrized case, consider N points on X parametrized by a Stein manifold W, we seek a family of automorphisms of X, parametrized by W, putting them into a "standard position" which does not depend on the parameter. This general transitivity will be shown to enjoy an Oka principle, to the effect that the obstruction to a holomorphic solution is of a purely topological nature. In the presence of a holomorphic volume form and of a corresponding volume density property, similar results for volume-preserving automorphisms will be obtained. Precise definitions will be given in Chapter 1.

Let X and W be connected complex manifolds. Let $Y_{X,N}$ be the configuration space of ordered N-tuples of points in X: $Y_{X,N} = X^N \setminus \Delta$, where

$$\Delta = \left\{(z^1, \dots, z^N) \in X^N; z^i = z^j \text{ for some } i \neq j\right\}$$

is the diagonal. Consider a holomorphic map $x : W \to Y_{X,N}$, that is, N holomorphic maps $x^j : W \to X$ such that for each $w \in W$, the N points $x^1(w), \dots, x^N(w)$ are pairwise distinct. Interpreting $x : W \to Y_{X,N}$ as a parametrized collection of points, the following property can be thought of as a strong type of N-transitivity.

Definition. *Assume that* $\mathrm{Aut}(X)$ *acts* N*-transitively on* X. *Fix* N *pairwise distinct points* $z^1, \ldots, z^N$ *in* X. *We say that the parametrized points* $x^1, \ldots, x^N$ *are* simultaneously standardizable *if there exists a "parametrized automorphism"* $\alpha \in \mathrm{Aut}_W(X)$, *where*

$$\mathrm{Aut}_W(X) = \{\alpha \in \mathrm{Aut}(W \times X); \alpha(w, z) = (w, \alpha^w(z))\},$$

with

$$\alpha^w(x^j(w)) = z^j$$

for all $w \in W$ *and* $j = 1, \ldots, N$.

By the transitivity assumption, the definition does not depend on the choice of the z^j's.

This notion was introduced by F. Kutzschebauch and S. Lodin in [KL13], where they proved that for $X = \mathbb{C}^n$ and $W = \mathbb{C}^k$, if $k < n - 1$, then any collection of parametrized points $W \to Y_{X,N}$ is simultaneously standardizable. Our main result is the following.

Theorem. *Let* W *be a Stein manifold and* X *a Stein manifold with the density property. Let* N *be a natural number and* $x : W \to Y_{X,N}$ *be a holomorphic map. Then, the parametrized points* $x^1, \ldots, x^N$ *are simultaneously standardizable by an automorphism lying in the path-connected component of the identity* $(\mathrm{Aut}_W(X))^0$ *of* $\mathrm{Aut}_W(X)$ *if and only if* x *is null-homotopic.*

Since being null-homotopic is a purely topological condition, this can be thought of as an Oka principle for a strong form of parametric infinite transitivity. In the particular case $W = \mathbb{C}^k$ and $X = \mathbb{C}^n$, any map $W \to Y_{X,N}$ is null-homotopic, so we recover the result of [KL13], without any restrictions on the dimension of W. Moreover, the problem of simultaneous standardization of parametrized points in $\mathbb{C}^n$ by automorphisms in $\mathrm{Aut}_W(\mathbb{C}^n)$ (not the connected component!) will be reduced to a purely topological problem in Section 2.5. This is a different Oka principle for a strong form of parametric infinite transitivity.

We also prove a similar result when X is a manifold with the ω-volume density property (for detailed definitions see Section 1.4) under the additional topological assumption that X is contractible. The proof requires a greater sophistication in the arguments, in particular for the application of the Andersén-Lempert theory in the presence of a volume form. The method of the proof in both cases is to show that $Y_{X,N}$ is "elliptic" in Gromov's sense and hence an Oka-Grauert-Gromov h-principle applies to maps $W \to Y_{X,N}$. This permits the use of a general, previously unavailable parametric version of the Andersén-Lempert theorem, which in the presence of a volume form is rather technical. The idea is then to define a countable sequence of automorphisms, each of which maps x closer to some constant $\hat{x}$ on a larger set, which converges to the desired standardization.

Concerning new manifolds with the density property

Even before the Andersén-Lempert theorem was proven, Andersén had considered in [And90] the situation where the vector fields preserve the standard holomorphic volume form $dz_1 \wedge \cdots \wedge dz_n$ on $\mathbb{C}^n$, obtaining a similar result: *all algebraic vector fields of zero divergence are finite sums of complete fields of zero divergence.* There is a corresponding *volume density property* for manifolds equipped with a holomorphic volume form, which has been substantially less studied. Beyond $\mathbb{C}^n$, only a few isolated examples were known to Varolin [Var99], including $(\mathbb{C}^*)^n$ and $\mathrm{SL}_2(\mathbb{C})$. It took around ten years until new instances of these manifolds were found in [KK10]: all linear algebraic groups equipped with the left invariant volume form, as well as some algebraic Danielewski surfaces. It is worth pointing out that it is much more difficult to establish the volume density property than the usual density property, for reasons that will become apparent in Section 1.4.3.

Our main result in Chapter 3 is about describing a new class of manifolds which enjoy this volume density property. We will prove a general result (see Theorem 3.11), from which we can deduce the following:

Theorem. *Let $n \geq 1$ and $f \in \mathcal{O}(\mathbb{C}^n)$ be a nonconstant holomorphic function with smooth reduced zero fiber X_0, such that $\tilde{H}^{n-2}(X_0) = 0$ if $n \geq 2$. Then the hypersurface $\overline{\mathbb{C}^n_f} = \{uv = f(z_1, \ldots, z_n)\} \subset \mathbb{C}^{n+2}$ has the volume density property with respect to the form $\bar{\omega}$ satisfying $d(uv - f) \wedge \bar{\omega} = du \wedge dv \wedge dz_1 \wedge \cdots \wedge dz_n$.*

This "corollary" was known in the special case where f is a polynomial: this is due to S. Kaliman and F. Kutzschebauch, see [KK10]. Their proof heavily depends on the use of a spectral sequence of Grothendieck and seems difficult to generalize to the non-algebraic case. The method of proof we will give in Chapter 3 is completely different. It relies on modifying and using a suitable criterion involving so-called "semi-compatible pairs" of vector fields, developed in [KK15a] for the algebraic setting. It also involves studying the topology and homogeneity of suspensions over a manifold, and lifting these fields in such a way that a technical but essential generating condition on the wedge product of the tangent space of the suspension is satisfied.

Until now all known manifolds with the volume density property were algebraic, and the tools used to establish this property are algebraic in nature. The examples presented in Chapter 3 are the first known non-algebraic manifolds with the volume density property.

It is still unknown whether a contractible Stein manifold with the volume density property has to be biholomorphic to $\mathbb{C}^n$. It is believed that the answer is negative, see [KK10]. For instance the affine algebraic submanifold of $\mathbb{C}^6$ given by the equation $uv = x + x^2y + s^2 + t^3$ is such an example. Another prominent one is the Koras-Russell cubic threefold, see [Leu]. In Section 3.4 we will show how to use Theorem 3.11 to produce a non-algebraic manifold with the volume density property which is diffeomorphic to $\mathbb{C}^n$, which to our knowledge is the first of this kind. In fact, we prove the following.

Theorem. *Let $\phi : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ be a proper holomorphic embedding, and consider the manifold defined by $\overline{\mathbb{C}^n_f} = \{uv = f(z_1, \ldots, z_n)\} \subset \mathbb{C}^{n+2}$, where $f \in \mathcal{O}(\mathbb{C}^n)$ generates the ideal of functions vanishing on $\phi(\mathbb{C}^{n-1})$. Then $\overline{\mathbb{C}^n_f}$ is diffeomorphic to $\mathbb{C}^{n+1}$ and has the volume density property with respect to the volume form $\bar{\omega}$ satisfying $d(uv-f)\wedge\bar{\omega} = du\wedge dv\wedge dz_1\wedge\cdots\wedge dz_n$. Moreover $\overline{\mathbb{C}^n_f}\times\mathbb{C}$ is biholomorphic to $\mathbb{C}^{n+2}$, and therefore is a potential counterexample to the Zariski Cancellation Problem if ϕ is a non-straightenable embedding.*

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Chapter 1

Preliminary notions & context

In this preliminary chapter, we present notions necessary as a background for the work contained in Chapters 2 and 3. It will also serve to introduce the notation, and to discuss the relevant previously known results and techniques. We rely on the classical texts [AMR88, GR09, Nar85, GH94, Hör90, God58, GR79, GR84], on original papers, as well as on more modern treatments such as [Zam08] and [For11]. This last reference is particularly worthy of mention, not only because of its exhaustive bibliography, but because it gives a clear and complete account of Oka theory and elliptic complex geometry, previously unavailable in book form.

1.1 Flows of holomorphic vector fields

We introduce holomorphic vector fields and their flows, along with some remarks that will be central in the sequel. Excellent sources for this material are, for example, [AMR88] in the differentiable category, and [GR09, Nar85] for complex manifolds.

Let X be a complex manifold, of complex dimension n. This is in particular a real manifold of dimension $2n$ without singularities. We denote by $(z_1, \ldots, z_n)$ its local complex coordinates, and $z_j = x_j + iy_j$ its local real coordinates. We may associate to X its real tangent bundle TX; a section of this real bundle, which we view as a derivation on the algebra of germs of smooth functions, is called a *vector field* on X. We may also associate its complexified tangent bundle $T^{\mathbb{C}}X = TX \otimes_{\mathbb{R}} \mathbb{C}$, whose sections are called *complex vector fields*. There is a unique $\mathbb{R}$-linear endomorphism of TX which is defined in local coordinates by $J(\partial_{x_j}) = \partial_{y_j}$ and $J(\partial_{y_j}) = -\partial_{x_j}$, where ∂_x is shorthand for $\frac{\partial}{\partial x}$, and

$$\{\partial_{x_1}, \partial_{y_1}, \ldots, \partial_{x_n}, \partial_{y_n}\}$$

is a basis of TX in a local coordinate chart. Extend J to $T^{\mathbb{C}}X$: we obtain a decomposition of $T^{\mathbb{C}}X$ in the i and $-i$ eigenspaces

$$T^{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X.$$

By introducing the symbols

$$\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j}), \quad \partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$$

we see that in local coordinates, $T^{1,0}X$ (resp. $T^{0,1}X$) is spanned by $\{\partial_{z_j}\}_j$ (resp. $\{\partial_{\bar{z}_j}\}_j$). Note that by the Cauchy-Riemann equations a smooth function f is holomorphic if and only if $\partial_{\bar{z}_j}(f) = 0$ for all j, and that $T^{1,0}X$ is a holomorphic vector bundle over X, that is, its fibers are $\mathbb{C}^n$ and its transition functions are in $\mathrm{GL}_n(\mathbb{C})$ (these are the complex Jacobians of the holomorphic transition maps between complex charts on X). We therefore define **holomorphic vector fields** to be holomorphic sections of the bundle $T^{1,0}X$, and we denote by $\mathrm{VF}(X)$ the set of all these fields. This is a module over the ring of holomorphic functions on X, hereafter denoted by $\mathcal{O}(X)$.

Moreover, it is possible to identify real vector fields on X with holomorphic vector fields. Namely, there is an $\mathbb{R}$-linear isomorphism $\Phi : TX \to T^{1,0}X$, which is given in local holomorphic coordinates by

$$\sum_j (a_j\partial_{x_j} + b_j\partial_{y_j}) \mapsto \sum_j (a_j + ib_j)\partial_{z_j}, \quad a_j, b_j \in \mathcal{O}(X).$$

We may therefore *define* a real vector field Θ on X to be holomorphic if $\Phi(\Theta)$ is a holomorphic section of $T^{1,0}X$; in local coordinates, the vector field $\Theta = \sum_j (a_j\partial_{x_j} + b_j\partial_{y_j})$ is holomorphic if and only if for all j, the functions $a_j + ib_j$ are holomorphic. In particular, since by the Cauchy-Riemann equations $\Phi(\Theta)f = \Theta(f)$ for any $f \in \mathcal{O}(X)$, a holomorphic vector field Θ may alternatively by defined as a derivation mapping $\mathcal{O}(X)$ into $\mathcal{O}(X)$. We will tacitly use this identification between TX and $T^{1,0}X$, about which we will not say more (details can be found, for example, in [GH94]). With the exception of this section, whenever we speak of vector fields (without further qualification), we actually mean *holomorphic* vector fields as just defined.

Given a holomorphic vector field Θ on a complex manifold X, the system of n ordinary differential equations on a local coordinate chart $U \subset X$

$$\frac{d}{dt}\phi_t(z) = \Theta(\phi_t(z)), \quad \phi_0(z) = z, \quad (z \in U, t \in \mathbb{C}) \tag{1.1}$$

has a unique local solution, which is holomorphic in all variables. The map $t \mapsto \phi_t(z)$ is called the (local) flow of Θ, and defines a one-parameter group of local biholomorphisms ϕ_t on X, which we call time t-maps. This maps satisfy the group law $\phi_t \circ \phi_{t'}(z) = \phi_{t+t'}(z)$. By uniqueness of trajectories, the flow $t \mapsto \phi_t(z)$ extends to a maximal interval around the origin. We say that the vector field Θ is $\mathbb{C}$-**complete** (sometimes also called completely integrable) if this interval is $\mathbb{C}$, or in other words, if the solution ϕ of Equation 1.1 exists for all $t \in \mathbb{C}$ and $z \in X$.

The usual notion, for real manifolds, is that of $\mathbb{R}$-completeness: consider the vector field Θ as a section of TX and study Equation 1.1, now in $2n$ variables, which is taken with $t \in \mathbb{R}$. By the Cauchy-Riemann equations a holomorphic vector field is $\mathbb{C}$-complete if and only if V and JV are $\mathbb{R}$-complete. In [For96a, 2.2], it is shown that every $\mathbb{R}$-complete holomorphic vector field on $\mathbb{C}^n$ is also $\mathbb{C}$-complete. More generally, this is known to hold for any connected Stein manifold without nonconstant bounded plurisubharmonic functions, such as Stein manifolds that are also Oka (see below for definitions). In the rest of this work we shall say "complete vector field" instead of "$\mathbb{C}$-complete holomorphic vector field". For us the crucial fact is that the flow of a complete vector field on X defines a one-parameter group of **automorphisms** of X, i.e., biholomorphic maps from X onto X.

We denote by $\mathrm{CVF}(X)$ the set of complete vector fields. A very important observation is that the sum of complete fields is in general *not* complete, so $\mathrm{CVF}(X)$ is not a vector space. However, if $f \in \mathcal{O}(X)$ also belongs to $\mathrm{Ker}(\Theta)$, where Θ is understood as a derivation, and Θ is complete, then $f\Theta$ *is* complete. In fact, even if $\Theta(f) \in \mathrm{Ker}(\Theta)$, then $f\Theta \in \mathrm{CVF}(X)$, see for example [Var99, Thm. 3.2].

We shall also consider time-dependent vector fields, which are one-parameter families of locally defined vector fields $\{\Theta_t\}$. More precisely, a time-dependent vector field Θ defined on $\Omega \subset \mathbb{C} \times X$ is a section of the pullback of the tangent bundle under the projection of the extended phase space $\pi : \mathbb{C} \times X \to X$. For fixed t, Θ_t is a vector field on a domain $\Omega_t \subset X$, and the solution of the differential equation associated to Θ will depend on the initial time; namely, given some z in some Ω_s, the analogue of the differential equation 1.1 is

$$\frac{d}{dt}\phi_t^s(z) = \Theta_t(\phi_t^s(z)), \quad \phi_s^s(z) = z$$

which has a unique local solution, and the flows satisfy the semigroup property

$$\phi_t^r \circ \phi_r^s = \phi_t^s$$

wherever this equation makes sense.

Assume that there is a domain $U_0 \subset \Omega_0$ such that the flow $\phi_t(z) = \phi_t^0(z)$ exists for all $t \in [0,1]$ and $z \in U_0$. Then $\phi_t : U_0 \to \phi_t(U_0)$ is a biholomorphic map for each t, ϕ_0 is the identity on U_0, and $\phi_t^s = \phi_t \circ \phi_s^{-1}$ on $\phi_s(U_0)$.

It is a consequence of Grönwall's inequality (see for example [AMR88]) that approximation of time-dependent vector fields leads to approximation of flows.

1.2 Automorphisms of complex affine space

Before moving on to more complicated manifolds, we will consider the holomorphic automorphism group $\mathrm{Aut}(\mathbb{C}^n)$. For $n \geq 2$, these groups are large and complicated, as we will see, and the understanding of its structure has emerged as a challenging but fundamental problem: see for example [Kra96] for questions concerning the group of *algebraic* automorphisms, that is, whose components are polynomials. Our point of departure will be the observation by E. Andersén and L. Lempert in the 1990's [AL92] that every polynomial vector field on $\mathbb{C}^n$ is a finite sum of *complete* polynomial fields. It was then observed by J-P. Rosay and F. Forstnerič [FR93] that this implies that every isotopy of biholomorphic maps between a certain kind of domains in $\mathbb{C}^n$ can be approximated by automorphisms of $\mathbb{C}^n$. These methods, which provide a way to obtain global objects with prescribed local behavior, compensate to a certain extent the lack of holomorphic partitions of unity, and have been used in a number of interesting constructions, some of which we will review in this section. They include the existence of non-straightenable embedded complex lines in $\mathbb{C}^n$, in sharp contrast to the Abhyankar-Moh-Suzuki theorem for algebraic embeddings (see also Chapter 3); of proper holomorphic embeddings with prescribed properties; of non-Runge Fatou-Bieberbach domains, among many others. A comprehensive account is available in the surveys by S. Kaliman and F. Kutzschebauch [KK11, KK15b].

Following this remarkable achievements, D. Varolin introduced a class of manifolds in [Var00, Var01] for which this phenomena hold, see Section 1.4.

1.2.1 Shears on $\mathbb{C}^n$ and the Andersén-Lempert theorem

Since every automorphism of $\mathbb{C}$ is linear, we focus on $\mathrm{Aut}(\mathbb{C}^n)$ for $n \geq 2$, which we endow with the topology of uniform convergence on compact subsets.

Arguably the simplest automorphisms of $\mathbb{C}^n$ are maps of the form

$$(z_1, \ldots, z_n) \mapsto (z_1 + f(z_2, \ldots, z_n), z_2, \ldots, z_n), \quad f \in \mathcal{O}(C^{n-1}) \tag{1.2}$$

whose inverse are of the same form with f replaced by $-f$. Observe that the Jacobian determinant of this map is 1. We also have maps of the form

$$(z_1, \ldots, z_n) \mapsto \left(z_1 e^f(z_2, \ldots, z_n), z_2, \ldots, z_n\right), \quad f \in \mathcal{O}(C^{n-1}). \tag{1.3}$$

The choice of coordinates being artificial, we allow for linear changes of variables: we call **additive shears** the maps of the form 1.2 and all of their $\mathrm{SL}_n(\mathbb{C})$-conjugates (to preserve the Jacobian determinant 1), and **multiplicative shears** those of the form 1.3 and their $\mathrm{GL}_n(\mathbb{C})$-conjugates. The term *shear* will refer to maps of both types. Using shears, it is very simple to show that the group of automorphisms of $\mathbb{C}^n$ (for $n \geq 2$) acts N-transitively on $\mathbb{C}^n$ for any N. In fact, more is true: as a preparation for results in Chapter 2, suppose that $x^1, \ldots, x^N \in \mathbb{C}^n$ are N *distinct* points in $\mathbb{C}^n$. We denote the space of such vectors by $Y = (\mathbb{C}^n)^N \backslash (\cup_{i \neq j} \Delta_{i,j})$, where $\Delta_{i,j} = \left\{(z^1, \ldots, z^N); z^i = z^j\right\}$. The following lemma makes precise that given two sufficiently close N-tuples of points in $\mathbb{C}^n$, we can find an automorphism, arbitrarily small on any large compact K, which maps one N-tuple to the other.

Lemma 1.1. *Let $N \geq 1$, $n \geq 2$, and $x = (x^1, \ldots, x^N) \in Y$. Let $\epsilon > 0$ and a compact set $K \subset \mathbb{C}^n$ containing all of the x^j's be given. Then there is a neighborhood U of x in Y and a holomorphic map $\Psi : U \to \mathrm{Aut}(\mathbb{C}^n)$ such that*

1. *for each $y \in U$, $d(\Psi_y(z), z) < \epsilon$ for all $z \in K$;*
2. *$\Psi_x = id$;*
3. *$\Psi_y(x) = y$, that is, $\Psi_y(x_j) = y_j$ for all $j = 1, \ldots, N$.*

In particular, given a holomorphic map $y : W \to Y$, where W is any complex manifold, such that $y(w) \in U$ for all w in some compact $L \subset W$, there exists a holomorphic[1] map $\Psi : L \to \mathrm{Aut}(\mathbb{C}^n)$ such that $\Psi^w(y(w)) = x$ and $d(\Psi^w(z), z) < \epsilon$ for all $w \in L$: just define $(w, z) \mapsto (w, \Psi^{-1}_{y(w)}(z))$ for all $(w, z) \in L \times \mathbb{C}^n$. The holomorphic dependence of Ψ on w is clear from the construction below.

Proof. Let U be a neighborhood of x in Y and K any compact containing x. We will show that if U is small enough, then we can find $\Psi : U \to \mathrm{Aut}(\mathbb{C}^n)$ satisfying properties 2 and 3 above, such that for all $y \in U$,

$$\max_{z \in K} |\Psi_y(z) - z| \leq h(x, y, K). \tag{1.4}$$

where h is a function tending to 0 when y tends to x. For then if U is small enough, the right-hand side is smaller than ϵ.

We take the subscript notation for the coordinates, namely $z^j_k = \Pi_k(z^j)$ where Π_k is the projection to the k-th coordinate. Choose coordinates such that $x^i_k \neq x^j_k$ for all $k = 1, \ldots, n$ and $i \neq j$. By shrinking U we can suppose that $U = \bigcup_{i=1}^N U^i$, where each U^i is a neighborhood of x^i in $\mathbb{C}^n$, is such that $\Pi_k(U^i) \cap \Pi_k(U^j) = \emptyset$ for all $i \neq j$ and all k. Then the map Ψ is given by a holomorphically-depending shear automorphism, with coefficients coming from an interpolation formula. What follows are the details of this formula.

[1]While $\mathrm{Aut}(\mathbb{C}^n)$ is not a complex manifold, not even an infinite-dimensional one, the notion of holomorphic map into $\mathrm{Aut}(\mathbb{C}^n)$ is rather straightforward: we say a map $\alpha : X \to \mathrm{Aut}(\mathbb{C}^n)$ is holomorphic if the map $X \times \mathbb{C}^n \to \mathbb{C}^n$ given by $(x, z) \mapsto \alpha_x(z)$ is holomorphic in the usual sense.

Let $y \in U$ and define the following functions $f_1, \ldots, f_n \in \mathcal{O}(\mathbb{C})$:

$$f_1(\zeta) = \sum_{i=1}^{N} \left\{ \prod_{\substack{k=1 \\ k \neq i}}^{N} \frac{\zeta - y_2^k}{y_2^i - y_2^k} \right\} \cdot (y_1^i - x_1^i);$$
$$f_s(\zeta) = \sum_{i=1}^{N} \left\{ \prod_{k \neq i}^{N} \frac{\zeta - x_1^k}{x_1^i - x_1^k} \right\} \cdot (y_s^i - x_s^i), \quad s = 2, \ldots, n.$$

Observe that the denominators appearing in these formulas are non-zero, by the choice of coordinates and the assumption on U; moreover, these polynomials depend holomorphically on y, and have the following interpolation property: for all $i = 1, \ldots, N$,

$$f_1(y_2^i) = y_1^i - x_1^i,$$
$$f_s(x_1^i) = y_s^i - x_s^i, \quad s = 2, \ldots, n.$$

Set $\Psi_y = \beta \circ \gamma$, where

$$\beta(z_1, \ldots, z_n) = (z_1 + f_1(z_2), z_2, \ldots, z_n)$$
$$\gamma(z_1, \ldots, z_n) = (z_1, z_2 + f_2(z_1), \ldots, z_n + f_n(z_1)).$$

We see that $\Psi_y \in \mathrm{Aut}(\mathbb{C}^n)$, and in fact, by changing the parameter y, we obtain a holomorphic map $\Psi : U \to \mathrm{Aut}(\mathbb{C}^n)$. The interpolation property above and the definition of Ψ imply that condition 3 in the Lemma is satisfied:

$$\Psi_y(x^j) = \beta(x_1^j, x_2^j + c_{j,2}, \ldots, x_n^j + c_{j,n}) = \beta(x_1^j, y_2^j, \ldots, y_n^j) = y^j.$$

Condition 2 is clearly satisfied; to check Equation (1.4), fix $y \in U$ and $z \in K$; we estimate the L_1-norm of $\Psi_y(z) - z$:

$$\|\Psi_y(z) - z\|_1 \leq |f_1(z_2 + f_2(z_1))| + \sum_{j=2}^{n} |f_j(z_1)|. \tag{1.5}$$

First bound each term under the summation sign above:

$$|f_j(z_1)| \leq \sum_{i=1}^{N} |(h(x,i))| \cdot |y_j^i - x_j^i| \leq \sum_{i=1}^{N} M_{x,K} \cdot |y_j^i - x_j^i|,$$

where $h(x,i) = \left\{ \prod_{k \neq i} \frac{z_1 - x_1^k}{x_1^i - x_1^k} \right\}$ and $M_{x,K}$ is the maximum of the suprema over K of all the $h(x,i)$, which depends only on K and x. Now estimate similarly the first summand in equation (1.5):

$$|f_1(z_2+f_2(z_1))| \leq \sum_{i=1}^{N} \left| \prod_{k\neq i} \frac{z_2 + f_2(z_1) - y_2^k}{y_2^i - y_2^k} \right| \cdot |y_1^i - x_1^i|$$

Let y tend to x. Then by the previous estimation, each term $\frac{z_2+f_2(z_1)-y_2^k}{y_2^i-y_2^k}$ tends to $\frac{z_2-x_2^k}{x_2^i-x_2^k}$, which is finite. Hence the whole right-hand side in equation (1.5) tends to 0, as we wanted to show. □

In Chapter 2, we shall study this sort of behavior on more general manifolds.

Shears allow to very explicitly construct automorphisms with prescribed behavior, as just seen above. The following striking result of Andersén and Lempert [And90, AL92] is therefore highly relevant.

Theorem 1.2. *Every automorphism α of $\mathbb{C}^n$ $(n \geq 2)$ can be approximated uniformly on compacts by a finite composition of shears. If α has Jacobian 1 then the approximation is achieved with additive shears. Moreover, there are automorphisms which are not equal to a finite composition of shears.*

For example,

$$(z, w) \mapsto (ze^{zw}, we^{-zw})$$

is not a composition of shears, answering a question of J-P. Rosay and W. Rudin (see [RR88]).

To explain the spirit of the proof, note that every shear map is the time 1-map of the flow of a *shear field.* Namely, the map in Equation 1.2 (resp. 1.3) is the flow at time 1 of the field

$$\Theta = f(z_2, \ldots, z_n)\partial_{z_1} \quad (\text{resp.}\Theta' = z_1 f(z_2, \ldots, z_n)\partial_{z_1})$$

Let us point out that the field Θ above is of zero divergence. It is a standard fact that this is the case if and only if its flow ϕ_t preserves the *standard volume form*

$$\omega = dz_1 \wedge \cdots \wedge dz_n \tag{1.6}$$

for all t, meaning that $(\phi_t)^*(\omega) = \omega$, or equivalently, that ϕ_t has Jacobian 1. We also say in this case that Θ is **volume-preserving**. In particular all additive shear fields are volume-preserving. We can now state the main theorem in [And90, AL92]. A simplified proof may be found in [Ros99].

Theorem 1.3. *Any polynomial vector field (resp. of zero divergence) in $\mathbb{C}^n$, $n \geq 2$, is the finite sum of complete polynomial vector fields (resp. of zero divergence), in fact shear fields.*

It is worth explaining how Theorem 1.3 implies Theorem 1.2. Let α be an automorphism of $\mathbb{C}^n$; we may assume after a linear change of coordinates that $\alpha'(0) = id$. This allows us to connect α to the identity by setting $\alpha_t = \frac{1}{t}\alpha(tz)$. Define the time-dependent vector field Θ_t by the differential equation

$$\frac{d}{dt}\alpha_t(z) = \Theta_t(\alpha_t(z)), \quad \alpha_0(z) = z;$$

then α_t is the flow at time t of the solution of this equation. On the other hand the solution to this equation can be approximated by solving time-independent differential equations over small intervals of time. Namely, for N large, solve

$$\frac{d}{dt}\beta_t(z) = \Theta_{k/N}(\beta_t(z)), \quad \frac{k}{N} < t < \frac{k+1}{N}, \quad \beta_0(z) = z.$$

We are therefore approximating α with the composition of N maps; we now explain how to approximate each of these maps by a composition of the flows of shear fields. On a compact set, approximate the corresponding $\Theta = \Theta_{k/N}$ with a polynomial vector field (its flow will approximate the flow of Θ), and decompose it into a finite sum $\sum_{j=1}^{M} \Theta_j$ using Theorem 1.3. The flow of the sum $\sum \Theta_j$ is in turn obtained, approximately, as the composition of time ϵ-maps of the flows of Θ_j's, as follows: flow Θ_1 for time $0 \le t \le \epsilon$, then flow along Θ_2 for $\epsilon \le t \le 2\epsilon$, and so on, until Θ_M is flown up to time $M\epsilon$; then repeat, by flowing along Θ_1, Θ_2... until t reaches 1. The convergence of this approximation can be shown using classical tools, see for example [AMR88, §4.1], or [For11, §4.8–4.9].

In fact this shows that the flow (wherever it is defined) of any vector field Θ on $\mathbb{C}^n$ can be approximated by an automorphism.

1.2.2 Isotopic and parametric Andersén-Lempert theorems

In [FR93], Forstnerič and Rosay further developed the approach in the previous subsection to isotopies of locally defined biholomorphic maps. This turns out to be better suited for applications, for instance as in Chapter 2. It seems necessary to ask for a Runge property on the domain on which we wish to approximate. In the classical theory of functions of several complex variables (see for example [Hör90]), a **Runge domain** is an open subset $U \subset \mathbb{C}^n$ (or more generally $U \subset X$, where X is any complex manifold) with the property that holomorphic functions on U can be approximated, uniformly on compacts of U, by functions holomorphic on X. This is motivated by the desire to generalize the classical one variable Runge approximation theorem to higher dimensional spaces. In dimension 1 an open set is Runge if and only if it is simply connected, whereas in higher dimensions there is no topological characterization of Runge domains.

What in fact Andersén and Lempert proved is the following.

Theorem 1.4. *If F is a biholomorphic map from a star-shaped domain D to a Runge domain in $\mathbb{C}^n$, then F can be approximated uniformly on compacts by finite compositions of shears. Moreover if $JF = 1$, then F can even be approximated by finite compositions of additive shears.*

The authors already remarked that such a star-shaped domain is Runge (see e.g. [El 88]), but it does not hold as stated for an arbitrary Runge domain D (see for example [For96b, p. 177]).

A $\mathcal{C}^k$ *isotopy of injective holomorphic maps* is a $\mathcal{C}^k$ map $F : U \times [0,1] \to \mathbb{C}^n$ such that for each fixed $t \in [0,1]$, the map $F_t : U \to X$ is an injective holomorphic map. The following generalization concerning isotopies of biholomorphic maps between Runge domains is the main theorem in [FR93].

Theorem 1.5. *Let U be an Runge domain in $\mathbb{C}^n$ ($n \geq 2$), and F be a $\mathcal{C}^2$ isotopy of injective holomorphic maps from U into $\mathbb{C}^n$. Assume that each domain $U_t = F_t(U)$ is Runge in $\mathbb{C}^n$. If F_0 can be approximated uniformly on compacts by automorphisms of $\mathbb{C}^n$, then for every $t \in [0,1]$ the map F_t can be approximated in the same sense. Moreover if U is a pseudoconvex domain and F_t is volume-preserving and if $H^{n-1}(U, \mathbb{C}) = 0$, then F_t is a limit of volume-preserving automorphisms.*

It was observed in [FR94] that both the cohomological condition and the pseudoconvexity of U are necessary. Pseudoconvexity is understood in the classical sense: a domain of holomorphy in $\mathbb{C}^n$. See Section 1.3 for more details. We will discuss the cohomological condition in more detail in Section 1.3.2.

Observe that this result implies Theorem 1.2 at once. Furthermore, by connecting the identity to an arbitrary biholomorphic map F defined on a star-shaped domain, we see that Theorem 1.4 itself follows from its isotopic version. Indeed, we may assume that D contains the origin. Set $G = F - F(0)$, so that $G(0) = 0$. The (well defined) maps $G_t(z) = \frac{1}{t}G(tz)$ form an isotopy connecting $G_1 = G$ to $G_0 = G'(0)$, and there is another isotopy connecting this linear map to the identity. By combining these isotopies and reparametrizing $[0,1]$, we obtain a smooth isotopy of biholomorphic maps connecting F to the identity. Moreover, the proof of Theorem 1.5 is similar to what we have already indicated. For each $t_0 \in [0,1]$, consider the following vector field on U_{t_0}:

$$\Theta_{t_0}(z) = \frac{d}{dt}\Big|_{t=t_0} F_t(F_{t_0}^{-1}(z)), \quad z \in U_{t_0}.$$

Hence Θ is a time-dependent vector field, and Φ_{t_0} is obtained by integrating the time-dependent field Θ_t from time 0 to t_0. We conclude that the composition $\Phi_{t_0} = F_t \circ F_0^{-1}$ is approximable, and since by hypothesis F_0 is approximable too, we are done.

We remind the reader (see also Section 1.3) that an open set $U \subset \mathbb{C}^n$ is **holomorphically convex** in U, or $\mathcal{O}(U)$-convex, if and only if the $\mathcal{O}(U)$-convex hull of every compact $K \subset U$

$$\widehat{K}_U = \left\{ z \in U; |f(z)| \leq \sup_K |f| \quad \forall f \in \mathcal{O}(U) \right\} \subset U$$

is compact. A compact K is called holomorphically convex (in $U \subset \mathbb{C}^n$) if $K = \widehat{K}_U$, and $K \subset \mathbb{C}^n$ is called **polynomially convex** if it is $\mathcal{O}(\mathbb{C}^n)$-convex. Runge's theorem can be rephrased as follows, see [Hör90, Thm 4.3.2]:

Theorem 1.6. *Let $U \subset \mathbb{C}^n$ be pseudoconvex, and $K \subset U$ be a $\mathcal{O}(U)$-convex compact. Then every holomorphic function defined in a neighborhood of K can be approximated uniformly on K by a function holomorphic in U.*

It may also be shown, combining the results of [Hör90] with [Nar61, Thm.3], that every compact $K \subset \Omega$ subset of a pseudoconvex Ω (such as $\mathbb{C}^n$) which is $\mathcal{O}(\Omega)$-convex admits a basis of pseudoconvex neighborhoods that are Runge.

Returning to the Andersén-Lempert theorem, this fact is used in [FR93] to obtain approximation near polynomially convex sets. In [For94] the required regularity is shown to be $\mathcal{C}^0$, and in [Kut05] a parametric version is shown to hold (see also [For03], where it is used to prove an approximation result for holomorphic submersions). Combining this we obtain this version:

Theorem 1.7. *Let $n \geq 2$ and U be an open set in $\mathbb{C}^k \times \mathbb{C}^n$. Let F be a $\mathcal{C}^p$ $(p \geq 0)$ isotopy of injective holomorphic maps from U into $\mathbb{C}^k \times \mathbb{C}^n$ of the form*

$$F_t(w, z) = (w, F_t^w(z)), \quad (w, z) \in U, \quad \text{and } F_0^w = id. \tag{1.7}$$

Suppose $K \subset U$ is a compact polynomially convex subset of $\mathbb{C}^k \times \mathbb{C}^n$, and assume that $F_t(K)$ is polynomially convex in $\mathbb{C}^k \times \mathbb{C}^n$ for each $t \in [0, 1]$. Then for all $t \in [0, 1]$, F_t can be approximated uniformly on K (in the $\mathcal{C}^p$ norm) by automorphisms $\alpha_t \in \mathrm{Aut}(\mathbb{C}^k \times \mathbb{C}^n)$ of the form 1.7; moreover α_t depends smoothly on t, and α_0 can be chosen to be the identity.

1.2.3 The push-out method: compositions of automorphisms

Many applications of the Andersén-Lempert theorem, such as the one in section 2.4, require constructing a countable sequence of automorphisms in such a careful way that their composition converges uniformly on compacts to an automorphism. Let $\{\alpha_j\}_j$ be a sequence of automorphisms of $\mathbb{C}^n$, and let $\Psi_m = \alpha_m \circ \cdots \circ \alpha_1$. Under some mild assumptions, the infinite composition $\lim_{m\to\infty} \Psi_m$ converges to a **Fatou-Bieberbach mapping**, that is, a biholomorphism $F : U \to \mathbb{C}^n$ from a proper subset U of $\mathbb{C}^n$. This phenomenon, the existence of such **Fatou-Bieberbach domains**, does not exists in dimension 1, because of the Riemann mapping theorem.

Classically, these domains arise as the basin of attraction of holomorphic automorphisms of $\mathbb{C}^n$, see [RR88]. It is interesting to note that each such map must be non-algebraic, since injective polynomial maps $\mathbb{C}^n \to \mathbb{C}^n$ are automorphisms ([Rud95] gives an elementary proof of this). A Fatou-Bieberbach domain U is Runge if and only if the associated Fatou-Bieberbach map $F : \mathbb{C}^n \to U$ is a locally uniform limit of automorphisms of $\mathbb{C}^n$, see [For11, §4.3]. Let us mention, as an application of considerable interest of the Andersén-Lempert theorem, that E. Wold [Wol08] has shown the existence of a non-Runge Fatou-Bieberbach domain in $\mathbb{C}^n$.

We now state the actual push-out method for $\mathbb{C}^n$ that we will adapt in Section 2.4.

Theorem 1.8. *Let D be a connected domain in $\mathbb{C}^n$ and $K_0 \subset K_1 \subset \cdots \subset \cup_{j=0}^{\infty} K_j = D$ be an exhaustion of D by compacts such that for all j, $K_j \subset int(K_{j+1})$. Pick numbers ϵ_j such that*

$$0 < \epsilon_j < d(K_{j-1}, \mathbb{C}^n \setminus K_j), \quad \sum_{j=1}^{\infty} \epsilon_j < \infty.$$

Suppose that $\alpha_j \in \mathrm{Aut}(\mathbb{C}^n)$, $j \in \mathbb{N}$, satisfy

$$|\alpha_j(z) - z| < \epsilon_j, \quad z \in K_j.$$

Set $\Psi_m = \alpha_m \circ \cdots \circ \alpha_1$. Then there is an open set $\Omega \subset \mathbb{C}^n$ such that $\lim_{m\to\infty} \Psi_m = \Psi$ exists uniformly on compacts in Ω, and Ψ is a biholomorphic map of Ω onto D. In fact, $\Omega = \cup_{m=1}^{\infty} \Psi_m^{-1}(K_m)$.

For a proof, see [For11, §4.4]. If we select $D = \mathbb{C}^n$, then Ω is the set of points $z \in \mathbb{C}^n$ where the sequence $\{\Psi_m(z); m \in \mathbb{N}\}$ is bounded, and we obtain the following corollary:

Corollary 1.9. *Let $\{\alpha_j\}_j \subset \mathrm{Aut}(\mathbb{C}^n)$ and $\{K_j\}_j$ be an exhaustion of $\mathbb{C}^n$ satisfying the conditions of the previous theorem. Then the limit Ψ exists uniformly on compacts of Ω and is a Fatou-Bieberbach map from Ω onto $\mathbb{C}^n$.*

The method goes back at least to P. Dixon and J. Eterle [DE86]. It was subsequently applied by J. Globevnik [Glo98] to construct Fatou-Bieberbach domains with certain properties on the boundary; this was is turn is used in [BKW10] to find topologically knotted proper holomorphic embeddings of the unit disc $\mathbb{D} \subset \mathbb{C}$ into $\mathbb{C}^2$. J. Globevnik and B. Stensønes [GS95] also use the push-out method to make a substantial contribution to the still open part of Forster's conjecture, to the effect that every non-compact Riemann surface can be properly, holomorphically embedded in $\mathbb{C}^2$ (they prove it for a certain class of finitely connected planar domains). There is also a series of papers by Forstnerič and Wold using the Andersén-Lempert theorem to extend the class of Riemann surfaces that can be shown to be embeddable in $\mathbb{C}^2$: for example, [Wol06, FW13]

Theorem 1.8 is Proposition 5.1 in [For99], where it is used, among other things, to construct non-straightenable proper holomorphic embeddings (see also 3.4). Recall that a proper holomorphic embedding $\phi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ is said to be **holomorphically straightenable** if there exists an automorphism α of $\mathbb{C}^n$ such that

$$\alpha(\phi(\mathbb{C}^k)) = \mathbb{C}^k \times \{0\}^{n-k}.$$

The existence of non-tame sets in $\mathbb{C}^n$ (see [RR88]), combined with an interpolation theorem, implies that there exists for each $k < n$ non-straightenable proper holomorphic embeddings $\phi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ (for $n = 2$ see also the earlier paper [FGR96]). Note that proper algebraic embeddings are the holomorphic analogue of polynomial embeddings, and that the "classical" algebraic situation is in sharp contrast to the holomorphic one: a famous result of Abhyankar and Moh [AM75] states that if $k = 1$ and $n = 2$, then every polynomial embedding is algebraically straightenable. More generally, the Abhyankar-Sathaye conjecture asks if every polynomial embedding $\mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ is algebraically straightenable. For every $n > 2k+1$, polynomial embeddings $\phi : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ are algebraically straightenable (see [Kal92]), and the case of real codimension 2 remains notoriously open.

1.3 Stein manifolds

We shall be interested mainly in Stein manifolds, of which we recall the definition and some properties. A classical reference is [Hör90]; in our opinion [Zam08] gives a well-written account of the theory in $\mathbb{C}^n$.

A distinguishing characteristic of the theory of functions of several complex variables, in contrast to functions of a single variable, is the existence of domains $\Omega \subset \mathbb{C}^n$ where all holomorphic functions extend to a larger set. We call a domain for which this does *not* happen a **domain of holomorphy**. In fact, for $\Omega \subset \mathbb{C}^n$, the following are equivalent (see [Zam08, 1.8.8]): there is a single holomorphic function which does not extend across any point on the boundary of Ω; Ω is a domain of holomorphy; and Ω is **holomorphically convex**, that is, for any $K \subset \Omega$,

$$\widehat{K}_\Omega = \left\{ z \in \Omega; |f(z)| \leq \sup_K |f| \quad \forall f \in \mathcal{O}(\Omega) \right\} \tag{1.8}$$

is still compact in Ω. This is also equivalent to Ω being pseudoconvex, which means that Ω has a continuous plurisubharmonic exhaustion function.

The concept of a domain of holomorphy in $\mathbb{C}^n$ evolved to that of a Stein manifold. These manifolds may be characterized by having a large number of holomorphic functions defined on it, and are the natural spaces to which the classical theorems of one dimensional analysis generalize. In 1951 K. Stein introduced the class of *holomorphically complete manifolds* [Ste51], now called **Stein manifolds**. The original definition was simplified by later developments. We now say a complex manifold X is **Stein** if if is holomorphically convex and if given $x, y \in X$, there is a $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$. Open sets in $\mathbb{C}^n$ are Stein if and only if they are domains of holomorphy. Clearly, a closed complex submanifold of a Stein manifold is Stein, and it is a deep theorem that every Stein manifold admits a proper holomorphic embedding in some Euclidean space $\mathbb{C}^N$. Hence, Stein manifolds are embedded in Euclidean spaces, and we have the following

Theorem 1.10. *A complex manifold is Stein if and only if it is biholomorphic to a closed complex submanifold of some $\mathbb{C}^N$. In fact, if n is the dimension of X, then N can be taken to be $2n+1$.*

This is called the embedding theorem of Stein manifolds of Remmert, Bishop and Narasimhan. However, this N is not minimal: Y. Eliashberg and M. Gromov have shown in [EG92], using a generalization of Oka's principle (to be discussed in Section 1.5.2), that N can be taken to be the smallest integer greater than $(3n+1)/2$. Compare with he previously mentioned conjecture of Forster: every open Riemann surface admits a proper holomorphic embedding into $\mathbb{C}^2$.

This property is sufficient to show that $\mathrm{Aut}(X)$ is a Baire space when X is Stein, which is a fact we will use in Chapters 2 and 3. Namely, since X embeds in some $\mathbb{C}^m$, any continuous map $f : X \to X$ is given by m coordinate functions $f_1, \dots, f_m$. Since X is a topological manifold, it admits an exhaustion $X = \bigcup_{j=1}^{\infty} K_j$ by compact sets $K_1 \subset K_2 \subset \dots$, which we now fix. Denote by $||g||_i$ the maximum over K_i of the norm of any continuous function $g : X \to \mathbb{C}$. Then the space $\mathrm{Aut}(X)$ admits a metric given by the following formula (see [KK08b])

$$d(\Phi, \Psi) = \sum_{j=1}^{\infty} \frac{\min\left(\max_i(||\Phi_i - \Psi_i||_j), 1\right)}{2^j} \quad (\Phi, \Psi \in \mathrm{Aut}(X)).$$

This generates the compact-open topology and makes $\mathrm{Aut}(X)$ into a *complete* metric space, which is therefore a Baire space by the Baire category theorem. In particular, every countable collection of open and dense sets has nonempty intersection.

Of particular interest to us are the generalization of classical approximation and interpolation theorems of $\mathbb{C}^n$. Namely, recall the *Weierstrass interpolation theorem*: given a domain X in $\mathbb{C}$ and discrete set $S \subset X$, every continuous function $f : S \to \mathbb{C}$ extends to a holomorphic function $X \to \mathbb{C}$. The *Cartan extension* *theorem*, proved with techniques in sheaf theory that we will soon recall, generalizes S to a closed analytic subset of a Stein manifold X, and says that any holomorphic function $S \to \mathbb{C}$ extends to a holomorphic function defined on X. Recall also the *Runge approximation theorem* : given an open set $X \subset \mathbb{C}$, if $K \subset X$ is compact and $X \setminus K$ is connected, then every holomorphic function $f : K \to \mathbb{C}$ can be approximated uniformly on K by a holomorphic function $X \to \mathbb{C}$. The *Oka-Weil approximation theorem* replaces the topological condition on K by (the non-topological) holomorphic convexity: if X is Stein and $K \subset X$ is $\mathcal{O}(X)$-convex, then every holomorphic function in an open neighborhood of K can be approximated uniformly on K by functions in $\mathcal{O}(X)$.

1.3.1 Sheaf cohomology and applications

The concept of a coherent analytic sheaf is of the greatest importance in complex analysis, in particular for Stein manifolds. The idea is that theorems about coherent sheaves on domains of holomorphy can as well be proved for Stein manifolds. In this regard the most celebrated results are the Theorems A and B of H. Cartan and J-P. Serre from the beginning of the 50's, about the sheaf cohomology of a Stein manifold. We first briefly recall the most basic required notions. An excellent reference for general sheaf-theoretical methods is [God58], or [GR84] for the theory of coherent sheaves in the complex analytical setting.

First denote by $\mathcal{O}_X$ the sheaf of germs of holomorphic section on X, called the **structure sheaf** of X. An **analytic sheaf** $\mathcal{F}$ on a complex manifold X is a sheaf of $\mathcal{O}_X$ modules, i.e., it is the data, for any open $U \subset X$, of an $\mathcal{O}(U)$-module $\mathcal{F}(U)$ called **section of** $\mathcal{F}$ at or over U, satisfying some natural restriction conditions which we will not spell out here. The stalk $\mathcal{F}_x$ over any $x \in X$ is a module over the local ring $\mathcal{O}_{X,x}$. The sheaf $\mathcal{F}$ is called *locally finitely generated* if for any $x_0 \in X$ there is a neighborhood U of x_0 and finitely many sections $f_1, \ldots, f_s \in \mathcal{F}(U)$ whose germs $(f_j)_x$ at any $x \in U$ generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module. In other words there is a surjective morphism

$$\mathcal{O}|_U^n \to \mathcal{F}|_U \to 0,$$

and we say that the analytic sheaf $\mathcal{F}$ is **coherent** if it is locally finitely generated, and if the kernel of any morphism $\mathcal{O}|_U^n \to \mathcal{F}|_U$ is finitely generated. It may be vaguely said that coherence is a local principle of analytic continuation, or as H. Grauert and R. Remmert put it in ([GR79]) "if it is locally free except possibly on some small set where it is still finitely generated with the ring of relations again being finitely generated".

Let $A \subset X$ be a closed complex subvariety of X. For each $x \in X$, let $\mathcal{I}_{A,x}$ be the ideal in $\mathcal{O}_X$ of holomorphic germs at x whose restriction to A vanishes, and let

$$\mathcal{I}_A = \bigcup_{x \in X} \mathcal{I}_{A,x}$$

be the **sheaf of ideals** of A in X. Then the restriction of the quotient sheaf $\mathcal{O}_X/\mathcal{I}_A$ to A, denoted $\mathcal{O}_A$, is called the structure sheaf of A. We record here the main examples of coherent sheaves, all of which will be used in the sequel.

Theorem 1.11. *The following analytic sheaves on a complex manifold X are coherent:*

- ***(Oka's coherence theorem)*** *the structure sheaf $\mathcal{O}_X$*
- ***(Cartan's coherence theorem)*** *the sheaf of ideals $\mathcal{I}_A$ of a subvariety $A \subset X$*
- *the sheaf of holomorphic sections of holomorphic vector bundles*
- *the kernel and image of a morphism $g : \mathcal{F} \to \mathcal{G}$ of coherent analytic sheaves.*

Without going into details, the sheaf cohomology $H^j(X, \mathcal{F})$ of X with values in $\mathcal{F}$ is the right derived functor of the global section functor; what this means for us is that given a short exact sequence of sheaves on X

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

there is a long exact sequence in cohomology

$$\cdots \to H^j(X, \mathcal{E}) \to H^j(X, \mathcal{F}) \to H^j(X, \mathcal{G}) \to \dots$$

It may be shown that if X is any differentiable manifold, then the sheaf cohomology of X with values in the constant sheaf of stalk $\mathbb{C}$ is equal to the de Rham cohomology $H^j_{dR}(X, \mathbb{C})$, computed with differential forms (see also Section 1.3.2). We also quickly mention the Leray theorem and Čech cohomology, as this is used in the proof of Proposition 2.6. Let $\mathfrak{U} = \{U_i\}_i$ be an open cover of X and $\mathcal{F}$ a sheaf (even of groups) on X, and consider the set $C^j(\mathfrak{U}, \mathcal{F})$ of functions σ associating to each $j+1$ tuple of sets chosen from $\mathfrak{U}$, $a = (U_{i_0}, \dots, U_{i_j})$, with nonempty intersection $|a|$, a value $\sigma(a) \in \mathcal{F}(|a|)$. There is a natural coboundary map $d : C^j \to C^{j+1}$

$$d\sigma(U_{i_0}, \dots, U_{i_{j+1}}) = \sum_{k=0}^{j+1} (-1)^k \sigma\left(U_{i_0}, \dots, \widehat{U_{i_k}}, \dots, U_{i_{j+1}}\right)\Big|_{U_{i_0} \cap \dots \hat{U}_{i_k} \dots \cap U_{i_{j+1}}}$$

which makes $(C^\bullet(\mathfrak{U}, \mathcal{F}), d)$ into a cochain complex, and whose cohomology (in the commutative algebra sense) is denoted by $\check{H}^\bullet(\mathfrak{U}, \mathcal{F})$.

Let $\mathfrak{U}$ be an open cover of X and $\mathcal{F}$ a sheaf on X; we call $\mathfrak{U}$ **acyclic** with respect to $\mathcal{F}$ if $H^j(U, \mathcal{F}) = 0$ for all $U \in \mathfrak{U}$ and $j \geq 1$; $\mathfrak{U}$ is called *locally finite* if any $x \in X$ has a neighborhood which intersects only finitely many of the sets in $\mathfrak{U}$. We can formulate Leray's theorem (see for example [GR79, p. 43]):

Theorem 1.12. *If $\mathfrak{U}$ is a locally finite covering of a complex manifold X which is acyclic with respect to a sheaf $\mathcal{F}$, then there is an isomorphism*

$$H^j(X, \mathcal{F}) \cong \check{H}^j(\mathfrak{U}, \mathcal{F}).$$

We now state Cartan's Theorem A and B.

Theorem 1.13. *Let X be a Stein manifold, $\mathcal{F}$ a coherent sheaf on X, and $x \in X$. Then*

(A) *the stalks $\mathcal{F}_x$ are generated as $\mathcal{O}_{X,x}$-modules by global sections of $\mathcal{F}$, and*

(B) $H^j(X, \mathcal{F}) = 0, \quad \forall j \geq 1.$

A typical application is the following. Suppose $h : \mathcal{F} \to \mathcal{G}$ is a morphism of coherent analytic sheaves on X which is surjective (that is, the induced map on stalks is surjective). Then there is an short exact sequence

$$0 \to \mathrm{Ker}(h) \to \mathcal{F} \to \mathcal{G} \to 0,$$

which induces an exact sequence in cohomology

$$\cdots \to \mathcal{F}(X) \to \mathcal{G}(X) \to H^1(X, \mathrm{Ker}(h)) \to \ldots$$

Since by Theorem 1.11 $\mathrm{Ker}(h)$ is coherent, the last term is 0 by Cartan's theorem B, and hence the map at the level of global sections $\mathcal{F}(X) \to \mathcal{G}(X)$ is surjective. We record a very important special case, namely the surjection $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_A$, where $\mathcal{I}_A$ is the sheaf of ideals a closed complex subvariety A of a Stein manifold X. This is called the **Cartan Extension Theorem**:

Corollary 1.14. *Every holomorphic function defined on a closed complex subvariety of a Stein manifold X extends to a holomorphic function on X.*

This is the generalization of the Weierstrass interpolation theorem promised above. We also have an analogue of the Runge approximation theorem, which in this more general setting is called the **Oka-Weil approximation theorem**, see [For11, 2.4.7]:

Theorem 1.15. *Let $\mathcal{F}$ be a coherent analytic sheaf on a Stein manifold X. If K is an $\mathcal{O}(X)$-convex compact in X, then any section over an open neighborhood of K can be uniformly approximated on K by sections in $\mathcal{F}(X)$.*

It was known to Cartan that it is possible to extend a holomorphic function f defined on a subvariety S of a Stein manifold X to a holomorphic function on X, while simultaneously approximating a function g defined on a neighborhood of a $\mathcal{O}(X)$-convex compact $K \subset X$ (and coinciding with f on S): see [Car58, Thm. 2.1 bis]. In fact the cited result is an Oka principle for sections of principal bundles, see Section 1.5.1.

1.3.2 Remarks on holomorphic volume forms

Let X be a complex manifold of dimension n, and let T^*X be its cotangent bundle, which is the dual of the real tangent bundle TX. The $\mathbb{R}$-linear endomorphism of $T^{\mathbb{C}}X$ from Section 1.1 induces a dual endomorphism of the complexified cotangent bundle, with a splitting

$$(T^{\mathbb{C}}X)^* = (T^{1,0}X)^* \oplus (T^{1,0}X)^*;$$

in local coordinates on $U \subset X$, $(T^{1,0}X)^*|_U = \mathrm{Span}_{\mathbb{C}}\{dz_1, \ldots, dz_n\}$. Consider the holomorphic vector bundle which is the j-fold exterior product of the bundle $\wedge^j(T^{1,0}X)^*$. Its holomorphic sections are called **holomorphic j-forms**, and are given in a local coordinate chart U by

$$\alpha = \sum_{|I|=j} a_I dz_{i_1} \wedge \cdots \wedge dz_{i_j}, \quad a_I \in \mathcal{O}(U).$$

Of course one can consider analogously antiholomorphic forms, or mixed (p,q)-forms, but this is not of interest for our purposes. Denote by $\Omega^j(X)$ the vector space of global sections of $\wedge^j(T^{1,0}X)^*$, i.e. global j-forms. Note that the pullback by a holomorphic mapping f of a j-form is still a holomorphic j-form. The exterior derivative d of usual differential forms splits into $d = \partial + \bar{\partial}$. The classical **Poincaré lemma** states that d is locally exact: for every closed j-form α, the equation $d\beta = \alpha$ has a solution on any contractible open set. It is a theorem of Grothendieck (see for example [For11, 1.7.1]) that a Poincaré lemma holds locally on $\mathbb{C}^n$ for both the ∂ and $\bar{\partial}$ operator. In fact we shall use this local version in Section 2.2.

Lemma 1.16 (Poincaré lemma). *Let U be a contractible open set in X and $j \geq 1$. Then given any closed holomorphic j-form α on U, there exists a holomorphic $(j-1)$-form β on U such that $d\beta = \alpha$.*

Proof. Given the usual Poincaré lemma, it suffices to prove that β is a $(j-1,0)$-form satisfying $\bar{\partial}\beta = 0$, i.e., that β is holomorphic. This follows at once from the fact that α is a $(j,0)$-form and that $d = \bar{\partial} + \partial$. □

We record also an important consequence of Cartan's theorem B, due to Serre [Ser53, Thm.1]. Recall that the de Rham cohomology of X, denoted $H^j_{dR}(X,\mathbb{C})$, is the cohomology of the complex of real differential forms with the coboundary operator d, and is isomorphic to the singular cohomology of X with complex coefficients.

Theorem 1.17. *Suppose X is Stein. Then each de Rham cohomology class of X can be represented by a closed holomorphic form.*

Assume now that X has dimension n and is equipped with a (holomorphic) **volume form** ω, that is, a $(n,0)$-form which is nowhere vanishing. We denote by $\mathrm{Aut}(X,\omega)$ the space of all automorphisms of X preserving ω. Just as in the case where $X=\mathbb{C}^n$ and $\omega = dz_1\wedge\cdots\wedge dz_n$ from Section 1.2.1, we say a vector field $\Theta\in \mathrm{VF}(X)$ is **volume-preserving** if its flow at any time t preserves the given volume form: $\phi_t^*\omega=\omega$. It is convenient to define this notion in an alternative way, namely via the vanishing of the divergence. Recall that the **divergence of a vector field** Θ on X with respect to ω is the unique complex-valued function $\mathrm{div}_\omega\,\Theta$ such that

$$(\mathrm{div}_\omega\,\Theta)\omega = \mathcal{L}_\Theta\omega$$

where $\mathcal{L}_\Theta$ is the Lie derivative in the direction of Θ:

$$\mathcal{L}_\Theta = (\phi_{-t})^*\frac{d}{dt}(\phi_t)^*\omega$$

It is then easily seen that vector fields of zero divergence with respect to ω are exactly those for which $\mathcal{L}_\Theta\omega=0$, which are exactly those whose flows preserve ω.

Denote $\mathrm{VF}_\omega(X)$ the vector space of all such fields, and $\mathrm{CVF}_\omega(X)$ those fields in $\mathrm{VF}_\omega(X)$ that are also complete. Then $\mathrm{VF}_\omega(X)$ is a vector space, but not an $\mathcal{O}(X)$-module anymore, because for any $f\in\mathcal{O}(X)$, $\mathrm{div}_\omega\, f\Theta = \Theta(f)$, which has no reason to vanish. Hence the theory of Andersén-Lempert, which uses Runge approximation for sections of the coherent tangent sheaf, cannot be copied directly for volume-preserving fields. There is however a trick that makes the problem approachable: recall that given $\Theta\in\mathrm{VF}(X)$, there is a degree -1 $\wedge$-antiderivation ι_Θ on the graded algebra of forms $\Omega(X)$ called **interior product**, defined by the relation

$$(\iota_\Theta\alpha)(\nu)=\alpha(\Theta\wedge\nu),\quad \alpha\in\Omega^{k+1}(X),\nu\in\Gamma(\wedge^k TX,X).$$

Its relationship to the exterior derivative d is expressed through Cartan's formula (see for example [AMR88])

$$\mathcal{L}_\Theta\alpha = d\iota_\Theta\alpha+\iota_\Theta d\alpha.$$

Other formulas which we will find useful in Section 1.4.3 are

$$\mathcal{L}_{f\Theta}\alpha = f\mathcal{L}_\Theta\alpha + df\wedge\iota_\Theta\alpha \qquad\text{and}\qquad [\mathcal{L}_\nu,\iota_\mu]=\iota_{[\nu,\mu]}, \tag{1.9}$$

which in particular imply that ι_Θ and $\mathcal{L}_\Theta$ commute.

The trick is then that the non-degeneracy of ω implies that vector fields and $(n-1)$-forms are in one-to-one correspondence via $\Theta \mapsto \iota_\Theta\omega$, which by Cartan's formula restricts to an isomorphism

$$\Phi : \mathrm{VF}_\omega(X) \to \mathcal{Z}^{n-1}(X).$$

where $\mathcal{Z}^j(X)$ denotes the vector space of d-closed j-forms on X. We can now sketch a proof of the Andersén-Lempert Theorem 1.5 in the volume-preserving case, and explain how the topological condition is used. We first consider the isotopy $F_t : U \to \mathbb{C}^n$ as the flow of a time-dependent vector field $\Theta_t \in \mathrm{VF}_\omega(U_t)$. We have already explained why it suffices to approximate Θ_t for all t uniformly on compacts of U_t by globally defined fields of zero divergence. To Θ_t, associate via Φ a closed $(n-1)$-form α_t on U_t. By the cohomological assumption it is exact, so there is some form β_t with $d\beta_t = \alpha_t$, which by Theorem 1.17 can be assumed to be a holomorphic $(n-2)$-form. Now these forms are defined on Runge sets U_t so by Theorem 1.15 and by the remark following Theorem 1.6, since sections of the vector bundle $\Omega^{n-2}(X)$ form a coherent sheaf, each β_t may be approximated by some globally defined β'_t. Now $\alpha'_t = d\beta'_t$ are globally defined approximations of α_t, which are closed, so by the isomorphism Φ we get globally defined divergence-free vector fields Θ'_t approximating Θ, and hence ultimately an approximation by automorphisms in $\mathrm{Aut}(X, \omega)$.

A more refined version of this argument will be given during the proof of Proposition 2.5.

1.4 Density properties

It follows from the discussion in Section 1.2.1 that the main feature of $\mathbb{C}^n$ that allows to prove the Andersén-Lempert theorem is the abundance of complete vector fields on $\mathbb{C}^n$. This fact is captured in Theorem 1.3: every polynomial vector field is the finite sum of complete polynomial automorphisms. We have used a consequence of this: every holomorphic vector field in $\mathbb{C}^n$ can be approximated uniformly on compacts by finite sums of complete holomorphic vector fields.

In an attempt to generalize the techniques to other manifolds, D. Varolin abstracted this feature and introduced in [Var00, Var01] the class of manifolds with the *density property*, whose definition we now motivate. Observe that in the proof of "Theorem 1.3 implies Theorem 1.2", we used the following: if $\Theta = \sum_{j=1}^N \Theta_j$ then its flow ϕ_t is given by

$$\phi_t(x) = \lim_{n\to\infty} \left(\phi^1_{t/n} \circ \cdots \circ \phi^N_{t/n}\right)^n (x).$$

This follows from the fact that if ϕ, ψ are the flows of X, Y, then

$$X + Y = \frac{d}{dt}\Big|_{t=0} \phi_t \circ \psi_t,$$

see [AMR88, Thm 2.1.26] (the same proof work for holomorphic fields and flows). Similarly, it is well known that the Lie bracket $[X, Y]$ (which is defined to act as on $\mathcal{O}(X)$ as a derivation by $X(Y(f)) - Y(X(f)))$ satisfies

$$[X, Y] = \frac{d}{dt}\Big|_{t=0} \Psi_{-\sqrt{t}} \circ \phi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \phi_{\sqrt{t}}.$$

The space of vector fields on a complex manifold X is a Lie algebra with respect to this bracket. We say that a **Lie combination** of elements of a subset S of $\mathrm{VF}(X)$ is a vector field which can be written as a finite sum of terms of the form

$$[[\ldots[[\Theta_1, \Theta_2], \Theta_3], \ldots, \Theta_{n-1}], \Theta_n], \quad \Theta_j \in S.$$

By the previous discussion, the following holds (see [For11, 4.8.3] or [Var01, p.7]):

Lemma 1.18. *Given a family of complete vector fields $\{\Theta_j\}$ on a complex manifold X, the local flow of any Lie combination of the Θ_j's can be approximated by automorphisms of X.*

Let $\mathrm{Lie}(X)$ be the Lie algebra generated by the subset $\mathrm{CVF}(X)$: these are all Lie combinations of complete (holomorphic) vector fields on X. A complex manifold X is said to have the **density property** if $\mathrm{Lie}(X)$ is dense in $\mathrm{VF}(X)$ in the compact-open topology. If additionally the manifold X is equipped with a volume form ω, denote by $\mathrm{Lie}_\omega(X)$ the Lie algebra generated by elements in $\mathrm{CVF}_\omega(X) = \mathrm{VF}_\omega(X) \cap \mathrm{CVF}(X)$, and define X to have the **ω-volume density property** is $\mathrm{Lie}_\omega(X)$ is dense in $\mathrm{VF}_\omega(X)$. We will sometimes write DP for density property and ω-VDP for the volume density property.

Since this is the central topic of this thesis, we will make a number of remarks, study some of the consequences of the definition in Section 1.4.1 (for a more exhaustive list, see [KK11] or [KK15b]), and review all known examples of these manifolds in Section 1.4.2. We will also review in Section 1.4.3 the effective criteria used in the literature to prove that a given manifold does in fact possess one of these properties.

We begin by emphasizing that the DP is trivially satisfied on compact manifolds, as all vector fields are complete. However, there are relatively few holomorphic vector fields on those manifolds ($\mathrm{VF}(X)$ is finite-dimensional). On the other hand, the DP is rather restrictive on non-compact manifolds. If X is Stein of dimension n, it follows from Cartan's theorem A that $\mathrm{VF}(X)$ (and $\mathrm{VF}_\omega(X)$ if

$n \geq 2$) is infinite-dimensional (see [Var01, §3.2]). Then the DP (resp. the VDP if $n \geq 2$) implies that $\mathrm{Aut}(X)$ (resp. and $\mathrm{Aut}(X, \omega)$) is infinite dimensional.

Of course, the theorem of Andersén [And90] can be restated as saying that $\mathbb{C}^n$, for $n \geq 2$, has the VDP with respect to the standard volume form (Equation 1.6). If $n = 1$, since $\mathrm{div}_\omega \Theta = 0$ implies that Θ is constant (and all constant fields are complete), $\mathbb{C}$ trivially has the VDP. The theorem of Andersén and Lempert [AL92] is that $\mathbb{C}^n$, $n \geq 2$, has the DP. Since all complete vector fields on $\mathbb{C}$ are affine linear, $\mathbb{C}$ does not have the DP.

A manifold may have a VDP with respect to a given volume form but it has no reason to have the VDP with respect to another form. Note that there is no obvious relation between the DP and the VDP: namely, a manifold may have the VDP and not the DP, like $\mathbb{C}$; less trivially, $(\mathbb{C}^*)^k$ for $k \geq 2$ has the VDP (see Theorem 1.26) but it is unknown if it has the DP. In fact this is an older open problem ([RR88]): it is not even known if there are any complete vector fields on $(\mathbb{C}^*)^2$ with non-zero $(\frac{1}{zw} dz \wedge dw)$-divergence.

There is a related notion of algebraic density properties on algebraic manifolds, introduced by Kutzschebauch and Kaliman in [KK08b, KK08a]. We say that an affine algebraic variety has the **algebraic density property** or ADP if the Lie algebra $\mathrm{Lie}_{alg}(X)$ generated by complete algebraic vector fields coincides with the algebra of all algebraic vector fields $\mathrm{VF}_{alg}(X)$. Kaliman and Kutzschebauch introduced in [KK10] the following definition: if the manifold has an algebraic volume form ω, then X is said to have the **algebraic $\omega-$volume density property** or AVDP if

$$\mathrm{Lie}_{alg,\omega}(X) = \mathrm{VF}_{alg,\omega}(X).$$

In the case of affine algebraic manifolds, the ADP clearly implies the (holomorphic) DP. This is important because it is usually simpler to establish the ADP, since more tools are available. For affine algebraic varieties, the AVDP also implies the VDP, although this is a nontrivial fact: see [KK10, Thm. 4.1].

1.4.1 Some consequences of the density property

Beyond those mentioned in the previous section, we record the most immediate consequence of the definition of the density properties: the analogue of the Andersén-Lempert theorem.

Theorem 1.19. *Let X be a Stein manifold with the DP (resp. with the ω-VDP). Let $U \subset X$ be an open set and $F_t : U \to X$ be a C^1 isotopy of injective holomorphic maps such that F_0 is the inclusion of U into X. Suppose that $K \subset U$ is a compact set such that $K_t = F_t(K)$ is $\mathcal{O}(X)$-convex for every $t \in [0, 1]$* (resp. *and $H^{n-1}(K, \mathbb{C}) = 0$). Then F_1 can be uniformly approximated on K by automorphisms of X, with respect to any Riemannian distance function on X.*

A very careful proof may be found in the appendix of [Rit13], at least for the case where X has the DP. In the volume preserving case, the observations in Section 1.3.2 make it clear that the same proof works.

Already Varolin in [Var00] showed that this implies the following interpolation result.

Theorem 1.20. *Let X be a Stein manifold of dimension $n \geq 2$ with the DP (resp. with an ω-VDP), $K \subset X$ compact, $x, y \in X \setminus \widehat{K}$, and $z_1, \ldots, z_N \in K$. Then there exists an automorphism $\alpha \in \mathrm{Aut}(X)$ (resp. $\alpha \in \mathrm{Aut}(X, \omega)$) such that $\alpha(z_j) = z_j$ for all j, $\alpha(x) = y$, and ψ is arbitrarily close to the identity on K.*

This of course implies the infinite transitivity of the action of the automorphism group:

Theorem 1.21. *If X is Stein and has the DP (resp. the ω-VDP and $\dim(X) \geq 2$), then for all $N \in \mathbb{N}$ and pairs of distinct tuples $(x_1, \ldots, x_N), (y_1, \ldots, y_N)$, there is an automorphism $\alpha \in \mathrm{Aut}(X)$ (resp. in $\mathrm{Aut}(X, \omega)$) such that $\alpha(x_i) = y_i$ for all i.*

In Chapter 2, we will prove a parametrized version of this.

Theorem 1.20 is in fact more general (there is some jet interpolation, which also works in the volume-preserving case), and it is used in [Var00] to show the following interesting phenomena.

Theorem 1.22. *Let X be a Stein manifold of dimension n with the DP. Then X has an open subset which is biholomorphic to $\mathbb{C}^n$, i.e., a Fatou-Bieberbach domain, and also a proper subset biholomorphic to X. If (X, ω) has the VDP, then the second statement holds.*

In Section 1.2.3 it was mentioned that there exist non-equivalent proper holomorphic embeddings $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$, for $k < n$. Another consequence of the density properties is that there are non-equivalent holomorphic embeddings of a given complex manifold M into a Stein manifold X with the DP or VDP. Precisely, the following holds.

Theorem 1.23. *Let X be a Stein manifold of dimension $n \geq 2$ with the DP or the VDP, and M be a Stein manifold of complex dimension $r < n$. Suppose there exists at least one proper holomorphic embedding $j : M \hookrightarrow X$. Then there exists another proper holomorphic embedding $j' : M \hookrightarrow X$ such that for any $\alpha \in \mathrm{Aut}(X)$,*

$$\alpha \circ j(M) \neq j'(M).$$

In connection to this, we mention a result due to R. Andrist, F. Forstnerič, T. Ritter and E. Wold [AFRW], which generalizes the particular case of the embedding theorem for Stein manifolds of Remmert, Narasimhan and Bishop (Theorem 1.10) to more general spaces.

Theorem 1.24. *Let X be a Stein manifold of dimension n satisfying the DP or the VDP. If S is a Stein manifold of dimension r and $2r+1 \leq n$, then any continuous map $f : S \to X$ is homotopic to a proper holomorphic embedding $F : S \hookrightarrow X$.*

There is another important consequence of Theorem 1.19, proven in [KK08b]:

Theorem 1.25. *Let X be a Stein manifold with the DP (resp. the ω-VDP). Then there exist $\Theta_1, \dots, \Theta_N \in \mathrm{CVF}(X)$ (resp. $\mathrm{CVF}_\omega(X)$) such that*

$$\mathrm{Span}_{\mathbb{C}}\{\Theta_j(x)\}_j = T_x X, \quad \forall x \in X.$$

This phenomenon is understood as a *flexibility* property, which we discuss briefly. In [AFK+13, AZK12], the authors make the following definition: a point p of a (reduced) algebraic variety X is called **algebraically flexible** if T_pX is generated by the tangent vectors to the orbit under the so-called "special automorphism group", which is a subgroup of $\mathrm{Aut}_{alg}(X)$ generated by all one-parameter unipotent subgroups (subgroups isomorphic to the additive group $\mathbb{C}_+$). The flexibility at every point of X is shown to be equivalent, at least in the case of smooth irreducible varieties, to the infinite transitivity of the action of this subgroup on X. The holomorphic version of this notion is the following: a point p of a complex manifold is said to be **holomorphically flexible** if complete fields span T_pX (i.e. holomorphic one-parameter subgroups of $\mathrm{Aut}(X)$), and the manifold X is called holomorphically flexible if every point $p \in X$ is. Define analogously the notion in the presence of a volume form (see [AFK+13, §6]). Holomorphic (volume) flexibility, for a Stein manifold, is equivalent with the transitive action of $\mathrm{Aut}(X)$ (resp. $\mathrm{Aut}(X, \omega)$) on X, but the equivalence of holomorphic flexibility with infinite transitivity is not known, see [Kut14, §3]. However, for a Stein manifold X, the (V)DP implies holomorphic (volume) flexibility (Theorem 1.25) – as well as infinite transitivity (Theorem 1.20) – and hence, as we will see in section 1.5, X admits a holomorphic spray, hence is elliptic, and is therefore an Oka-Forstnerič manifold.

1.4.2 Known examples

The theory of the density property has been developing in the last 15 years, and the list of examples is growing. Chapter 3 is about expanding this list, albeit rather modestly. It therefore seems pertinent to give an exhaustive list of manifolds known to enjoy either the (A)DP or some (A)VDP, which we present in a more or less chronological order.

We start with a theorem on general facts and implications.

Theorem 1.26. *Let X and Y be Stein manifolds, and G a complex Lie group with any left invariant holomorphic volume form ω_G.*

- *[Var01, 3.2] If X and Y have the DP, then so has $X \times Y$.*
- *[KK10, 4.1,4.4] If (X, ω_X) and (Y, ω_Y) have the AVDP, then $(X \times Y, \omega_X \wedge \omega_Y)$ has the AVDP, and hence the VDP.*
- *[Var01, 3.7] If X has the DP, then so has $X \times \mathbb{C}$ and $X \times \mathbb{C}^*$.*
- *[Var01, 4.4] If G has the VDP, then $G \times \mathbb{C}^*$ has the VDP (unknown for $\mathbb{C}$).*
- *[Var01, 4.2] $(G \times \mathbb{C}, \omega_G \wedge dz)$ as the VDP. If moreover G is Stein of positive dimension then $G \times \mathbb{C}$ has the DP.*
- *[Var01, 3.8] If (X, ω) has the VDP, and $(X \times \mathbb{C}, \omega \wedge dz)$ has the VDP, then $X \times \mathbb{C}$ has the DP.*

The first explicit manifolds with the DP or VDP were:

- [Var99]: The space $\mathrm{SL}_2(\mathbb{C})$ of 2×2 matrices with determinant 1 has the DP, and the VDP with respect to any left invariant holomorphic 3-form.
- [Var99]: $M = \mathbb{C}^2_{x,y} \setminus \{xy = 1\}$ has the VDP with respect to the form $\frac{1}{xy-1} dx \wedge dy$. By Theorem 1.26, $M \times \mathbb{C}$ has the DP.
- [TV00] Every complex semisimple Lie group has the DP. Semisimple means its Lie algebra is a direct sum of simple Lie algebras. For example, $PSL_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm id\}$ has the DP and the VDP with respect to any any right invariant volume form. Also the quadric $\{x^2 + y^2 + z^2 = 1\}$ has the DP and the VDP with respect to $xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$.
- [TV06] Homogeneous spaces $X = G/R$, where G is a complex semisimple Lie group of adjoint type and R is a reductive group, have the DP. "Adjoint type" means that G has trivial center; and "reductive" is a technical condition that is equivalent to X being Stein. This includes affine quadrics of the sort $\{\sum_{j=0}^{N} x_j^2 = 1\}$ with $N \geq 1$.
- [KK08b] Let $p \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial (resp. a holomorphic function) with smooth reduced zero fiber, i.e., the partial derivatives $\frac{\partial p}{\partial x_j}$ have no common zeros on the zero fiber of p. Then the hypersurface (resp. the Stein manifold)

  $$X_p = \big\{(x_1, \ldots, x_n, u, v) \in \mathbb{C}^n \times \mathbb{C}^2_{u,v};\, uv = p(x_1, \ldots, x_n)\big\} \tag{1.10}$$

  has the ADP (resp. the DP).

Kaliman and Kutzschebauch introduced a powerful criterion in [KK08a] to establish the DP, using compatible fields (see Section 1.4.3), and were able to considerably extend the class of manifolds with the A(V)DP.

Theorem 1.27. *The following manifolds have the ADP:*

- $\mathbb{C}^k \times (\mathbb{C}^*)^l$*, with* $k \geq 1$ *and* $k + l \geq 2$*.*
- $\mathrm{SL}_n(\mathbb{C})$*, the space of* $n \times n$ *matrices with determinant* 1*.*
- *Generally, all complex linear algebraic groups, whose connected components are different that tori* $(\mathbb{C}^*)^n$ *or* $\mathbb{C}$*. Note that* $(\mathbb{C}^*)^n$ *for* $n \geq 2$ *does not have the ADP, see [And00].*
- *Even more generally, homogeneous spaces of the form* $X = G/R$ *where* G *is a linear algebraic group and* R *is a closed proper reductive subgroup, such that* X *has connected components different from* $\mathbb{C}$ *or tori (due to F. Donzelli, A. Dvorski and S. Kaliman [DDK10]).*

In [KK10], the following manifolds are shown to have an AVDP:

- *If* p *is a polynomial in* n *variables with a smooth reduced zero fiber* Z *such that, if* $n \geq 2$*,* $\tilde{H}^{n-2}(Z, \mathbb{C}) = 0$*, then the hypersurface given by Equation* 1.10 *has the AVDP with respect to the volume form* ω *satisfying*

$$\omega \wedge dp = \omega_{std}.$$

- *All linear algebraic groups* G *have the AVDP with respect to the left invariant volume form.*

In [KK15a], a new criterion for the AVDP was introduced, using semi-compatible fields (see 1.4.3). This greatly simplifies the proof of the last item, and allows to pass to homogeneous spaces, among other results:

- Let G be a linear algebraic group, R a closed reductive subgroup of G, and $X = G/R$ the homogeneous space. Suppose that X has a G-invariant algebraic volume form ω. Then X has the ω-AVDP.
- The surface in $\mathbb{C}^3$ given by

$$p(x) + q(y) + xyz = 1,$$

where p and q are polynomials with $p(0) = q(0) = 0$ such that $1 - p(x)$ and $1 - q(x)$ have simple roots only, has the AVDP with respect to the form

$$\omega = \frac{dx \wedge dy}{xy}.$$

Finally, we mention some examples due to Donzelli and to M. Leuenberger.

- ([Don12]) Danilov-Gizatullin surfaces have ADP. We refer to the original article for the very technical definition.
- ([Leu]) The Koras-Russel cubic threefold defined by the equation

$$x + x^2y + s^2 + t^3 = 0$$

in $\mathbb{C}^4$ has the ADP and the AVDP with respect to the volume form

$$\frac{dx}{x^2} \wedge ds \wedge dt.$$

Besides the examples given in Chapter 3, these are to our knowledge the only manifolds with the DP or VDP.

1.4.3 Algebraic criteria

As noted above, an effective criterion for the algebraic density property was found by Kaliman and Kutzschebauch in [KK08a]. The idea is to find a nonzero $\mathbb{C}[X]$-module in $\mathrm{Lie}_{alg}(X)$, which can be "enlarged" in the presence of a certain homogeneity condition to the whole $\mathrm{VF}_{alg}(X)$. The module can be found as soon as there is a pair of complete fields which is "compatible" in a certain technical sense. The algebraic VDP was first thoroughly studied in [KK10], and a corresponding criterion was subsequently developed in [KK15a], wherein the notion of "semi-compatible" vector fields is central. In this section we present these notions and criteria in the algebraic setting, and we postpone to Section 3.2 the more general non-algebraic criteria.

Let X be an algebraic manifold and $x_0 \in X$. A finite subset M of the tangent space $T_{x_0}X$ is called a **generating set** if the image of M under the action of the isotropy subgroup of x_0 in $\mathrm{Aut}_{alg}(X)$ spans $T_{x_0}X$ as a complex vector space.

Theorem 1.28. *Let X be an affine algebraic manifold such that $\mathrm{Aut}_{alg}(X)$ acts transitively, and let $\mathbb{C}[X]$ denote its ring of regular functions. Let L be a $\mathbb{C}[X]$-submodule of $\mathrm{VF}_{alg}(X)$ such that $L \subset \mathrm{Lie}_{alg}(X)$. If for some $x_0 \in X$ the fiber $L_{x_0} = \{\Theta(x_o); \Theta \in L\}$ contains a generating set, then X has the ADP.*

We give a sketch of the proof (see [KK08a, Thm. 1] for more details). The module $\mathrm{Lie}_{alg}(X)$ generates a subsheaf $\mathcal{F}'$ of the tangent sheaf, and the sum of finitely many of the translates of $\mathcal{F}'$ by pushforwards (using automorphisms) is a coherent subsheaf $\mathcal{F}$, whose sections span the tangent space $T_{x_0}X$ at some x_0, since L_{x_0} contains a generating set. By the transitivity of $\mathrm{Aut}_{alg}(X)$ this holds for every

$x \in X$, so global sections of $\mathcal{F}$ span the tangent space everywhere, and by Cartan's theorem B every algebraic vector field on X is a $\mathbb{C}[X]$-linear combination of sections of $\mathcal{F}$, which are also complete (since pushing forward preserves completeness of fields).

The problem is now reduced to finding a nontrivial module L in $\mathrm{Lie}_{alg}(X)$, which is done using so-called compatible pairs, which we now define. A pair of nontrivial algebraic vector fields Θ, η is said to be a **compatible pair** if Θ is locally nilpotent[2] and η is either locally nilpotent or semisimple (i.e. its flow generates an algebraic $\mathbb{C}^*$-action), and satisfies the following properties:

(i) the vector space $\mathrm{Span}_{\mathbb{C}}(\mathrm{Ker}\,\Theta \cdot \mathrm{Ker}\,\eta)$ generated by elements from the set $\mathrm{Ker}\,\Theta \cdot \mathrm{Ker}\,\eta$ contains a nonzero ideal in $\mathbb{C}[X]$

(ii) there is some element $a \in \mathrm{Ker}\,\eta$ such that $\Theta(a) \in (\mathrm{Ker}\,\Theta) \setminus \{0\}$.

The point is that the existence of a compatible pair implies the existence of a nonzero $\mathbb{C}[X]$-module in $\mathrm{Lie}_{alg}(X)$. Indeed, let (Θ, η) be a compatible pair. Choose a as in condition (ii) and set $b = \Theta(f)$. If $f_j \in \mathrm{Ker}\,\Theta_j$ $(j = 1, 2)$ then the vector field

$$[af_1\Theta, f_2\eta] - [f_1\Theta, af_2\eta] = -bf_1f_2\eta$$

is a Lie combination of complete vector fields, i.e. belongs to $\mathrm{Lie}_{alg}(X)$. By part (i) of the definition, there is a nonzero ideal $I \subset \mathbb{C}[X]$, and the formula above shows that $I\eta$ generates a nonzero $\mathbb{C}[X]$-module contained in $\mathrm{Lie}_{alg}(X)$.

The criterion, used to prove Theorem 1.27, now takes the following form:

Theorem 1.29. *Let X be an affine algebraic manifold where $\mathrm{Aut}_{alg}(X)$ acts transitively. If there is a finite collection of compatible pairs $\{(\Theta_j, \eta_j)\}_j$ such that for some $x_0 \in X$, $\{\eta_j(x_0)\}_j \subset T_{x_0}X$ is a generating set, then X has the ADP.*

By the argument above applied to every compatible pair, there is a nonzero $\mathbb{C}[X]$-module L in $\mathrm{Lie}_{alg}(X)$. Since $\{\eta_j(x_0)\}_j$ is a generating set, $\{\Theta(x_0); \Theta \in L\}$ contains a generating set, we are done by the previous theorem.

We now turn to the volume case. Let X be an affine algebraic manifold of dimension n with an algebraic volume form ω. It is clear that there is no straightforward generalization of the above ideas, since there is no way to find a $\mathbb{C}[X]$-module in $\mathrm{Lie}_{alg,\omega}(X)$: if $f \in \mathbb{C}[X]$ and $\Theta \in VF_\omega(X)$, then $\mathrm{div}_\omega(f\Theta) = \Theta(f)$ is nonzero in general. We have mentioned that the introduction of a new criteria in [KK15a], partially analogous to the one just given, allowed the authors to greatly simplify

[2]A derivation (or vector field) Θ on X is locally nilpotent if for every $f \in \mathbb{C}[X]$ there is an integer k such that $\Theta^k(f) = 0$, where Θ^k means iteration of the derivation. This is equivalent to the flow ϕ_t of Θ being an algebraic $\mathbb{C}$-action on X. Observe that in general complete fields generate $\mathbb{C}$-actions that are not necessarily algebraic, as the example $z\partial_z$ shows.

the proof of the AVDP in Theorem 1.27. The idea is to search for a module in a different space, namely in $d^{-1} \circ \Phi(\mathrm{Lie}_{alg,\omega}(X))$, where $\Phi : \mathrm{VF}_{alg,\omega}(X) \to \mathcal{Z}^{n-1}$ is the interior product map defined in Section 1.3.2 and $d : \Omega^{n-2}(X) \to \mathcal{Z}^{n-1}$ is exterior differentiation. The key is the following formula, whose proof is elementary and follows from Equation 1.9: for any vector fields Θ, η, we have

$$\iota_{[\Theta,\eta]}\omega = d\left(\iota_\Theta \circ \iota\right). \tag{1.11}$$

Suppose Θ and η are complete, and let $f \in \operatorname{Ker}\Theta$, $g \in \operatorname{Ker}\eta$. Replace Θ by $f\Theta$ and η by $g\eta$ in the equation above. By the linearity of ι, we get that $fg\iota_\Theta\iota_\eta \in d^{-1}\Phi(\mathrm{Lie}_{alg,\omega}(X))$. We define two vector fields Θ, η to be a **semi-compatible pair** if Θ, η are complete nontrivial algebraic vector fields such that condition (i) of the definition of compatible pairs holds, i.e. there is a nonzero ideal of $\mathbb{C}[X]$ contained in $\mathrm{Span}_{\mathbb{C}}(\operatorname{Ker}\Theta \cdot \operatorname{Ker}\eta)$. The previous discussion therefore gives a proof of the following (see also [KK15a, §3], where it is proved in a more general setting, well-suited for the applications for homogeneous spaces):

Lemma 1.30. *Let X be an algebraic variety of dimension n equipped with a holomorphic volume form ω. Let (Θ, η) be a semi-compatible pair of divergence-free fields on X. Then $d^{-1} \circ \Phi(\mathrm{Lie}_{alg,\omega}(X))$ contains a nonzero $\mathbb{C}[X]$-submodule L of the module $\Omega^{n-2}(X)$.*

It is then rather straightforward, using the ideas already discussed, to derive the following theorem. We chose to omit here the proof, since we will in Section 3.2 give the details of the proof of an analogous result in the holomorphic category that generalizes this.

Theorem 1.31. *Let X be an algebraic variety of dimension n equipped with an algebraic volume form ω. Suppose there are finitely many semi-compatible pairs of divergence-free vector fields (Θ_j, η_j) with associated ideals I_j that have the property that for any $x \in X$, the set*

$$\{f(x)\Theta_j(x) \wedge \eta_j(x); f \in I_j\}_j$$

generates the fiber $T_xX \wedge T_xX$ of $TX \wedge TX$ over x. Suppose also that the restriction of de Rham homomorphism

$$\mathcal{Z}^{n-1}(X) \to H^{n-1}(X, \mathbb{C})$$

to $\Phi(\mathrm{Lie}_{alg,\omega}(X))$ is surjective. Then X has the AVDP with respect to ω.

1.5 Oka theory

The aim of this section is to put the notions of the density property and flexibility discussed above in a broader perspective, and to introduce the notions of an Oka-Forstnerič manifold and of the Oka property, which we will use in Chapter 2. For this we will first give a brief historical overview, and then proceed with the modern definitions.

1.5.1 Historical Oka-Grauert principle

The concept of an Oka-Forstnerič manifold, to be defined below, is relatively new. It was formally introduced by Forstnerič in [For09] (he called them Oka manifolds), after F. Lárusson highlighted their theoretical importance in [Lár04]. However, the developments leading up to this definition are much older and go back as far the 30's and to the problem of generalizing the theorem of Weierstrass about finding entire functions in $\mathbb{C}$ with prescribed poles and zeros. This naturally leads to the so-called Cousin multiplicative problem (see e.g. [For11]), which asks for the existence of a globally defined meromorphic function that is specified in terms of local data. K. Oka proved in 1939 that this problem is solvable on a domain of holomorphy with holomorphic functions if and only if it is solvable by continuous functions. This implies that in a holomorphic fiber bundle with fiber $\mathbb{C}^*$ over a domain of holomorphy, every continuous section can be continuously deformed to a holomorphic section. With the advent of sheaf theory and the theorems of Cartan and Serre, this was generalized to a Stein manifold X, since it is possible to formulate the problem purely in cohomological terms. Moreover, this implies that a holomorphic line bundle (vector bundle of rank 1) over a Stein manifold is holomorphically trivial if it is topologically trivial,

Grauert [Gra58] generalized Oka's theorem to vector bundles of arbitrary rank over Stein manifolds (in fact Stein spaces), to the effect that the topological classification of these bundles coincides with the holomorphic classification. In fact, this follows from the following theorem of his: if $Z \to X$ is a holomorphic vector bundle, then the inclusion of the **space of holomorphic sections** $\Gamma_{\mathcal{O}}(X, Z)$ into the **space of continuous sections** $\Gamma_{\mathcal{C}}(X, Z)$ is a **weak homotopy equivalence:** this means that the induced maps of homotopy groups

$$\pi_j(\Gamma_{\mathcal{O}}(X, Z)) \to \pi_j(\Gamma_{\mathcal{C}}(X, Z))$$

are isomorphisms, for all j. This implies in particular that every continuous section can be deformed (via a homotopy) to a holomorphic section, and that any two homotopic holomorphic sections are also homotopic through holomorphic sections. Cartan called this *the Oka-Grauert principle*, see [Car58]. Bearing in mind that a vector bundle is a fiber bundle with affine fiber and structure group $\mathrm{GL}_n(\mathbb{C})$, there are some generalizations proven around the same time, namely for holomorphic fiber bundles with fibers which are complex Lie groups; or even complex homogeneous fibers, whose structure group is also a Lie group with the additional property that it acts transitively by automorphisms on the fiber.

Modern Oka theory begins with the seminal paper of Gromov [Gro89], wherein he achieves a tremendous generalization of these results: he shows, among many other things, that the existence of a so-called dominating spray on the fiber Y of a holomorphic fiber bundle $h : Z \to X$ over a Stein space X suffices to prove that the inclusion $\Gamma_{\mathcal{O}}(X, Z) \hookrightarrow \Gamma_{\mathcal{C}}(X, Z)$ is a weak homotopy equivalence. This notion of ellipticity is very general, and this theorem includes all previously known versions of the Oka-Grauert principle. In contrast to the results of Oka and Grauert (and his contemporaries), for Gromov it is only some properties of the fiber, and not anymore of the structure group and of the transition maps, which are sufficient to prove the principle. We will give a simple overview of these ideas below.

1.5.2 Gromov's ellipticity and Oka manifolds

The main result of Gromov in [Gro89] is the generalization of the Oka-Grauert principle (or h-principle as he calls it) to fiber bundles where in particular the fiber may not be a "linear space". The nonlinear Oka principle of Gromov has important applications: the proof of Forster's conjecture about the optimal dimension in which Stein manifolds of dimension n can be properly embedded, by Eliashberg and Gromov [EG92]; and the solution of the Gromov-Vaserstein problem by B. Ivarsson and F. Kutzschebauch [IK12] about the possibility of factoring a matrix in $SL_m(\mathcal{O}(\mathbb{C}))$ into a product of elementary matrices. To explain at least a simple version of it, we first make some definitions.

Let Y be a complex manifold. A **holomorphic spray** on Y is a holomorphic vector bundle $\pi : E \to Y$, called spray bundle, together with a holomorphic map $s : E \to Y$, called spray map, such that for every $y \in Y$ the zero of the fiber over y, denoted 0_y, is mapped by s to y: $s(0_y) = y$. The spray is called **dominating** if , for all $y \in Y$,

$$d_{0_y} s : T_{0_y} E \to T_y Y$$

maps the fiber E_y (seen as a linear subspace of $T_{0_y} E$) surjectively onto $T_y Y$. The manifold Y is **elliptic**, in this sense of Gromov, if it admits a dominating spray.

A simplified version of Gromov's theorem can be formulated as follows.

Theorem 1.32. *Let $h : Z \to X$ be a holomorphic fiber bundle over a Stein manifold X with elliptic fiber Y. Then the sections satisfy the Oka-Grauert principle, i.e.*

$$\Gamma_{\mathcal{O}}(X, Z) \hookrightarrow \Gamma_{\mathcal{C}}(X, Z)$$

is a weak homotopy equivalence. Moreover, given an $\mathcal{O}(X)$-convex compact $K \subset X$, any continuous section σ which is holomorphic in a neighborhood of K is homotopic to a holomorphic section which is uniformly close to σ on K.

This simplified version nonetheless includes the Oka-Grauert principles discussed previously, since every complex homogeneous manifold is elliptic. Indeed, if a Lie group G (with Lie algebra $\mathfrak{g}$) is acting transitively on Y by automorphisms, then the map $s : Y \times \mathfrak{g} \to Y$ given by

$$s(y, v) = \exp(v)y \in Y$$

(where $\exp : \mathfrak{g} \to G$ is the exponential map of the Lie group), is a dominating spray, hence Y is elliptic.

Proposition 1.33. *Holomorphically flexible Stein manifolds are elliptic. In particular, by Theorem* 1.25*, Stein manifolds with the (V)DP are elliptic.*

Recall that a holomorphically flexible manifold (Section 1.4.1) is defined to be a complex manifold where complete vector fields span the tangent spaces. The proposition follows from the fact that if Y is a complex manifold (Stein is not needed) and $\Theta_1, \ldots, \Theta_N$ span T_yY at some y, then the composition of the flows ϕ_t^j is a dominating spray. Indeed, the map $s : \mathbb{C}^N \times Y \to Y$ given by

$$s(t_1, \ldots, t_N, y) = \phi_{t_N}^N \circ \cdots \circ \phi_{t_1}^1(y)$$

is of full rank at $t = 0$ for any y, so is a dominating spray map from the trivial bundle. It can also be shown that if the manifold is additionally Stein, then a finite collection of complete fields will suffice to generate the tangent space at every point in Y, see e.g. [Kut14, Lemma 25].

Returning to Gromov's theorem, consider the particular case where $h : X \times Y \to Y$ is the trivial fibration. It follows that any continuous map $f_0 : X \to Y$ from a Stein manifold X to an elliptic manifold Y can be homotoped to a holomorphic map f_1. Moreover if f_0 is already holomorphic on a $\mathcal{O}(X)$-compact, then f_1 (in fact, each f_t) can be chosen to be arbitrarily close to f_0 on K. It can also be shown, although we have not included it in our formulation of Gromov's theorem, that if f_0 was already holomorphic on a closed complex subvariety S, then f_1 (in fact, each f_t) can be chosen to equal f_0 on S. This generalizes the Oka-Weil and Cartan extension theorem (Theorems 1.14, 1.15) to maps into a manifold more general than $\mathbb{C}$. The emphasis may therefore now be shifted to those properties of a complex manifold Y that would ensure that, for any Stein manifold X, the inclusion of the space of holomorphic maps from X to Y into that of the continuous maps from X to Y be a weak homotopy equivalence, with these natural additions concerning approximation and interpolation.

Elliptic manifolds therefore form a large class of manifolds that are, loosely speaking, natural targets for holomorphic maps originating on Stein manifolds; they are in some sense "dual" to Stein manifolds, which are the natural sources of holomorphic maps. This was made precise by means of abstract homotopy theory by F. Lárusson [Lár04]. Building upon this and Gromov's results, subsequent work by Forstnerič established the equivalence of over a dozen properties that somehow indicate that a manifold is the target of many holomorphic maps. He introduced in [For09] the formal definition of what here call Oka-Forstnerič manifolds, which we give below.

Let us say that a complex manifold Y enjoys the **Oka property** (or Oka property with approximation) if for every Stein manifold X, every compact $\mathcal{O}(X)$ convex subset K of X, and every continuous map $f_0 : X \to Y$ which is holomorphic near K, there exists a homotopy $f_t : X \to Y$ of continuous maps ($t \in [0, 1]$) such that $f_t|_K$ is holomorphic and uniformly close to f_0, and the map $f_1 : X \to Y$ is holomorphic. Forstnerič gave in [For06] a strikingly simple characterization of these manifolds in terms of a Runge-type approximation property for holomorphic maps from Euclidean spaces. Namely, if Y is a complex manifold such that any holomorphic map from a neighborhood of a compact convex (in the elementary sense!) set $K \subset \mathbb{C}^n$ can be approximated uniformly on K by an entire map, then Y satisfies the Oka property; the analogous conclusion also follows for sections of holomorphic fiber bundles with Stein base X and fiber Y. The hypothesis is called the **Convex approximation property** and is obviously implied by the Oka property. Another equally striking result from [For05] is that the convex approximation property of Y is equivalent to the Oka property of Y with approximation *and extension from complex subvarieties* (the reader should now understand what this means), and implies as well the analogous result for sections of bundles with fiber Y and Stein base X. We define a complex manifold Y to be an **Oka-Forstnerič manifold** if is satisfies these equivalent properties. There are a number of more sophisticated properties (with jet interpolation, parameters, more general stratified bundles, etc) that we will not review and that are also shown to be equivalent to those just mentioned. In particular, *sections of holomorphic fiber bundles with Stein bases and Oka-Forstnerič fibers satisfy the Oka principle with approximation and interpolation.*

In this new terminology, the theorem of Gromov can be reformulated:

Theorem 1.34. *An elliptic manifold satisfies the convex approximation property, and is therefore Oka-Forstnerič manifold.*

Corollary 1.35. *A Stein manifold with the (V)DP is an Oka-Forstnerič manifold.*

According to a theorem of Lárusson [Lár05], a Stein manifold which is Oka-Forstnerič must be elliptic. It is therefore not surprising that the main source of Oka-Forstnerič manifolds are elliptic manifolds. In fact, there are no known examples of Oka-Forstnerič manifolds that are not elliptic.

Chapter 2

An Oka principle for a parametric infinite transitivity property

The contents of this chapter consist essentially of the results of the paper of the same name, authored by F. Kutzschebauch and myself: [KR].

2.1 Summary of results

We have already motivated the problem in the introduction (p. 4). Recall however that while it is elementary that $\mathrm{Aut}(\mathbb{C}^n)$ acts infinitely transitively on $\mathbb{C}^n$, the parametrized situation is challenging: consider the simplest case, where a collection of N distinct points in $\mathbb{C}^2$ is allowed to vary holomorphically (with the varying parameter ranging over $\mathbb{C}$). It is unclear how to construct a family of automorphisms of $\mathbb{C}^2$ depending holomorphically on the parameter and mapping, for each parameter, the given N-tuple into a given fixed "standard" tuple in $\mathbb{C}^2$.

Let X and W be connected complex manifolds (hereafter all manifolds are assumed connected). Let $Y_{X,N}$ be the configuration space of ordered distinct N-tuples of points in X, and consider a holomorphic map $x : W \to Y_{X,N}$, that is, N holomorphic maps $x^j : W \to X$ such that for each $w \in W$, the N points $x^1(w), \ldots, x^N(w)$ are pairwise distinct. Recall from p. 4 that if $\mathrm{Aut}(X)$ acts transitively on X, the parametrized points $x^1, \ldots, x^N$ are called *simultaneously standardizable* if there exists a "parametrized automorphism" $\alpha \in \mathrm{Aut}_W(X)$ with

$$\alpha^w(x^j(w)) = z^j$$

for all $w \in W$ and $j = 1, \ldots, N$, for a fixed tuple $(z^1, \ldots, z^N)$.

We can state the main theorem of this chapter.

Theorem 2.1. *Let W be a Stein manifold and X a Stein manifold with the density property. Let N be a natural number and $x : W \to Y_{X,N}$ be a holomorphic map.*

Then the parametrized points $x^1, \dots, x^N$ are simultaneously standardizable by an automorphism lying in the path-connected component of the identity $(\mathrm{Aut}_W(X))^0$ *of* $\mathrm{Aut}_W(X)$ *if and only if x is null-homotopic.*

Theorem 2.1 is an Oka principle for a strong form of parametric infinite transitivity. Since any map $\mathbb{C}^k \to Y_{\mathbb{C}^n,N}$ is null-homotopic, we recover the result of [KL13], without any restrictions on the dimension of W. Moreover, Theorem 2.1 reduces the problem of simultaneous standardization of parametrized points in $\mathbb{C}^n$ by automorphisms in $\mathrm{Aut}_W(\mathbb{C}^n)$ (not the connected component!) to a purely topological problem as explained in Section 2.5, Corollary 2.17. This is a (slightly different) Oka principle for a strong form of parametric N-transitivity.

We are also able to prove a similar result when X is a manifold with the ω-volume density property under an additional topological assumption.

Theorem 2.2. *Let X be a Stein manifold with a volume form ω which satisfies the ω-volume density property. Assume X has dimension greater than* 1 *and that is contractible. Then similarly, for any natural number N a holomorphically parametrized collection of points $x : W \to Y_{X,N}$ can be simultaneously standardized by a volume-preserving automorphism lying in the path-connected component of the identity* $(\mathrm{Aut}_W(X, \omega))^0$ *of* $\mathrm{Aut}_W(X, \omega)$ *if and only if x is null-homotopic.*

The dimension assumption is obviously necessary, as we have observed in Chapter 1. However we do not know if contractibility can be relaxed for the conclusion to hold, see Section 3.4. It is presently unknown whether a contractible Stein manifold with the volume density property has to be biholomorphic to $\mathbb{C}^n$. It is believed that there are plenty of them not biholomorphic to $\mathbb{C}^n$, but that the tools for distinguishing them biholomorphically from $\mathbb{C}^n$ have yet to be developed: this will be discussed in greater detail in Chapter 3.

In what follows the dependence of an automorphism on a parameter is always understood to be a *holomorphic* dependence as just described. A homotopy connecting two maps f_0 and f_1 between any two complex manifolds $W \to X$ is only assumed to be a *continuous* mapping $f : W \times [0, 1] \to X$. If each f_t is holomorphic, we speak of a *homotopy through holomorphic maps*, and if furthermore the function is $\mathcal{C}^k$ (resp. $\mathcal{C}^\infty$), it is a $\mathcal{C}^k$ (resp. *smooth*) *homotopy* between f_0 and f_1. Finally if the variable t is allowed to vary in a complex disc $D_r \subset \mathbb{C}$ $(r > 1)$, and f is holomorphic, we speak of an *analytic* homotopy.

The structure of this chapter, and therefore of the proof, is as follows. In Section 2.2 we prove a general parametric version of the Andersén-Lempert theorem, and we discuss the approximation of local holomorphic phase flows by volume-preserving automorphisms in the parametric case (which turns out to be more elusive). In Section 2.3 we establish that $Y_{X,N}$ is elliptic in Gromov's sense and hence an Oka-Grauert-Gromov h-principle applies to maps $W \to Y_{X,N}$; this will allow us to use the Andersén-Lempert theorem. Section 2.4 contains the details of the proof of Theorem 2.2, from which Theorem 2.1 also follows. The idea is to define a countable sequence of automorphisms, each of which maps x closer to some constant $\hat{x}$ on a larger set, which converges to the desired standardization. In Section 2.5 we make explicit a homotopy-theoretical point of view, and prove a version of Grauert's Oka principle for principal bundles. Finally we consider two examples which illustrate cases in which the topological obstruction of Corollary 2.17 vanishes, and does not vanish, respectively.

2.2 A parametric Andersén-Lempert theorem

In Section 1.2.2 we discussed the Andersén-Lempert theorem for complex affine space:

Theorem 2.3 (Andersén-Lempert Theorem)**.** *Let $n \geq 2$ and U be an open set in $\mathbb{C}^k \times \mathbb{C}^n$. Let F be a $\mathcal{C}^p$ ($p \geq 0$) isotopy of injective holomorphic maps from U into $\mathbb{C}^k \times \mathbb{C}^n$ of the form*

$$F_t(w, z) = (w, F_t^w(z)), \quad (w, z) \in U, \qquad \text{and } F_0^w = id. \tag{$\star$}$$

Suppose $K \subset U$ is a compact polynomially convex subset of $\mathbb{C}^k \times \mathbb{C}^n$, and assume that $F_t(K)$ is polynomially convex in $\mathbb{C}^k \times \mathbb{C}^n$ for each $t \in [0, 1]$. Then for all $t \in [0, 1]$, F_t can be approximated uniformly on K (in the $\mathcal{C}^p$ norm) by automorphisms $\alpha_t \in \mathrm{Aut}_{\mathbb{C}^k}(\mathbb{C}^n)$; moreover α_t depends smoothly on t, and α_0 can be chosen to be the identity.

As noted in Section 1.4, the main point of the definition of manifolds with the DP is that a generalized Andersén-Lempert theorem holds. In fact, the proof of Theorem 1.19 can be closely followed by carrying a parameter. The only apparent difficulty arises when the density property is used to construct a vector field in the Lie algebra generated by complete fields, for the holomorphic dependence of these new fields on w is not obvious. However Lemma 3.5 in [Var01] shows precisely that if V_w is a vector field on X depending holomorphically on a Stein parameter w, then V_w can be approximated locally uniformly on $W \times X$ by Lie combinations of complete vector fields which depend holomorphically on the parameter. This proves the following parametric version of the Andersén-Lempert theorem in manifolds with the density property.

Theorem 2.4. *Let W be a Stein manifold and X a Stein manifold with the DP. Let $U \subset W \times X$ be an open set and $F_t : U \to W \times X$ be a smooth isotopy of injective holomorphic maps of the form ($\star$). Suppose $K \subset U$ is a compact set such that $F_t(K)$ is $\mathcal{O}(W \times X)$-convex for each $t \in [0, 1]$. Then for all $t \in [0, 1]$, F_t can* *be approximated uniformly on K (with respect to any distance function on X) by automorphisms $\alpha_t \in \mathrm{Aut}_W(X)$ which depend smoothly on t, and moreover we can choose $\alpha_0 = id$.*

Let now X be a complex manifold equipped with a holomorphic volume form ω. We have already discussed in Section 1.4 the non-parametric case. We also know from [FR94] that the approximation cannot be on arbitrary $\mathcal{O}(X)$-compact subsets K of X: there are topological obstructions. In the parametric case, the crucial condition is an extension property, which we now state. Let W be a Stein manifold and denote by $\pi_W : W \times X \to W$ the projection, and for a subset U of $W \times X$, denote the "w-slices" by

$$U_w = (\{w\} \times X) \cap U,$$

and its projection to W by

$$U' = \pi_W(U).$$

Recall from Section 1.3.2 that $\Omega^k(X)$ (resp. $Z^k(X)$) is the space of holomorphic k-forms on X (resp. closed forms). We want to consider holomorphic mappings of the form

$$w \mapsto \beta^w \in \Omega^k(U_w), \quad w \in U' = \pi_W(U). \tag{2.1}$$

For this consider the pullback of the bundle $\Omega^k(X)$ by the projection $\pi_X : W \times X \to X$ and denote this bundle over $W \times X$ by $\Omega^k_W(X)$. Its global sections are forms on $W \times X$ which locally are of type $\sum_I h(w,z)dz_I$. Denote the local sections on $U \subset W \times X$ by $\Omega^k_W(U)$; they correspond to a coherent sheaf on $W \times X$, and we identify a local section $\beta \in \Omega^k_W(U)$ to a holomorphic mapping as in Equation (2.1). We can define parametric vector fields, $\mathrm{VF}_W(X)$, analogously.

Because divergence-free vector fields do not form an analytic subsheaf of $\mathrm{VF}(X)$ (in fact they do not even form an $\mathcal{O}(X)$-module!), the difficulty of proving an obvious analogue of Theorem 2.4 lies in a Runge-type approximation of a locally defined divergence-free vector field by a global divergence-free field.

Definition. *Let $U \subset W \times X$. We say a local section $\Theta \in \mathrm{VF}_W(U, \omega)$, that is, a holomorphic map*

$$w \mapsto \Theta^w \in \mathrm{VF}(U_w, \omega), \quad w \in U'$$

is globally approximable *if there exists a global section $\widehat{\Theta} \in \mathrm{VF}_W(W \times X, \omega)$, that is a holomorphic map*

$$w \mapsto \widehat{\Theta}^w \in \mathrm{VF}(X, \omega), \quad w \in W$$

approximating Θ uniformly on compacts of U.

Next we explain what our sufficient condition is. Assume that X is a Stein manifold with the ω-VDP, and let $U \subset W \times X$ be open. Let $F_t : U \to W \times X$ be a smooth isotopy of injective, volume-preserving holomorphic maps of the form $(\star)$. Consider now the ω-divergence-free vector fields

$$\Theta_t^w = \frac{dF_s^w}{ds}\bigg|_{s=t} \circ (F_t^w)^{-1}.$$

If each Θ_t can be globally approximated in the sense just defined, with smooth dependence on t, then the ω-VDP can be used exactly as in the proof of Theorem 2.4 to show that each F_t can be approximated uniformly on compacts of U by volume-preserving automorphisms $\alpha_t \in \mathrm{Aut}_W(X, \omega)$ which depend smoothly on t, with $\alpha_0 = id$.

We will now give two instances where such a global approximation is possible, both of which will be used below. We fix from now on a distance function d on X.

Proposition 2.5. *Let W and X be Stein manifolds and assume that X has an ω-VDP. Let $f : W \times [0,1] \to X$ be a smooth homotopy through holomorphic maps between f_0 and f_1. If $L \subset W$ is a $\mathcal{O}(W)$-convex compact, then given $\epsilon > 0$ there exists $A_t \in \mathrm{Aut}_W(X, \omega)$, with $A_0 = id$, depending smoothly on t, such that*

$$d(A_t^w \circ f_0(w), f_t(w)) < \epsilon \quad \forall (w,t) \in L \times [0,1].$$

Observe that a similar result holds for maps into X^N and therefore into $Y_{X,N}$ (see the proof of Corollary 2.13 with the notation preceding Lemma 2.8). Furthermore, note that if X has the DP, then the proof below can be considerably simplified to obtain the same result with $A_t \in \mathrm{Aut}_W(X)$ only.

Proof. We claim that there is a suitable neighborhood $U \subset W \times X$ of the graph

$$\Gamma_L(f_0) = \{(w, f_0(w)); w \in L\},$$

with contractible fibers U_w, and on it an isotopy of injective volume-preserving holomorphic maps F of the form $(\star)$ extending the definition of f_t, i.e.

$$F_t^w(f_0(w)) = f_t(w) \quad \forall (w,t) \in U' \times [0,1].$$

Since f can be thought of as a section of the trivial holomorphic fibration $W \times X$, the pullback by f of the normal bundle on $W \times X$ is trivial over L. Hence for each $w \in L$, there is a (contractible) coordinate neighborhood $U_0(w) \subset X$ of $f_0(w)$ with chart

$$\phi_0^w : U_0(w) \to B \subset \mathbb{C}^n$$

mapping $f_0(w)$ to 0, depending holomorphically on w, and such that the restriction of ω to $U_0(w)$ is $(\phi_0^w)^*(\tilde{\omega}_w)$, where $\tilde{\omega}_w$ is some volume form on B:

$$\tilde{\omega}_w(z) = g(z,w)dz_1 \wedge \cdots \wedge dz_n, \quad g \in \mathcal{O}(B \times L).$$

By the Moser trick and compactness of L we may shrink B and $U_0(w)$ in order that for all $w \in L$,

$$\tilde{\omega}_w(z) = g(0,w)dz_1 \wedge \cdots \wedge dz_n \quad \forall z \in B.$$

Note that this z-independent formula for $\tilde{\omega}_w$ holds with respect to a possibly different family of coordinate charts, which we still denote by ϕ_0^w. Again by compactness there exist coordinate neighborhoods $U_0(w), U_{t_1}(w), \ldots, U_1(w)$ of

$$f_0(w), f_{t_1}(w), \ldots, f_1(w)$$

respectively, each of which are equivalent to B with a constant volume form, covering $\{f_t(w); t \in [0,1]\} \subset X$ for each $w \in L$. On $U_0(w)$ we define, for each $t \in [0, \tau_0(w))$ (where $\tau_0(w)$ is such that $f_t(w) \in U_0(w)$ for all $t < \tau_0(w)$), a ω-divergence-free field $\Theta_t^{(0)}(w)$ depending holomorphically on w by pulling back the field on B which is constantly equal to

$$\left.\frac{d}{ds}\right|_{s=t} \phi_0^w \circ f_s(w).$$

Similarly on $U_{t_1}(w)$ there is such a family of fields $\Theta_t^{(1)}(w)$ $(\tau_1'(w) < t < \tau_1(w))$, so by using a suitable smooth cut-off function $\chi^w(t)$ one can further define on $U_0(w) \cup U_{t_1}(w)$ the fields

$$\chi^w(t)\Theta_t^{(0)}(w) + (1-\chi^w(t))\Theta_t^{(1)}(w), \quad t \in [0, \tau_1(w)),$$

which are still divergence-free and restrict to $\frac{d}{ds}\big|_{s=t} f_s(w)$. For fixed w, a small enough neighborhood of $f_0(w)$ will flow entirely inside of $U_0(w) \cup U_{t_1}(w)$ under the flow of the above time-dependent vector field. The claim is proved by repeating this construction until the last intersection with $U_1(w)$: we get a neighborhood U of $\Gamma_L(f_0)$ and the desired isotopy F_t consists of the time-t maps of the flow of the described time-dependent field. Note that U can also be chosen with the property that $F_t(U)$ is Runge for all $t \in [0,1]$: since $F_t(\Gamma_W(f_0)) = \Gamma_W(f_t)$ is an analytic set in $W \times X$, its $\mathcal{O}(W \times X)$-convexity easily follows from the Cartan extension theorem on Stein manifolds; proceed then as in the proof of Lemma 2.2 in [FR93].

We have seen that it suffices to show that each field

$$\Theta_t^w = \left.\frac{dF_s^w}{ds}\right|_{s=t} \circ (F_t^w)^{-1}$$

can be globally approximated with smooth dependence on t. Let $\eta_t \in Z_W^{n-1}(F_t(U))$ be a section of $Z_W^{n-1}(X)$ defined by

$$\eta_t^w = \iota_{\Theta_t^w}\omega \in Z^{n-1}(F_t(U_w)), \quad w \in U'.$$

The set $F_t(U)$ is fiberwise contractible, and in fact there is a contraction of each $F_t^w(U_w)$ depending holomorphically on w, hence Poincaré's lemma gives an explicit section $\beta_t \in \Omega_W^{n-2}(F_t(U))$ satisfying

$$d\beta_t^w = \eta_t^w.$$

By the Runge property and Cartan's theorem A, the forms β_t can be approximated by global sections in this coherent sheaf, i.e., there a holomorphic map $\widehat{\beta}_t : W \to \Omega^{n-2}(X)$ approximating β_t, and by the Cauchy estimates we can even ensure that $d\beta_t$ approximates $d\widehat{\beta}_t$. In fact, it is classical that these approximations can be taken to be smooth on t (see [Bun64]). There is a unique vector field $\widehat{\Theta}_t^w$ given by the duality $\iota_{\widehat{\Theta}_t^w}\omega = d\widehat{\beta}_t^w$ (since ω is non-degenerate), which is then divergence-free. By standard theory of differential equations, it approximates Θ_t^w. $\square$

Proposition 2.6. *Let W be Stein and suppose that X is a contractible Stein manifold with an ω-VDP and of dimension at least two. Suppose that the compact $K \subset W \times X$ has the following form: $K' = \pi_W(K)$ is $\mathcal{O}(W)$-convex, and there is a $\mathcal{O}(W)$-convex compact $L' \subseteq K'$ such that*

$$K = \Gamma_{K'}(g) \cup (L' \times S),$$

where S is a compact $\mathcal{O}(X)$-convex subset of X and $g : W \to X$ is holomorphic. Let $F_t : U \to W \times X$ be an isotopy of injective volume-preserving holomorphic maps of the form ($\star$) defined on a neighborhood U of K which has the property that $U_w = X$ for all $w \in V$ where V is a neighborhood of L'. Then for any $\epsilon > 0$ there exists $A_t \in \mathrm{Aut}_W(X, \omega)$, with $A_0 = id$, depending smoothly on t, such that

$$d(A_t^w(z), F_t^w(z)) < \epsilon \quad \forall (w, z, t) \in K \times [0, 1].$$

Proof. We may assume without loss of generality that U is a neighborhood of K of the form $U = A \cup B$ where A is a fiberwise contractible neighborhood of $\Gamma_{U'}(g)$ and $B = V \times X$, such that that $F_t(U)$ is Runge (because K is easily seen to be $\mathcal{O}(W \times X)$-convex: see then argument in the previous proof) and $F_t(A)$ has contractible fibers for all $t \in [0, 1]$. Define as before

$$\Theta_t^w = \frac{dF_s^w}{ds}\bigg|_{s=t} \circ (F_t^w)^{-1} \ \text{ and } \ \eta_t^w = \iota_{\Theta_t^w}\omega \in Z^{n-1}(F_t(U_w)), \quad w \in U'.$$

By the Poincaré lemma and the contractibility of X and of the fibers of $F_t(A)$, there are local sections $\beta_{A,t} \in \Omega_W^{n-2}(F_t(A))$ and $\beta_{B,t} \in \Omega_W^{n-2}(F_t(B))$ such that $d\beta^w_{A,t} = \eta^w_t$ on $F_t(A)$ and $d\beta^w_{B,t} = \eta^w_t$ on $F_t(B)$. It now suffices to find a single family $\beta_t \in \Omega_W^{n-2}(F_t(U))$ depending smoothly on t and satisfying $d\beta^w_t = \eta^w_t$ for all $w \in U'$: we would then conclude as in the previous proof.

For simplicity fix $t = 0$. The $(n-2)$-form $\beta_A - \beta_B$ is closed on $A \cap B$, a set with contractible fibers, so again $\beta^w_A - \beta^w_B = d\delta^w_{AB}$ for a section $\delta_{AB} \in \Omega_W^{n-3}(A \cap B)$. Consider the covering $\mathcal{U} = \{A, B\}$ of U and let $\check{\mathrm{H}}^1(\mathcal{U}, \Omega_W^{n-3}(U))$ be the first Čech cohomology group of the covering $\mathcal{U}$ with values in the sheaf $\Omega_W^{n-3}(U)$. By Cartan's theorem B, the sheaf $\Omega_W^{n-3}(U)$ is acyclic on A, B and $A \cap B$, so by Leray's theorem (see section 1.3.1)

$$\check{\mathrm{H}}^1(\mathcal{U}, \Omega_W^{n-3}(U)) = \mathrm{H}^1(U, \Omega_W^{n-3}(U)).$$

But again the right-hand side is trivial. The vanishing of the Čech cohomology group yields a splitting

$$\delta^w_{AB} = \delta^w_B - \delta^w_A,$$

where δ_A (resp. δ_B) is a section of $\Omega_W^{n-3}(X)$ on A (resp. B). Now since

$$\beta^w_A + d\delta^w_A = \beta^w_B + d\delta^w_B \quad \forall w \in V,$$

the "glueing" property of the sheaf gives the desired β^w. In the case that $n = 2$, replace the above formula by $\beta^w_A = c + \beta^w_B$, where c is a constant.

When t is allowed to vary smoothly in $[0, 1]$, the form $\delta_{AB,t}$ above can be chosen depending smoothly on t. We can consider the sheaf of smooth maps from $[0, 1]$ into Ω_W^{n-3}, whose cohomology is shown in [Bun64] to vanish (as in Cartan's theorem B), so the argument above carries to this new sheaf and we obtain a smoothly depending family of sections $\beta_t \in \Omega_W^{n-2}(F_t(U))$ as desired. □

2.3 Space of configurations is Oka

Theorem 2.7. *If X is a Stein manifold with the DP or a Stein manifold of dimension greater than one with the ω-VDP, then $Y_{X,N}$ is an Oka-Forstnerič manifold for any N.*

We remark that if X has the ω-VDP and dimension 1, then it must be either $(\mathbb{C}, dz)$ or $(\mathbb{C}^*, z^{-1}dz)$. If $N \geq 4$, or in the case $X = \mathbb{C}^*$, then $X^N \setminus \Delta$ is a projective space with too many hyperplanes removed, and this cannot be Oka-Forstnerič by Theorem 3.1 in [Han14]. The same result shows that if $X = \mathbb{C}$ and $N = 2$ or 3, then $X^N \setminus \Delta$ is indeed Oka-Forstnerič.

Theorem 2.7 will be deduced from the following lemma. First we introduce some more notation. Let $Y = Y_{X,N}$ and define the linear map

$$\oplus : \mathrm{VF}(X) \to \mathrm{VF}(Y)$$

as follows: for each $V \in \mathrm{VF}(X)$ let $\oplus V \in \mathrm{VF}(X^N)$ be the vector field in X^N defined by $\oplus V(z^1, \dots, z^N) = (V(z^1), \dots, V(z^N)) \in T_{(z^1,\dots,z^N)}X^N$. Clearly $\oplus V \in \mathrm{VF}(Y)$, and since in fact this field is tangent to Δ, the image of a point in Y under the flow of $\oplus V$ remains in Y. It is clear that $\oplus$ restricts to a map between complete fields. Similarly, we have an obvious map

$$\oplus : \mathrm{Aut}(X) \to \mathrm{Aut}(Y).$$

Lemma 2.8. *Let X be a Stein manifold with the DP (resp. with the ω-VDP and dimension greater than one) and let $N \geq 1$. Then there exist complete (resp. and divergence-free) vector fields $V_1, \dots, V_m$ on X such that*

$$T_yY = \mathrm{Span}\{\oplus V_j(y)\}_j \quad \forall y \in Y = Y_{X,N}.$$

In particular, the statement for $N = 1$ holds, as we have seen in Section 1.3. We give a proof based on the techniques in [KK11].

Proof. We give the proof for a manifold with the DP and only give indications of the modifications required for the ω-VDP case. Let $x^1, \dots, x^N \in X$ be N pairwise distinct points in X. Since X is Stein we can pick a Runge open set around $\{x^1\} \cup \dots \cup \{x^N\}$ of the form $U = \dot{\bigcup}_{j=1}^N U^j$, so small that a chart $U^j \to \mathbb{C}^n$ exists for each j, where n is the dimension of X (as in the proof of Proposition 2.5 $\omega|_{U^j}$ is the pullback of the standard volume form $\omega_{std} = dz_1 \wedge \dots \wedge dz_n$). By pulling back the coordinate vector fields in $\mathbb{C}^n$ we obtain, for each $j = 1, \dots, N$,

$$V_1^j, \dots, V_n^j \in \mathrm{VF}(U^j) \text{ such that } \mathrm{Span}\{V_i^j(x^j)\}_i = T_{x^j}X.$$

For each fixed j, define n vector fields on U as follows: for $i = 1, \dots, n$, let $\Theta_i^j \in \mathrm{VF}(U)$ be the trivial extension of V_i^j to U, that is, extend it as the zero field outside of U^j. (Note these are are divergence-free in the other case). Consider the vector fields $\oplus\Theta_i^j$ defined on $U^1 \times \dots \times U^N \subset Y$. They span the tangent space to $y_0 = (x^1, \dots, x^N)$:

$$T_{y_0}Y = \mathrm{Span}\{\oplus\Theta_i^j(y_0)\}_{i,j}.$$

Since U is Runge in X, there exists $\eta_i^j \in \mathrm{VF}(X)$ approximating Θ_i^j on $\overline{U}$. Similarly in the volume case, a field $\Theta \in \mathrm{VF}(U, \omega)$ over a Runge open set with $\mathrm{H}_{DR}^{n-1}(U) = 0$ can be approximated by a global field $\eta \in \mathrm{VF}(X, \omega)$, as seen in the discussion in Section 2.2. This implies that $\oplus\eta_i^j$ approximates $\oplus\Theta_i^j$, so we can assume that

$$T_yY = \mathrm{Span}\{\oplus\eta_i^j(y)\}_{i,j}$$

holds for all y in a neighborhood of y_0 in Y. By the density property, we can further approximate each η_i^j by a finite *sum* of complete vector fields $\eta_i^{j,k}$ on X. Indeed, given *complete* fields $V, W \in \mathrm{VF}(X)$, $[V,W] = \lim_{t\to 0^+} \frac{(V^t)^*W - W}{t}$, where V^t is the time-t map of the flow of V; observe that multiplication by $1/t$ and the pullback by a global automorphism preserves the completeness of a field. Let $\eta_k \in \mathrm{VF}(X)$ be the collection of the complete fields just obtained. Then the complete fields $V_k = \oplus\eta_k \in \mathrm{VF}(Y)$ span T_yY for all y in a neighborhood of y_0. Exactly the same holds in presence of the ω-VDP.

We now enlarge this family in order to generate the tangent spaces at any $y \in Y$. Notice that the fields V_k span T_yY on Y minus a proper analytic set A, which we decompose into its (possibly countably many) irreducible components A_i ($i \geq 1$). It suffices to show that there exists $\Psi \in \mathrm{Aut}(X)$ (resp. $\mathrm{Aut}(X,\omega)$) such that $(\oplus\Psi)(Y \setminus A) \cap A_i \neq \emptyset$ for all i. Indeed, this would imply that the family $\{(\oplus\Psi)_*(V_k)\}_k$ of complete vector fields spans $T_{a_i}Y$ (where $a_i \in (\oplus\Psi)(Y \setminus A) \cap A_i$) for each i, so the enlarged finite collection $\{\oplus\Psi_*(V_k)\}_k \cup \{V_k\}_k$ of complete fields would fail to span the tangent space in an exceptional variety of lower dimension. The conclusion follows from the finite iteration of this procedure.

To obtain this automorphism, consider

$$B_i = \{\Psi \in \mathrm{Aut}(X); \oplus\Psi(Y \setminus A) \cap A_i \neq \emptyset\}.$$

Each B_i is clearly an open set. To verify that it is also dense, let $\alpha \in \mathrm{Aut}(X)$ and $y^* \in A_i$. As above, there are finitely many complete fields Θ_k on X such that $\oplus\Theta_k$ span the tangent space of Y at y^*. So there is some complete $V \in \mathrm{VF}(X)$ such that $\oplus V$ is not tangent to A_i. Thus $V^t \circ \alpha$ is an element in B_i for small t (where V^t is the flow of the field V). As noted in Section 1.3, we can apply the Baire category theorem, so there exists $\Psi \in \bigcap B_i$ and we are done. $\square$

The conclusion is obviously false for the ω-VDP and $dim(X) = 1$: the only divergence-free vector fields on $(\mathbb{C}, dz)$ are constant.

We have just showed that there exist finitely many complete vector fields on Y spanning the tangent space everywhere. This provides a dominating spray on Y, and so Y is elliptic, hence Oka-Forstnerič, see Section 1.5. This proves Theorem 2.7.

Let us derive another easy but important consequence of the above lemma, which is in analogy to the result in Section 1.2.1.

Lemma 2.9. *Let X be a Stein manifold with the DP (or ω-VDP and $dim(X) > 1$) and fix a metric d on it. Let $y_0 = (x^1, \dots, x^N) \in Y$, $\epsilon > 0$ and a compact $K \subset X$ containing each x^j be given. Then there is a neighborhood U_ϵ of y_0 in Y with the following property: given a complex manifold W and an analytic homotopy $f : W \times D_r \to Y$ ($r > 1$) satisfying*

$$f_t(W) \subset U_\epsilon \text{ for all } t \in D_r,$$

there exists a holomorphic map $\Psi : D_r \to \mathrm{Aut}_W(X)$ *(or* $\mathrm{Aut}_W(X,\omega)$*) such that* $\Psi_0^w = id_X$*, and such that for all* $(w,t) \in W \times D_r$*,*

1. $d(\Psi_t^w, id) < \epsilon$ *and* $d((\Psi_t^w)^{-1}, id) < \epsilon$ *on* K*;*
2. *and* $(\oplus\Psi_t^w) \circ f_0(w) = f_t(w)$*.*

Proof. By the previous lemma, there are complete (divergence-free) vector fields $V_1, \dots, V_m$ on X such that $\{\oplus V_j(y_0)\}_j$ span $T_{y_0}Y$. By discarding linearly dependent elements of the generating set, we can assume that $m = nN$, where n is the dimension of X. Let ϕ_j be the flow of V_j. By completeness its time-t map, denoted ϕ_j^t, is a (volume-preserving) automorphism of X. Define two holomorphic maps $\phi, \phi_- : \mathbb{C}^m \times X \to X$ by

$$\phi(\mathbf{t}, z) = \phi_1^{t_1} \circ \cdots \circ \phi_m^{t_m}(z) \quad \phi_-(\mathbf{t}, z) = \phi_m^{-t_m} \circ \cdots \circ \phi_1^{-t_1}(z).$$

and consider the holomorphic map $\varphi : \mathbb{C}^m \to \mathrm{Aut}(X)$ (or $\mathrm{Aut}(X,\omega)$) given by

$$\varphi(\mathbf{t}) = \phi(\mathbf{t}, \cdot) : X \to X;$$

define φ_- analogously. By continuity there exists a ball $B_R \subset \mathbb{C}^m$ around $\mathbf{0}$ such that for each $\mathbf{t} \in B_R$,

$$d(\varphi(\mathbf{t}), id) < \epsilon/2 \text{ and } d(\varphi_-(\mathbf{t}), id) < \epsilon/2 \text{ on } K^\epsilon,$$

where K^ϵ is a compact containing the ϵ-envelope $\{x \in X; d(x,K) < \epsilon\}$ of K. Consider now the map $s : \mathbb{C}^m \to Y$ defined by

$$s(\mathbf{t}) = (\phi(\mathbf{t}, x^1), \dots, \phi(\mathbf{t}, x^N)).$$

Then $s(\mathbf{0}) = y_0$ and, for all $j = 1, \dots, m$,

$$\frac{\partial s}{\partial t_j}(\mathbf{0}) = (V_j(x^1), \dots, V_j(x^N)) = \oplus V_j(y_0).$$

Since $\mathrm{Span}\{\oplus V_j(y_0)\}_j = T_{y_0}Y$, by the implicit mapping theorem s is locally biholomorphic on a neighborhood (which we assume contained in B_R) of $\mathbf{0}$ onto a neighborhood U_ϵ of y_0 in Y. Then the holomorphic mapping $\psi = \varphi \circ s^{-1} : U_\epsilon \to \mathrm{Aut}(X)$ (or $\mathrm{Aut}(X,\omega)$) clearly satisfies, for each $y \in U_\epsilon$, $(\oplus\psi(y))(y_0) = y$, and

$$d(\psi(y), id) < \epsilon/2 \text{ and } d(\psi^{-1}(y), id) < \epsilon/2 \text{ on } K^\epsilon.$$

Now set

$$\tilde{\Psi}_t^w(x) = \psi(f_t(w))(x)$$

and define $\Psi : D_r \to \mathrm{Aut}_W(X)$ (or $\mathrm{Aut}(X,\omega)$) by $\Psi_t^w = \tilde{\Psi}_t^w \circ \left(\tilde{\Psi}_0^w\right)^{-1}$. $\square$

We call such a map $\Psi : D_r \to \mathrm{Aut}_W(X)$ satisfying $\Psi_0^w = id$ an *analytic isotopy of parametrized automorphisms*. Let us point out two consequences that will be of use. In the first place, note that an analogous result holds if f_t is a homotopy with t varying smoothly in $[0,1]$ instead of a complex disc: we obtain a *smooth* isotopy of parametrized automorphisms $\Psi : [0,1] \to \mathrm{Aut}_W(X)$ satisfying the same properties. As a second remark, observe that the map s in the above proof is defined independently of ϵ, and hence so is U. Therefore, if W is compact and a single map $f : W \to Y$ satisfies $f(W) \subset U' \subset U$, since $s^{-1}(f(W))$ is compact in $B_{R'} \subset B_R$, for $\eta > 0$ small enough and $r = 1 + \eta$ the function

$$S_t(w) = s(t \cdot (s^{-1} \circ f(w))), \quad (w,t) \in W \times D_r$$

takes values in U' and defines an analytic homotopy between the constant $S_0 = y_0$ and $S_1 = f : W \to Y$. We end this section with a corollary of Theorem 2.7.

Corollary 2.10. *Let W and X be as in Theorem* 2.1 *or* 2.2*. Then any two holomorphic maps $f_0, f_1 : W \to Y_{X,N}$ which are homotopic are smoothly homotopic through holomorphic maps.*

Proof. Let Y be any Oka-Forstnerič manifold. We prove that if $f : W \times [0,1] \to Y$ is a homotopy between two holomorphic maps f_0 and f_1, then they are in fact homotopic via an analytic homotopy, so in particular they are smoothly homotopic through holomorphic maps.

Let $r > 1$ and $R : D_r \to [0,1] \subset \mathbb{C}$ be any continuous retraction of the disc $D_r \subset \mathbb{C}$ onto the interval. Then

$$\begin{aligned} F : W \times D_r &\to Y \\ (w,t) &\mapsto f_{R(t)}(w) \end{aligned}$$

is a continuous map extending f from $W \times [0,1]$ to $W \times D_r$. Now $T = W \times \partial[0,1]$ is a closed complex submanifold of the Stein manifold $S = W \times D_r$. The map F is holomorphic when restricted to $W \times \partial[0,1]$, so according to the Basic Oka Property with interpolation (but no approximation, see Section 1.5.2) it can be deformed to a holomorphic map $H : W \times D_r \to Y$, which equals F on $W \times \partial[0,1]$. $\square$

Note that this proof does not allow to obtain additionally approximation over a $\mathcal{O}(W)$-convex compact piece L. Compare with Corollary 2.13 below.

2.4 Proof of the main theorem

We will prove Theorem 2.2. A similar and simpler proof for Theorem 2.1 can be extracted, by ignoring the complications arising from the preservation of the volume form. We first go through some technicalities to prepare for the proof.

Let X, Y and W be as in Theorem 2.2. Fix from now on a distance function d on X, an N-tuple $\hat{x} = (\hat{x}^1, \dots, \hat{x}^N) \in Y$, a holomorphic map $x_0 = (x^1, \dots, x^N) : W \to Y$, and a homotopy $x : W \times [0,1] \to Y$ between x_0 and $x_1 = \hat{x}$. The metric d induces a natural distance in Y: for $\eta, \zeta \in Y$, let

$$d_Y(\eta, \zeta) = \max_{j=1,\dots,N} d(\eta^j, \zeta^j).$$

We will now construct automorphisms $\alpha_j \in \mathrm{Aut}_W(X, \omega)$ and verify that they converge to an element in $\mathrm{Aut}_W(X, \omega)$. For this we apply the following criterion, which generalizes Theorem 1.8.

Lemma 2.11. *Let X be a Stein manifold with metric d and W be any manifold. Suppose W is exhausted by compact sets L_j ($j \geq 1$), and X by compacts K_j ($j \geq 0$). For each $j \geq 1$, let ϵ_j be a real number such that*

$$0 < \epsilon_j < d(K_{j-1}, X \setminus K_j) \quad \textit{and} \quad \sum \epsilon_j < \infty.$$

For each $j \geq m \geq 1$, let $\alpha_j \in \mathrm{Aut}_W(X)$, and let $\beta^w_{j,m} \in \mathrm{Aut}(X)$ be defined by

$$\beta^w_{j,m} = \alpha^w_j \circ \cdots \circ \alpha^w_m.$$

Assume that for each $w \in L_j \setminus L_{j-1}$ (take $L_0 = \emptyset$),

$$d(\alpha^w_j, id) < \epsilon_j \qquad \textit{on } K_j \tag{2.2}$$

$$d(\alpha^w_{j+s}, id) < \epsilon_{j+s} \qquad \textit{on } K_{j+s} \cup \beta^w_{j+s-1,j}(K_{j+s}) \quad \forall s \geq 1. \tag{2.3}$$

Then $\beta = \lim_{m\to\infty} \beta_{m,1}$ exists uniformly on compacts and defines an element in $\mathrm{Aut}_W(X)$, or in $\mathrm{Aut}_W(X, \omega)$ if each $\alpha_j \in \mathrm{Aut}(X, \omega)$.

Proof. Let $w \in L_1$. The remark which is the content of [Rit13, Prop. 1] shows that if Equation 2.2 holds for all j, then the limit β^w is injective holomorphic map onto X defined on the set which consists exactly of the points z in X such that the sequence $\left\{\beta^w_{m,1}(z); m \in \mathbb{N}\right\}$ is bounded. If we assume furthermore that

$$d(\alpha^w_s, id) < \epsilon_s \text{ on } K_s \cup \beta^w_{s-1,1}(K_s) \quad \forall s \geq 2,$$

which is Equation 2.3 for $j = 1$, we can ensure that the set of convergence for β^w is X. Hence $\{\beta^w_{m,1}\}_m$ converges to an automorphism of X if $w \in L_1$. For $w \in L_j \setminus L_{j-1}$ and $j \geq 2$, the same reasoning shows that $\lim_{m\to\infty} \beta^w_{m+j,j}$ is an automorphism and we obtain $\beta^w \in \mathrm{Aut}(X)$ by precomposing it with the automorphism $\beta^w_{j-1,1}$. It is clear from the construction that β depends holomorphically on w, since the convergence is uniform on compacts. ☐

In practice we will construct the automorphisms α_j for $j \geq 1$ inductively. Observe that when defining α_j, there are only j constraints to satisfy: $d(\alpha_j^w, id) < \epsilon_j$ should hold

- on K_j if $w \in L_j \setminus L_{j-1}$, according to Equation 2.2;
- on $K_j \cup \beta^w_{j-1,m}(K_j)$ if $w \in L_m \setminus L_{m-1} (1 \leq m \leq j-1)$, according to Equation 2.3.

By Corollary 2.10, we can assume that x_0 and $\hat{x}$ are smoothly homotopic through holomorphic maps $x_t : W \to Y$, so by Proposition 2.5 (see remarks preceding its proof) we could obtain $\alpha \in \mathrm{Aut}_W(X, \omega)$ mapping x_0 close to $\hat{x}$ over some $L \subset W$. Over L we have a "small homotopy" which sends $\Gamma_L(\oplus\alpha \circ x_0)$ to $\hat{x}$ and on the rest of W some homotopy is given. So Proposition 2.5 should instead be applied to some motion coming from the "glueing" of these homotopies, whose holomorphic dependence on w relies on the Oka property. Its existence follows from this technical lemma.

Lemma 2.12. *Let L be a $\mathcal{O}(W)$-convex compact set, and $f_t : W \to Y$ be a smooth homotopy between some holomorphic map f_0 and the constant $f_1 = \hat{x}$. Then there exists an $\varepsilon > 0$ depending on f and L with the following property: for every $\varepsilon' \leq \varepsilon$, every smooth $F : W \times [0,1] \to Y$ with $F_t = f_{2t-1}$ for $t \geq 1/2$ satisfying*

$$d_Y(F_t(w), F_{1-t}(w)) < \varepsilon'/2 \quad \forall (w,t) \in L \times [0,1], \tag{2.4}$$

and every $\mathcal{O}(W)$-convex compact $L^- \subset \mathrm{int}(L)$, there exists an analytic homotopy $H : W \times D_r \to Y$ between F_0 and $\hat{x}$ such that

$$d_Y(H_t(w), \hat{x}) < \varepsilon' \quad \forall (w,t) \in L^- \times D_r.$$

Proof. The injectivity radius for the metric d_Y is bounded from below by a positive constant on the compact $f(L \times [0,1])$. We let ε be the minimum of this constant and of the radius (in the metric d_Y) of the open set U mentioned in the second remark following Lemma 2.9. Fix $\varepsilon' \leq \varepsilon$ and let $F : W \times [0,1] :\to Y$ be as above. Then $F_0(L) \subset Y$ lies in a certain $B_{\varepsilon'/2} \subset U$, so according to that remark there is an analytic homotopy $S : L \times D_R \to Y$ between $S_0 = F_0$ and $S_1 = \hat{x}$ satisfying

$$d_Y(S_t(w), \hat{x}) < \varepsilon'/2 \quad \forall (w,t) \in L \times D_R.$$

Denote by σ the restriction of S to $L \times [0,1]$. We claim that there is a continuous $\mathbf{h} : L \times [0,1]_s \times [0,1]_t \to Y$ such that

$$\begin{array}{ll} \mathbf{h}(w,0,t) = F_t(w) & \mathbf{h}(w,s,0) = \sigma_0(w) \\ \mathbf{h}(w,1,t) = \sigma_t(w) & \mathbf{h}(w,s,1) = \sigma_1(w). \end{array}$$

Consider $w \in L$ fixed. By the definition of ε, for each $s \in [0,1]$ there is a unique geodesic path in Y from $F_s(w)$ to $F_{1-s}(w)$. By following it at constant speed, the parametrization $\gamma_s^w : [0,1]_t \to Y$ is uniquely determined. Let $h^w : [0,1]_s \times [0,1]_t \to Y$ be defined by

$$h_s^w(t) = \begin{cases} F_t(w) & \text{if } 0 \le t \le s/2 \\ \gamma_{s/2}^w(l(t)) & \text{if } s/2 \le t \le 1 - s/2 \\ F_t(w) & \text{if } 1 - s/2 \le t \le 1, \end{cases}$$

where l is the linear function of t taking values 0 at $s/2$ and 1 at $1-s/2$. This is a well-defined homotopy between F and the geodesic segment h_0^w going from $F_0(w)$ to the constant $\hat{x}$; it is uniquely defined for each w. By letting w vary in L, all the elements in the definition of $h_s(t)$ vary continuously, so $h : L \times [0,1] \times [0,1] \to Y$ provides a homotopy of homotopies between F and the geodesic segment h_0. Now it suffices to connect $\sigma : L \times [0,1] \to Y$ to $h_0 : L \times [0,1] \to Y$ and compose; this is achieved in a similar way, and the claim is proved.

Let $\rho : D_R \times [0,1] \to D_R$ be a homotopy between the identity ρ_0 and a continuous retraction $\rho_1 : D_R \to [0,1]$, and extend $\mathbf{h}$ to $L \times [0,1]_s \times D_R$ by defining

$$\mathbf{H}(w,s,t) = \begin{cases} \mathbf{h}(w, 2s, \rho_1(t)) & \text{if } 0 \le s \le 1/2 \\ S(w, \rho_{2-2s}(t)) & \text{if } 1/2 \le s \le 1. \end{cases}$$

Let U be a neighborhood of L^- such that $\overline{U} \subset \operatorname{int}(L)$. Then there exists a smooth function $\chi : W \to [0,1]$ such that $\chi|_U = 1$ and $\chi|_{W \setminus L} = 0$. Define $\tilde{H} : W \times D_R \to Y$ by

$$\tilde{H}(w,t) = \begin{cases} \mathbf{H}(w, \chi(w), t) & \text{if } w \in L, \\ F_{\rho_1(t)}(w) & \text{if } w \notin L. \end{cases}$$

Consider the inclusion of the closed complex submanifold $T = W \times \partial[0,1]$ into the Stein manifold $\mathfrak{S} = W \times D_R$. The map $\tilde{H}$ is continuous on $\mathfrak{S}$, restricts to the holomorphic maps σ_0 and σ_1 on T, and is equal to the holomorphic S on a neighborhood of the $\mathcal{O}(\mathfrak{S})$-convex set $L^- \times \overline{D_r}$ (for some $1 < r < R$). By the Oka Property (see Section 1.5.2), there is a holomorphic map $H : \mathfrak{S} \to Y$ which restricts to $\tilde{H}$ on T and approximates $\tilde{H}$ on $L^- \times D_r$. □

Let us illustrate how we will use this lemma.

Corollary 2.13. *Let $f : W \times [0,1] \to Y$ be a smooth homotopy through holomorphic maps connecting f_0 to some constant $f_1 = \hat{x}$. Given a $\mathcal{O}(W)$-convex compact L and $\epsilon > 0$ small enough, there exists $\alpha \in \operatorname{Aut}_W(X, \omega)$ and a smooth homotopy through holomorphic maps $h : W \times [0,1] \to Y$ with $h_0 = \oplus\alpha \circ f_0$, $h_1 = \hat{x}$, and*

$$d_Y(h_t(w), \hat{x}) < \epsilon \quad \forall (w,t) \in L \times [0,1].$$

Proof. Pick a $\mathcal{O}(W)$-convex compact L^+ such that $L \subset \text{int}(L^+)$ and let $0 < \epsilon < \varepsilon(f, L^+)$ where ε is as in the lemma. By Proposition 2.5, there is $A_t \in \text{Aut}_W(X, \omega)$ depending smoothly on t such that $A_0 = id$ and

$$d_Y(\oplus A_t^w \circ f_0(w), f_t(w)) < \epsilon/2 \quad \forall (w,t) \in L^+ \times [0,1].$$

Let $\alpha = A_1$. Define $F : W \times [0,1] \to Y$ by

$$F_t(w) = \begin{cases} \oplus A_{1-2t}^w \circ f_0(w) & \text{if } t \leq 1/2 \\ f_{2t-1}(w) & \text{if } t \geq 1/2. \end{cases}$$

This is a smooth homotopy between the holomorphic map $\oplus\alpha \circ f_0$ and $\hat{x}$. By the above inequality F_t satisfies Equation 2.4, so the lemma yields the desired homotopy by restricting H to $[0,1]$. □

We now prove the main technical tool, which roughly said allows us to iterate the approximations over a growing sequence of compacts in W.

Proposition 2.14. *Let $\eta > 0$ and K a compact in X containing each $\hat{x}^j$ be given. Then there exists a real number $\delta(K, \eta) > 0$ with the following property. If $h : W \times [0,1] \to Y$ is a smooth homotopy through holomorphic maps, with $h_1 = \hat{x}$ and approximation*

$$d_Y(h_t(w), \hat{x}) < \delta(K, \eta) \quad \forall (w,t) \in L_1 \times [0,1],$$

where $L_1 \subset W$ is a $\mathcal{O}(W)$-convex compact, then:
(a) There exists a smooth isotopy of parametrized automorphisms $\Psi : [0,1] \to \text{Aut}_{L_1}(X, \omega)$, such that for all $(w,t) \in L_1 \times [0,1]$,

$$\begin{aligned} \oplus\Psi_t^w \circ h_0(w) &= h_t(w), \\ d(\Psi_t^w, id) &< \eta \text{ on } K. \end{aligned} \tag{2.5}$$

(b) Given $\epsilon > 0$, L_2 a $\mathcal{O}(W)$-convex compact containing L_1, and a $\mathcal{O}(X)$-convex compact C, there exists a smooth isotopy $A_t \in \text{Aut}_W(X, \omega)$ with $A_0 = id$ such that

$$d(A_t^w(z), \Psi_t^w(z)) < \eta \quad \forall (w,z,t) \in L_1 \times C \times [0,1] \tag{2.6}$$

and

$$d_Y(\oplus A_t^w \circ h_0(w), h_t(w)) < \epsilon \text{ on } L_2 \times [0,1].$$

Proof. (a) The existence of $\delta(K, \eta)$ and the volume-preserving Ψ_t with these properties follows immediately from the first remark following Lemma 2.9.
(b) Define a time-dependent vector field on $L_1 \times X$ by

$$\Theta_s^w(x) = \left.\frac{d}{dt}\right|_{t=s} \Psi_t^w\left((\Psi_s^w)^{-1}(x)\right).$$

It satisfies, for each $j = 1, \dots, N$ and $s \in [0,1]$,

$$\Theta_s^w(h_s^j(w)) = \frac{d}{dt}\bigg|_{t=s} h_t^j(w),$$

which implies that $\Theta_s^w(x)$ is a vector field on $(L_1 \times X) \cup \Gamma_W(h_s)$. We will show that, for each s, this field can be extended to a neighborhood of $(L_1 \times X) \cup \Gamma_W(h_s)$ with approximation on $L_1 \times C$.

There is a smooth isotopy of parametrized automorphisms $\beta_t \in \mathrm{Aut}_{L_2}(X, \omega)$ such that $\beta_1 = id$ and $(\oplus\beta_t) \circ h_t = \hat{x}$. Indeed, Proposition 2.5 applied to h_t provides $\tilde{B}_t \in \mathrm{Aut}_W(X, \omega)$, depending smoothly on t, with the property that $B_t = \tilde{B}_1 \circ \tilde{B}_t^{-1}$ maps $\Gamma_{L_2}(h_t)$ arbitrarily close to $\Gamma_{L_2}(\hat{x})$. Hence Lemma 2.9 applied to $\oplus B_t \circ h_t : W \to Y$ gives elements $\Phi_t \in \mathrm{Aut}_{L_2}(X, \omega)$, depending smoothly on t, such that

$$\oplus(\Phi_t^w \circ B_t^w) \circ h_t(w) = \oplus B_0^w \circ h_0(w) \quad \forall w \in L_2.$$

Then ${\Phi_1}^{-1} \circ \Phi_t \circ B_t \in \mathrm{Aut}_{L_2}(X, \omega)$ is the desired β_t.
The pushforwards $(\beta_t)_*(\Theta_t)$ define together a divergence-free time-dependent vector field on $(L_1 \times X) \cup \Gamma_{L_2}(\hat{x})$. Just as in the proof of Proposition 2.5, this can be extended from the analytic subvariety $\Gamma_{L_2}(\hat{x})$ to a neighborhood of it, and moreover it is a classical result of E. Bishop [Bis62] following from Cartan's theorem A and B that this can be done with smooth dependence on the t parameter and with arbitrary approximation on a large $\mathcal{O}(W \times X)$ compact of the form $L_1 \times \tilde{K}$, where $\tilde{K}$ contains

$$\beta_{[0,1]}^{L_1}(C) = \{\beta_t^w(x); w \in L_1, x \in C, t \in [0,1]\} \subset X. \tag{2.7}$$

Its pullback is an approximate extension of the time-dependent vector field Θ above, whose flow provides an isotopy of injective volume-preserving holomorphic maps $F_t : \Omega \to W \times X$, where Ω is a neighborhood of $\Gamma_{L_2}(h_0)$ containing $L_1 \times X$, and such that

$$d(F_t^w(z), \Psi_t^w(z)) < \eta/2 \quad \forall (w, z, t) \in L_1 \times C \times [0,1] \tag{2.8}$$

$$\oplus F_t^w \circ h_0(w) = h_t(w) \quad \forall (w, t) \in L_2 \times [0,1]. \tag{2.9}$$

Observe that in fact Ψ^w is defined for w in a neighborhood of L_1, so we may apply Proposition 2.6. We obtain $A_t \in \mathrm{Aut}_W(X, \omega)$ such that

$$d(A_t^w(z), F_t^w(z)) < \min(\epsilon, \eta/2)$$

on $(L_1 \times C) \cup \Gamma_{L_2}(h_0)$. This and (2.8) show that (2.6) holds. Furthermore, by (2.9),

$$d_Y(\oplus A_t^w \circ h_0(w), h_t(w)) < \epsilon \quad \forall (w, t) \in L_2 \times [0,1]. \qquad \square$$

Proof. (of Theorem 2.2) The "only if" part follows from the definition of the path connected component; we have to prove the "if" part. Fix a compact exhaustion of $W \times X$, of the form $W = \bigcup_{j=1}^{\infty} L_j$ and $X = \bigcup_{j=0}^{\infty} K_j$, where each L_j (resp. K_j) is a $\mathcal{O}(W)$-convex (resp. $\mathcal{O}(X)$-convex) compact set, and such that $L_j \subset \operatorname{int}(L_{j+1})$. Fix also real numbers ϵ_j ($j \geq 1$) such that $0 < \epsilon_j < d(K_{j-1}, X \setminus K_j)$ and $\sum \epsilon_j < \infty$. We can suppose that K_0 contains $\hat{x}^j$ for all $j = 1, \dots, N$.

By Corollary 2.10, x_0 and $\hat{x}$ are smoothly homotopic through holomorphic maps. Hence Corollary 2.13 gives $\alpha_0 \in \operatorname{Aut}_W(X, \omega)$ and a smooth homotopy of holomorphic maps $h : W \times [0,1] \to Y$ between $h_0 = \oplus\alpha_0 \circ x_0$ and $h_1 = \hat{x}$ with

$$d_Y(h_t(w), \hat{x}) < \delta(K_1, \epsilon_1/2) \quad \forall (w,t) \in L_1 \times [0,1],$$

where $\delta > 0$ is as in Proposition 2.14. Apply part (a) of it to h_t: we obtain some $\Psi : [0,1] \to \operatorname{Aut}_{L_1}(X, \omega)$. Consider the compact

$$\Psi_{[0,1]}^{L_1}(K_2)$$

(recall the notation from Equation 2.7) and define C_1 to be a $\mathcal{O}(X)$-convex compact containing its $(\epsilon_1/2)$-envelope. By part (b) of Proposition 2.14, we obtain a smooth isotopy of automorphisms $A_t \in \operatorname{Aut}_W(X, \omega)$ with $A_0 = id$ such that

$$d_Y(\oplus A_t^w \circ h_0(w), h_t(w)) < \min(\epsilon_1, \delta(C_1, \epsilon_2/2), \varepsilon(h, L_3))/2 \quad \forall (w,t) \in L_3 \times [0,1],$$

where ε is as in Lemma 2.12. Combining (2.5) and (2.6) shows

$$d(A_t^w(z), z) < \epsilon_1 \quad \forall (w, z, t) \in L_1 \times K_1 \times [0,1].$$

We let $\alpha_1 = A_1$. Then in particular

$$d(\alpha_1^w, id) < \epsilon_1 \text{ on } L_1 \times K_1.$$

Thus α_1 satisfies the only condition imposed by (the remark following) Lemma 2.11. Observe finally that by Equation 2.6, $\alpha_1^w(K_2) \subset C_1$ for $w \in L_1$.

We now construct inductively α_j for $j \geq 2$. Fix $k \geq 1$ and assume that we have defined $C_j \subset X$ and $\alpha_j \in \operatorname{Aut}_W(X, \omega)$, for all $1 \leq j \leq k$, such that the following conditions hold (recall that $\beta_{j,m}^w = \alpha_j^w \circ \cdots \circ \alpha_m^w$ and $\beta_j = \beta_{j,0}$):

(a) α_j is smoothly isotopic to the identity through some $A_t \in \operatorname{Aut}_W(X, \omega)$;

(b) $d_Y(\oplus A_t^w \circ h_0(w), h_t(w)) < \min(\epsilon_j, \delta(C_j, \epsilon_{j+1}/2), \varepsilon(h, L_{j+2}))/2$ for all $(w,t) \in L_{j+2} \times [0,1]$, where $h : W \times [0,1] \to Y$ is a smooth homotopy between $\oplus\beta_{j-1} \circ x_0$ and $\hat{x}$;

(c) C_j contains K_{j+1}, and $\left\{\beta_{j,m}^w(K_{j+1}); w \in L_m \setminus L_{m-1}\right\}$ for every $1 \leq m \leq j$;

(d) and every A_t^w satisfies the j conditions of Lemma 2.11, that is, for every $1 \leq m \leq j$, if $w \in L_m \setminus L_{m-1}$, then $d(A_t^w, id) < \epsilon_j$ on $K_j \cup \beta_{j-1,m}^w(K_j)$.

We have just verified that these conditions hold for $k = j = 1$. Let $j \geq 1$. It suffices to show that α_{j+1} and C_{j+1} can be constructed satisfying the above conditions: indeed, by condition (d), Lemma 2.11 would imply that $\beta = \lim_{j\to\infty} \beta_{j,1} \in \mathrm{Aut}_W(X, \omega)$ exists, and by construction (since $\epsilon_j \to 0$) $\oplus\beta$ maps $\alpha_0 \circ x_0$ to $\hat{x}$, so $\beta \circ \alpha_0 \in \mathrm{Aut}_W(X, \omega)$ would be the simultaneous standardization. Further, by conditions (a) and (d), $\beta \circ \alpha_0$ lies in $(\mathrm{Aut}_W(X, \omega))^0$.

So let A and h be as in conditions (a) and (b) at step j. By the inequality in condition (b), and since $L_{j+1} \subset \mathrm{int}(L_{j+2})$, we can apply Lemma 2.12 to

$$F_t(w) = \begin{cases} \oplus A_{1-2t}^w \circ h_0(w) & \text{if } t \leq 1/2 \\ h_{2t-1}(w) & \text{if } t \geq 1/2. \end{cases}$$

We obtain a smooth homotopy through holomorphic maps $H : W \times [0, 1] \to Y$, such that $H_0 = \oplus\beta_j \circ x_0$ and $H_1 = \hat{x}$ and for all $t \in [0, 1]$,

$$d_Y(H_t(w), \hat{x}) < \delta(C_j, \epsilon_{j+1}/2) \quad \forall w \in L_{j+1}.$$

By the first part of Proposition 2.14 there is a smooth isotopy

$$\Psi : [0, 1] \to \mathrm{Aut}_{L_{j+1}}(X, \omega)$$

with $\oplus\Psi_t^w \circ H_0(w) = H_t(w)$ and

$$d(\Psi_t^w, id) < \epsilon_{j+1}/2 \text{ on } L_{j+1} \times C_j. \tag{2.10}$$

Define C_{j+1} to be a $\mathcal{O}(X)$-convex compact containing the $(\epsilon_{j+1}/2)$-envelope of

$$C_j \cup \Psi_{[0,1]}^{L_{j+1}}(K_{j+2}) \cup \bigcup_{1\leq m\leq j} \Psi_{[0,1]}^{L_m}(\beta_{j,m}(K_{j+2})). \tag{2.11}$$

By the second part of Proposition 2.14, there are $A_t^w \in \mathrm{Aut}_W(X, \omega)$ smoothly depending on t and with $A_0 = id$ such that

$$d(A_t^w, \Psi_t^w) < \epsilon_{j+1}/2 \text{ on } L_{j+1} \times C_{j+1} \times [0, 1] \tag{2.12}$$

$$d_Y(\oplus A_t^w \circ H_0(w), H_t(w)) < \min(\epsilon_{j+1}, \delta(C_{j+1}, \epsilon_{j+2}/2), \varepsilon(H, L_{j+3}))/2 \text{ on } L_{j+3} \times [0, 1]. \tag{2.13}$$

Define $\alpha_{j+1} = A_1$, so condition (a) of the induction is met at step $j + 1$. Equation 2.13 means that condition (b) is also satisfied.

Let us check that condition (d) holds at step $j+1$. Note that by (2.12) and (2.10)

$$d(A_t^w, id) < d(A_t^w, \Psi_t^w) + d(\Psi_t^w, id) < \epsilon_{j+1} \text{ on } L_{j+1} \times C_j.$$

By condition (c), C_j contains $K_{j+1} \cup \beta_{j,m}^w(K_{j+1})$ for any $w \in L_m \setminus L_{m-1}$, where $1 \leq m \leq j+1$, so $d(A_t^w, id) < \epsilon_{j+1}$ on K_{j+1}.

It remains to show that C_{j+1} satisfies condition (c). Since $\Psi_0 = id$, it contains K_{j+2}. Let $1 \leq m \leq j+1$, $w \in L_m \setminus L_{m-1}$ and $z \in K_{j+2}$. By the definition of C_{j+1}, it suffices to check that

$$d(\beta_{j+1,m}^w(z), z') < \epsilon_{j+1}/2 \tag{2.14}$$

where z' is some element of the compact (2.11). If $m = j+1$, pick $z' = \Psi_1^w(z)$. Then (2.14) follows from (2.12). If $m < j+1$, let $z' = \Psi_1^w(\beta_{j,m}^w(z))$, which belongs to (2.11). Then

$$d(\beta_{j+1,m}^w(z), z') = d(\alpha_{j+1}(\beta_{j,m}^w(z)), \Psi_1^w(\beta_{j,m}^w(z))) < \epsilon_{j+1}/2$$

where the inequality again follows from (2.12), since $\beta_{j,m}^w(z) \in C_{j+1}$. The induction is complete. □

2.5 Examples and homotopical viewpoint

In this section we change slightly our point of view. With W and X as before, we consider $\mathrm{Hol}(W, Y_{X,N})$, the space of N parametrized points in X. We identify the group $\mathrm{Aut}_W(X)$ with the group of holomorphic mappings from W to $\mathrm{Aut}(X)$, which we denote by $G = \mathrm{Hol}(W, \mathrm{Aut}(X))$. We naturally get an identification between G_0, the path-connected component of the identity in G, with $(\mathrm{Aut}_W(X))^0$. The group G acts on the space $\mathrm{Hol}(W, Y_{X,N})$ by

$$(\alpha \cdot x)(w) = (\oplus \alpha(w)) \circ x(w)$$

where $x = (x^1, \ldots, x^N) \in \mathrm{Hol}(W, Y_{X,N})$ as before. It also acts on the space of homotopy classes (or path-connected components), which we denote here by $[\mathrm{Hol}(W, Y_{X,N})]$. Since the path-connected component G_0 of the identity in G acts trivially, we get an action of G/G_0, the space of homotopy classes $[\mathrm{Hol}(W, \mathrm{Aut}(X))]$ of holomorphic maps from W to $\mathrm{Aut}(X)$, on $[\mathrm{Hol}(W, Y_{X,N})]$. Then an immediate consequence of Theorem 2.1 can be phrased as follows.

Corollary 2.15. *Any $x \in \mathrm{Hol}(W, Y_{X,N})$ is simultaneously standardizable if and only if G/G_0 acts transitively on $[\mathrm{Hol}(W, Y_{X,N})]$.*

By Theorem 2.7, $Y_{X,N}$ is an Oka-Forstnerič manifold. Hence the Oka principle, or *weak homotopy equivalence principle* (see Section 1.5.2) applies: $[\mathrm{Hol}(W, Y_{X,N})]$ is isomorphic to the space of homotopy classes $[\mathrm{Cont}(W, Y_{X,N})]$ of continuous maps from W to $Y_{X,N}$. Thus we deduce:

Corollary 2.16. *Any $x \in \mathrm{Hol}(W, Y_{X,N})$ is simultaneously standardizable if and only if G/G_0 acts transitively on $[\mathrm{Cont}(W, Y_{X,N})]$.*

Let us consider the special case $X = \mathbb{C}^n$, $n > 1$. The group of holomorphic automorphisms $\mathrm{Aut}(\mathbb{C}^n)$ admits a strong deformation retract onto $\mathrm{GL}_n(\mathbb{C})$. Therefore

$$[\mathrm{Hol}(W, \mathrm{Aut}(\mathbb{C}^n))] \cong [\mathrm{Hol}(W, \mathrm{GL}_n(\mathbb{C}))] \tag{2.15}$$

as well as

$$[\mathrm{Cont}(W, \mathrm{Aut}(\mathbb{C}^n))] \cong [\mathrm{Cont}(W, \mathrm{GL}_n(\mathbb{C}))].$$

By the Oka principle (since $\mathrm{GL}_n(\mathbb{C})$ is Oka-Forstnerič),

$$[\mathrm{Hol}(W, \mathrm{GL}_n(\mathbb{C}))] \cong [\mathrm{Cont}(W, \mathrm{GL}_n(\mathbb{C}))]. \tag{2.16}$$

As a consequence, the following purely topological characterization of simultaneous standardization can be deduced from our main theorem.

Corollary 2.17. *Any $x \in \mathrm{Hol}(W, Y_{\mathbb{C}^n,N})$ is simultaneously standardizable if and only if $[\mathrm{Cont}(W, \mathrm{GL}_n(\mathbb{C}))]$ acts transitively on $[\mathrm{Cont}(W, Y_{\mathbb{C}^n,N})]$.*

We also see from Equations 2.15 to 2.16 that

$$[\mathrm{Cont}(W, \mathrm{Aut}(\mathbb{C}^n))] \cong [\mathrm{Hol}(W, \mathrm{Aut}(\mathbb{C}^n))], \tag{2.17}$$

which is a partial Oka principle of the infinite-dimensional manifold $\mathrm{Aut}(\mathbb{C}^n)$. We can ask the following question: is it true that for any Stein manifold X with the density property, we have that

$$[\mathrm{Cont}(W, \mathrm{Aut}(X))] \cong [\mathrm{Hol}(W, \mathrm{Aut}(X))]?$$

Continuing with the case $X = \mathbb{C}^n$, we give another interpretation of our results which is a generalization of Grauert's Oka principle to principal bundles for certain infinite-dimensional subgroups of $\mathrm{Aut}(\mathbb{C}^n)$, in the spirit of Section 1.5.1. First note that simultaneous standardization is the same as lifting the map in the following diagram, where $(z^1, z^2, \ldots, z^N)$ is a fixed N-tuple of points in $X = \mathbb{C}^n$:

$$\begin{tikzcd} & \mathrm{Aut}(\mathbb{C}^n) \arrow[d] & \alpha \arrow[d, mapsto] \\ W \arrow[ur] \arrow[r, "x"] & Y_{\mathbb{C}^n,N} & (\alpha(z^1), \alpha(z^2,) \ldots, \alpha(z^N)) \end{tikzcd}$$

Since $Y_{\mathbb{C}^n,N}$ is homogeneous under $G = \mathrm{Aut}(\mathbb{C}^n)$, we can write it as $Y_{\mathbb{C}^n,N} = G/H_{n,N}$, where $H_{n,N}$ is the (isotropy) subgroup of $G = \mathrm{Aut}(\mathbb{C}^n)$ fixing the N-tuple $(z^1, z^2, \ldots, z^N)$ of points in $\mathbb{C}^n$. The above diagram in this notation becomes

$$\begin{tikzcd} & G \arrow[d, "\pi"] \\ W \arrow[ur] \arrow[r, "x"] & G/H_{n,N} \end{tikzcd}$$

where $\pi : G \to G/H_{n,N}$ is the natural $H_{n,N}$-principal bundle. The existence of a holomorphic (resp. continuous) lift in this diagram is equivalent to the fact that the pullback bundle P_x with projection $x^*(\pi) : x^*(G) \to W$ (which is an $H_{n,N}$-principal bundle over W) is holomorphically (resp. topologically) trivial. Suppose the bundle P_x is topologically trivial: then there exists $\alpha_{cont} : W \to \mathrm{Aut}(\mathbb{C}^n)$ lifting x. By Equation (2.17) there is a holomorphic map $\alpha_{hol} : W \to \mathrm{Aut}(\mathbb{C}^n)$ homotopic to α_{cont}. It follows that $\alpha_{hol}^{-1} \circ x : W \to Y_{\mathbb{C}^n,N} = G/H_{n,N}$, defined by

$$w \mapsto \left((\alpha_{hol}^w)^{-1} \circ x^1(w), \ldots, (\alpha_{hol}^w)^{-1} \circ x^N(w)\right),$$

is null-homotopic, and therefore lifts by Theorem 2.1. This shows that x lifts holomorphically, i.e., the bundle P_x is holomorphically trivial. We have then proven following version of Grauert's Oka principle for principal bundles under the groups $H_{n,N}$:

Corollary 2.18. *For any holomorphic map $x : W \to Y_{\mathbb{C}^n,N}$ from any Stein manifold W, the $H_{n,N}$-principal bundle P_x, which is the pullback by x of the canonical $H_{n,N}$-principal bundle $\pi : \mathrm{Aut}(\mathbb{C}^n) \to \mathrm{Aut}(\mathbb{C}^n)/H_{n,N}$, is holomorphically trivial if and only if it is topologically trivial.*

We end this section with two examples. The first shows the difference between simultaneous standardization using automorphisms in the path-connected component of the identity $(\mathrm{Aut}_W(X))^0$ and using the whole group $\mathrm{Hol}(W, \mathrm{Aut}(X))$. In this example the map $x \in \mathrm{Hol}(W, Y_{X,N})$ is not null-homotopic, so the standardization cannot be achieved by automorphisms in $(\mathrm{Aut}_W(X))^0$; however standardization is possible by elements in $\mathrm{Hol}(W, \mathrm{Aut}(X))$.

The second is an example where the topological obstruction from Corollary 2.17 *does* prevent from simultaneous standardization, i.e., in this example

$$[\mathrm{Cont}(W, \mathrm{GL}_n(\mathbb{C}))]$$

does not act transitively on $[\mathrm{Cont}(W, Y_{\mathbb{C}^n,N})]$.

Example 2.19. *Let W be any Stein manifold. Then any $x \in \mathrm{Hol}(W, Y_{\mathbb{C}^2,2})$ is simultaneously standardizable.*

Proof. Let

$$x = (x_1(w), x_2(w)) = \left(\begin{pmatrix} z_1(w) \\ \eta_1(w) \end{pmatrix}, \begin{pmatrix} z_2(w) \\ \eta_2(w) \end{pmatrix}\right),$$

and define

$$\alpha_1^w(z, \eta) = (z - z_1(w), \eta - \eta_1(w))$$
$$\alpha_2^w(z, \eta) = \begin{pmatrix} z_2(w) - z_1(w) & f(w) \\ \eta_2(w) - \eta_1(w) & g(w) \end{pmatrix} \cdot \begin{pmatrix} z \\ \eta \end{pmatrix}$$

Observe that $z_2(w)-z_1(w)$ and $\eta_2(w)-\eta_1(w)$ have no common zeros. Hence, since W is Stein, Cartan's theorem B implies that there are $f, g \in \mathcal{O}(W)$ such that

$$\begin{pmatrix} z_2(w) - z_1(w) & f(w) \\ \eta_2(w) - \eta_1(w) & g(w) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

Hence $\alpha_1, \alpha_2 \in \mathrm{Aut}_W(\mathbb{C}^2)$ and $(\alpha_2^{-1})^w \circ \alpha_1^w$ maps $x_1(w)$ to $(0,0)$ and $x_2(w)$ to $(1,0)$, which gives the simultaneous standardization. □

As a consequence, by Corollary 2.17, $[\mathrm{Cont}(W, \mathrm{GL}_2(\mathbb{C}))]$ acts transitively on $[\mathrm{Cont}(W, Y_{\mathbb{C}^2,2})]$. In order to find an example of this form where standardization by elements of $(\mathrm{Aut}_W(X))^0$ is not possible, consider the special case $W = \mathrm{SL}_2(\mathbb{C})$. Then there exists a non null-homotopic $x \in \mathrm{Hol}(\mathrm{SL}_2(\mathbb{C}), Y_{\mathbb{C}^2,2})$ which can be standardized with an element not in $(\mathrm{Hol}(\mathrm{SL}_2(\mathbb{C}), \mathrm{Aut}(\mathbb{C}^2)))^0$. Indeed, the holomorphic map $\mathrm{SL}_2(\mathbb{C}) \to Y_{\mathbb{C}^2,2}$ given by

$$A \mapsto \left(A\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix} \right)$$

induces the identity mapping on the 3-sphere (by projection to the first factor of $Y_{\mathbb{C}^2,2}$), so is not a null-homotopic map.

Example 2.20. *Let W be a small (so that the map below gives pairwise different points) Grauert tube around SU_2, i.e., a Stein neighborhood of SU_2 in $\mathrm{SL}_2(\mathbb{C})$ which contracts onto the 3-sphere SU_2. Then $x \in \mathrm{Hol}(W, Y_{\mathbb{C}^2,3})$ defined by*

$$A \mapsto \left(A\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix} \right)$$

is not simultaneously standardizable.

Proof. Consider the map $\phi : Y_{\mathbb{C}^2,3} \to S^3 \times S^3$ given by

$$(x_1, x_2, x_3) \mapsto \left(\frac{x_2 - x_3}{|x_2 - x_3|}, \frac{x_1 - x_2}{|x_1 - x_2|}\right)$$

Since W contracts to $\mathrm{SU}_2 \cong S^3$ the composition $\phi \circ x : W \to S^3 \times S^3$ gives a map from $S^3 \to S^3 \times S^3$. It has bidegree $(0,1)$ and applying any element in $[\mathrm{Hol}(W, \mathrm{Aut}(\mathbb{C}^2))] \cong [\mathrm{Cont}(W, \mathrm{GL}_2(\mathbb{C}))]$ to it, changes both degrees by the same amount, so the corresponding bidegree will never be $(0,0)$. Therefore no application of an element in $[\mathrm{Hol}(W, \mathrm{Aut}(\mathbb{C}^2))]$ to x can lead to a null-homotopic map. □

Chapter 3

Non-algebraic manifolds with the volume density property

This chapter consists of the contribution from the preprint [Ram]. There are slight differences with that version, mostly due to the fact that some of the material has already been provided in Chapter 1.

3.1 Summary of results

We have already pointed out in the introduction and in Section 1.4.2 that all known manifolds with the VDP are algebraic, and the tools used to establish this property are algebraic in nature. In this chapter, we will show how to adapt the results in Section 1.4.3 to the holomorphic case, and give the first known examples of non-algebraic manifolds with the VDP: they arise as suspensions or pseudo-affine modifications over Stein manifolds satisfying some technical properties. As an application, it may be shown that there are such manifolds that are potential counterexamples to the Zariski Cancellation Problem, a variant of the Tóth-Varolin conjecture, and the problem of linearization of $\mathbb{C}^*$-actions on $\mathbb{C}^3$.

The main result is the following.

Theorem 3.1. *Let X be a Stein manifold of dimension $n \geq 2$ such that $H^n(X) = H^{n-1}(X) = 0$. Let ω be a volume form on X and suppose that $\mathrm{Aut}(X, \omega)$ acts transitively. Assume that there is a finite collection S of semi-compatible pairs (α, β) of volume-preserving vector fields such that for some $x_0 \in X$, $\{\alpha(x_0) \wedge \beta(x_0); (\alpha, \beta) \in S\}$ spans $\wedge^2 T_{x_0} X$. Let $f : X \to \mathbb{C}$ be a nonconstant holomorphic function with smooth reduced zero fiber X_0 and $\tilde{H}^{n-2}(X_0) = 0$. Then the suspension $\overline{X} \subset \mathbb{C}^2_{u,v} \times X$ of X along f has the VDP with respect to a natural volume form $\bar{\omega}$ satisfying $d(uv - f) \wedge \bar{\omega} = (du \wedge dv \wedge \omega)|_{\overline{X}}$.*

Since $X = \mathbb{C}^n$ satisfies the required hypothesis, we have as a simple corollary:

Theorem 3.2. *Let $n \geq 1$ and $f \in \mathcal{O}(\mathbb{C}^n)$ be a nonconstant holomorphic function with smooth reduced zero fiber X_0, such that $\tilde{H}^{n-2}(X_0) = 0$ if $n \geq 2$. Then the hypersurface $\overline{\mathbb{C}^n_f} = \{uv = f(z_1, \ldots, z_n)\} \subset \mathbb{C}^{n+2}$ has the volume density property with respect to the form $\bar{\omega}$ satisfying $d(uv - f) \wedge \bar{\omega} = du \wedge dv \wedge dz_1 \wedge \cdots \wedge dz_n$.*

For $n = 1$ this manifold is called a *Danielewski surface*. Theorem 3.2 was known in the special case where f is a polynomial, see Section 1.4.2. Their proof heavily depends on the use of Grothendieck's spectral sequence and seems difficult to generalize to the non-algebraic case. The method of proof proposed here is completely different. It relies on modifying and using the criterion of semi-compatible pairs of algebraic vector fields discussed in Section 1.4.3. This will be explained in Section 3.2. In Section 3.3 we study the suspension $\overline{X}$ (or pseudo-affine modification) of rather general manifolds X along $f \in \mathcal{O}(X)$. After some results concerning the topology and homogeneity of $\overline{X}$, we will see that that the structure of $\overline{X}$ makes it possible to lift compatible pairs of vector fields from X to $\overline{X}$, in such a way that a technical but essential generating condition on $T\overline{X} \wedge T\overline{X}$ is guaranteed (Theorem 3.9).

It is still unknown whether a contractible Stein manifold with the volume density property has to be biholomorphic to $\mathbb{C}^n$. It is believed that the answer is negative, see [KK10]. For instance the affine algebraic submanifold of $\mathbb{C}^6$ given by the equation $uv = x + x^2y + s^2 + t^3$ is such an example. Another prominent one is the Koras-Russell cubic threefold, see [Leu]. In Section 3.4 we will show how to use Theorem 3.11 to produce a non-algebraic manifold with the volume density property which is diffeomorphic to $\mathbb{C}^n$. As far as we know, this is the first of this kind. In fact, we prove the following.

Theorem 3.3. *Let $\phi : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ be a proper holomorphic embedding, and consider the manifold defined by $\overline{\mathbb{C}^n_f} = \{uv = f(z_1, \ldots, z_n)\} \subset \mathbb{C}^{n+2}$, where $f \in \mathcal{O}(\mathbb{C}^n)$ generates the ideal of functions vanishing on $\phi(\mathbb{C}^{n-1})$. Then $\overline{\mathbb{C}^n_f}$ is diffeomorphic to $\mathbb{C}^{n+1}$ and has the volume density property with respect to the volume form $\bar{\omega}$ satisfying $d(uv-f)\wedge\bar{\omega} = du\wedge dv\wedge dz_1\wedge\cdots\wedge dz_n$. Moreover $\overline{\mathbb{C}^n_f}\times\mathbb{C}$ is biholomorphic to $\mathbb{C}^{n+2}$, and therefore is a potential counterexample to the Zariski Cancellation Problem if ϕ is not straightenable.*

We end Section 3.4 with two examples which are related to the problem of linearization of holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^n$.

3.2 A criterion for volume density property

Let X be a complex manifold of dimension n, and assume X is equipped with a volume form ω. We use the notations from the previous chapters: only holomorphic vector fields are considered, meaning sections of $T^{1,0}X$. They form a $\mathcal{O}(X)$-module denoted by $\mathrm{VF}(X)$, and the volume-preserving ones a vector space $\mathrm{VF}_\omega(X)$. Similarly, global holomorphic sections of the bundle $\wedge^j T^*X$ are called holomorphic j-forms, and we denote by $\Omega^j(X)$ the vector space of all such forms. Let $\mathcal{Z}^j(X)$ (resp. $\mathcal{B}^j(X)$) denote the vector space of d-closed (resp. d-exact) j forms on X. In this chapter we denote by $\mathrm{Lie}_\omega(X)$ the Lie algebra generated by elements in $\mathrm{CVF}_\omega(X) = \mathrm{VF}_\omega(X) \cap \mathrm{CVF}(X)$.

We now give a holomorphic version of the criterion for the VDP in Section 1.4.3. Recall that in Section 1.3.2 we described the isomorphism

$$\Phi : \mathrm{VF}_\omega(X) \to \mathcal{Z}^{n-1}(X).$$

In the same spirit, there is an isomorphism of $\mathcal{O}(X)$-modules

$$\Psi : \mathrm{VF}(X) \wedge \mathrm{VF}(X) \to \Omega^{n-2}(X), \quad \nu \wedge \mu \mapsto \iota_\nu \iota_\mu \omega \tag{3.1}$$

and it is straightforward that $\iota_\mu \iota_\nu \omega = \iota_{\nu\wedge\mu}\omega$. We can deduce from the easily verified relation $[\mathcal{L}_\nu, \iota_\mu] = \iota_{[\nu,\mu]}$ that for $\nu, \mu \in \mathrm{VF}_\omega(X)$,

$$\iota_{[\nu,\mu]}\omega = d\iota_\nu \iota_\mu \omega. \tag{3.2}$$

Hence by restricting the isomorphism in Equation 3.1 to $\wedge^2 \mathrm{CVF}_\omega(X)$ and composing with the exterior differential $d : \Omega^{n-2} \to \mathcal{B}^{n-1}$ we obtain a mapping

$$d \circ \Psi : \mathrm{CVF}_\omega(X) \wedge \mathrm{CVF}_\omega(X) \to \mathcal{B}^{n-1}, \quad \nu \wedge \mu \mapsto \iota_{[\mu,\nu]}\omega,$$

whose image is in fact contained in $\Phi(\mathrm{Lie}_\omega(X))$.

Suppose we want to approximate $\Theta \in \mathrm{VF}_\omega(X)$ on $K \subset X$ by a Lie combination of elements in $\mathrm{CVF}_\omega(X)$. Consider the closed form $\iota_\Theta \omega$ and assume for the time being that it is exact. Then by Equation 3.1 there is $\gamma \in \mathrm{VF}(X) \wedge \mathrm{VF}(X)$ such that $\iota_\Theta \omega = d(\Psi(\gamma))$. It now suffices to approximate γ by a sum of the form $\sum \alpha_i \wedge \beta_i \in \mathrm{Lie}_\omega(X) \wedge \mathrm{Lie}_\omega(X)$. Indeed, by Equation 3.2, $\iota_\Theta \omega = d \circ \Psi(\gamma)$ would then be approximated by elements

$$d \circ \Psi\left(\sum \alpha_i \wedge \beta_i\right) = \sum \iota_{[\alpha_i,\beta_i]}\omega \in \Phi(\mathrm{Lie}_\omega(X)),$$

which implies that Θ is approximated uniformly on K by elements of the form $\sum [\alpha_i, \beta_i] \in \mathrm{Lie}_\omega(X)$, as desired. We therefore concentrate on this approximation on $\mathrm{VF}(X) \wedge \mathrm{VF}(X)$. We will assume that *(a)* there are $\nu_1, \ldots, \nu_k, \mu_1, \ldots, \mu_k \in \mathrm{CVF}_\omega(X)$ such that the submodule of $\mathrm{VF}(X) \wedge \mathrm{VF}(X)$ generated by the elements $\nu_j \wedge \mu_j$ is contained in the closure of $\mathrm{Lie}_\omega(X) \wedge \mathrm{Lie}_\omega(X)$. We may assume K to be $\mathcal{O}(X)$-convex, and let us suppose *(b)* that for all p in a Runge Stein neighborhood U of K, $\nu_j(p) \wedge \mu_j(p)$ generate the vector space $T_pX \wedge T_pX$. We then proceed with standard methods in sheaf cohomology: let $\mathfrak{F}$ be the coherent sheaf corresponding to the wedge of the tangent bundle. Condition *(b)* translates to the fact that the images of $\nu_j \wedge \mu_j$ generate the fibers of the sheaf, so by Nakayama's Lemma the lift to a set of generators for the stalks $\mathfrak{F}_p$ for all $p \in U$. Therefore, by Cartan's Theorem B, the sections of $\mathfrak{F}$ on U are of the form

$$\sum h_i(\nu_j \wedge \mu_j) \tag{3.3}$$

with $h_j \in \mathcal{O}(U)$. Since U is Runge, we conclude that every element $\gamma \in \mathrm{VF}(X) \wedge \mathrm{VF}(X)$ may be uniformly approximated on K by elements as in Equation 3.3 with $h_j \in \mathcal{O}(X)$. By assumption *(a)* γ may be approximated uniformly on K by elements in $\mathrm{Lie}_\omega(X) \wedge \mathrm{Lie}_\omega(X)$.

To find the pairs $\nu_j \wedge \mu_j$, observe that if $\nu, \mu \in \mathrm{CVF}_\omega(X)$, and $f \in \operatorname{Ker} \nu, g \in \operatorname{Ker} \mu$, then $f\nu, g\mu \in \mathrm{CVF}_\omega(X)$. By linearity, any element in the span of $(\operatorname{Ker} \nu \cdot \operatorname{Ker} \mu) \cdot (\nu \wedge \mu)$ lies in $\mathrm{Lie}_\omega(X) \wedge \mathrm{Lie}_\omega(X)$. By considering the closures, we see that if I is a nonzero ideal contained in the closure of $\overline{\mathrm{Span}\,(\operatorname{Ker} \nu \cdot \operatorname{Ker} \mu)}$, then $I \cdot (\nu \wedge \mu)$ generates a submodule of $\mathrm{VF}(X) \wedge \mathrm{VF}(X)$ which is contained in $\overline{\mathrm{Lie}_\omega(X) \wedge \mathrm{Lie}_\omega(X)}$. This motivates the following definition.

Definition. *Let ν, μ be nontrivial complete vector fields on X. We say that the pair (ν, μ) is* semi-compatible *if the closure of the span of $\operatorname{Ker} \nu \cdot \operatorname{Ker} \mu$ contains a nonzero ideal of $\mathcal{O}(X)$. We call the largest ideal $I \subset \overline{\mathrm{Span}\,(\operatorname{Ker} \nu \cdot \operatorname{Ker} \mu)}$ the ideal of the pair (ν, μ).*

To reduce to the special case just treated (where $\iota_\Theta \omega$ is exact), we must further assume that given $\Theta \in \mathrm{VF}_\omega(X)$, it is possible to obtain the zero class in $H^{n-1}(X)$ by subtracting an element of $\Phi(\mathrm{Lie}_\omega(X))$; however, Equation 3.2 implies that Lie brackets represent the zero class in $H^{n-1}(X)$, so it is enough to subtract elements from $\Phi(\mathrm{CVF}_\omega(X))$. The preceding discussion then shows that the existence of "enough" semi-compatible pairs of volume-preserving vector fields, along with this condition, suffices to establish the VDP. We have thus proved the following criterion:

Proposition 3.4. *Let X be a Stein manifold of dimension n with a holomorphic volume form ω, satisfying the following condition:*

every class of $H^{n-1}(X)$ contains an element in the closure of $\Phi(\mathrm{CVF}_\omega(X))$

Suppose there are finitely many semi-compatible pairs of volume-preserving vector fields (ν_j, μ_j) with ideals I_j such that for all $x \in X$,

$$\{I_j(x)(\nu_j(x) \wedge \mu_j(x))\}_j \text{ generates } \wedge^2 T_x X.$$

Then X has the ω-VDP.

It is also possible to adapt the criterion for the ADP (see Section 1.4.3). By an obvious variant of the above discussion, we obtain the following.

Proposition 3.5. *Let X be a Stein manifold. Suppose there are finitely many compatible pairs of vector fields (ν_j, μ_j) such that $I_j(x)\nu_j(h_j(x))$ generate T_xX for all $x \in X$. Then X has the DP.*

3.3 Suspensions

Let X be a connected Stein manifold of dimension n, and let $f \in \mathcal{O}(X)$ be a nonconstant holomorphic function with a smooth reduced zero fiber X_0 (this means that df is never 0 on X_0). To it we associate the space $\overline{X}$, called the *suspension* over X along f, which is defined as

$$\overline{X} = \{(u, v, x) \in \mathbb{C}^2 \times X; uv - f(x) = 0\}.$$

Since X_0 is reduced, $d(uv - f) \neq 0$ everywhere, so $\overline{X}$ is smooth. Hence $\overline{X}$ is a Stein manifold of dimension $n + 1$.

Suppose X has a volume form ω. Then $\Omega = du \wedge dv \wedge \omega$ is a volume form on $\mathbb{C}^2 \times X$. There exists a canonical volume form $\overline{\omega}$ on $\overline{X}$ such that

$$d(uv - f) \wedge \overline{\omega} = \Omega|_{\overline{X}}.$$

Moreover, any vector field $\overline{\Theta}$ on $\overline{X}$ has an extension Θ to $\mathbb{C}^2 \times X$ with $\Theta(uv - f) = 0$, and we have $\operatorname{div}_{\overline{\omega}} \overline{\Theta} = \operatorname{div}_\Omega \Theta|_{\overline{X}}$ (see [KK08b, 2.2,2.4]). In view of our criterion we now investigate the existence of sufficient semi-compatible fields, as well as the topology of $\overline{X}$.

Let $\Theta \in \mathrm{VF}(X)$. There exists an extension $\tilde{\Theta} \in \mathrm{VF}(\mathbb{C}^2 \times X)$ such that $\tilde{\Theta}(u) = \tilde{\Theta}(v) = 0$ and $\tilde{\Theta}(\tilde{g}) = \Theta(g)$ for all $g \in \mathcal{O}(X)$ (here $\tilde{g}$ is an extension of g not depending on u, v). Clearly, $\operatorname{div}_\Omega \tilde{\Theta} = \pi^*(\operatorname{div}_\omega \Theta)$, where $\pi : \mathbb{C}^2 \times X \to X$ is the natural projection. We may "lift" Θ to a field in $\overline{X}$ in two different ways. Consider the fields on $\mathbb{C}^2 \times X$

$$\Theta_u = v \cdot \tilde{\Theta} + \widetilde{\Theta(f)} \frac{\partial}{\partial u} \qquad \Theta_v = u \cdot \tilde{\Theta} + \widetilde{\Theta(f)} \frac{\partial}{\partial v},$$

which are clearly tangent to $\overline{X}$; we may therefore consider the corresponding fields (restrictions) on $\overline{X}$, which we denote simply Θ_u and Θ_v.

Lemma 3.6. *If Θ is ω-volume-preserving, then Θ_u and Θ_v are of $\overline{\omega}$-divergence zero. Moreover, if Θ is complete, then Θ_u and Θ_v are also complete.*

Proof. The completeness of the lifts is clear, but it will be useful for the sequel to compute explicitly their flows. Denote by $\phi^t(x)$ the flow of Θ on X, and let $g : X \times \mathbb{C} \to \mathbb{C}$ be the first order approximation of $f \circ \phi$ with respect to t; in other words, let g satisfy

$$f(\phi^t(x)) = f(x) + tg(x,t). \tag{3.4}$$

Since f is holomorphic, g is well defined and holomorphic on $X \times \mathbb{C}$. The claim is that $\Phi : \overline{X} \times \mathbb{C}_t \to \overline{X}$ defined by

$$\Phi^t(u,v,x) = (u + tg(x,tv), v, \phi^{tv}(x)) \tag{3.5}$$

is the flow of Θ_u, which therefore exists for all t. Indeed, we compute

$$\Theta_u(\Phi^t(u,v,x)) = v \cdot \Theta(\phi^{tv}(x)) + \Theta(f)(\phi^{tv}(x))\frac{\partial}{\partial u},$$

while on the other hand

$$\frac{\partial}{\partial t}\Phi^t(u,v,x) = \frac{\partial}{\partial t}(tg(x,tv))\frac{\partial}{\partial u} + \frac{\partial}{\partial t}(\phi^{tv}(x)) = \frac{\partial}{\partial t}(tg(x,tv))\frac{\partial}{\partial u} + v \cdot \Theta(\phi^{tv}(x)).$$

The equality $\frac{\partial}{\partial t}(tg(x,tv)) = \Theta(f)(\phi^{tv}(x))$ follows by differentiating Equation 3.4 at (x,tv).

Since $\operatorname{div}_{\overline{\omega}} \Theta_u = \operatorname{div}_\Omega \Theta_u|_{\overline{X}}$, and because divergence (with respect to any volume form) is linear and satisfies $\operatorname{div}(h \cdot \Theta) = h \operatorname{div} \Theta + \Theta(h)$, we get

$$\operatorname{div}_{\overline{\omega}} \Theta_u = v \cdot \operatorname{div}_\Omega \tilde{\Theta}|_{\overline{X}} + \tilde{\Theta}(v) + \Theta(f) \operatorname{div}_\Omega\left(\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial u}(\Theta(f)) = v \cdot \operatorname{div}_\Omega \tilde{\Theta}|_{\overline{X}}$$

and as noted above $\operatorname{div}_\Omega \tilde{\Theta} = \pi^*(\operatorname{div}_\omega \Theta) = 0$. □

Lemma 3.7. *Suppose (ν,μ) is a semi-compatible pair of vector fields on X. Then (ν_u,μ_v) and (ν_v,μ_u) are semi-compatible pairs on $\overline{X}$.*

Proof. By Lemma 3.6, the lifted and extended fields are complete. It then suffices to show that (ν_u,μ_v) is a semi-compatible pair in $\mathbb{C}^2 \times X$, because we may restrict the elements in the ideal to $\overline{X}$: by the Cartan extension theorem, this set forms an ideal in $\mathcal{O}(\overline{X})$.

Let I be the ideal of the pair (ν,μ). For any function $h \in \mathcal{O}(X)$, denote

$$\tilde{I} = \left\{\tilde{h} \cdot F(u,v); h \in I, F \in \mathcal{O}(\mathbb{C}^2)\right\} \subset \mathcal{O}(\mathbb{C}^2 \times X),$$

where $\tilde{h}$ is the trivial extension as above. This is clearly a nonzero ideal. An element in $\tilde{I}$ can be approximated uniformly on a given compact of $\mathbb{C}^2 \times X$ by a finite sum

$$(\sum_k \tilde{n}_k \tilde{m}_k) \sum_{i,j} a_{i,j} u^i v^j = \sum a_{i,j} (\tilde{n}_k v^j)(\tilde{m}_k u^i)$$

where $n_k \in \mathrm{Ker}(\nu), m_k \in \mathrm{Ker}(\mu)$ for all k. Since $\tilde{n}_k v^j \in \mathrm{Ker}(\nu_u)$ for all $j, k \geq 0$ and $\tilde{m}_k u^i \in \mathrm{Ker}(\mu_v)$ for all $i, k \geq 0$, it follows that $\tilde{I}$ is contained in the closure of $\mathrm{Span}(\mathrm{Ker}(\nu_u) \cdot \mathrm{Ker}(\mu_v))$. □

The topology of the suspension $\overline{X}$ is of course closely related to that of X. In the case where X is the affine space, this relationship is computed in detail in [KZ99, §4]. For more general X we have the following.

Proposition 3.8. *Assume X has dimension $n \geq 2$. If the complex de Rham cohomology groups satisfy $H^n(X) = H^{n-1}(X) = 0$ and $\tilde{H}^{n-2}(X_0) = 0$, where $\tilde{H}$ denotes reduced cohomology, then $H^n(\overline{X}) = 0$.*

Proof. Consider the long exact sequence of the pair $(\overline{X}, \overline{X} \setminus U_0)$ in cohomology, where U_0 is the subspace of $\overline{X}$ where u vanishes:

$$\cdots \to H^n(\overline{X}, \overline{X} \setminus U_0) \to H^n(\overline{X}) \to H^n(\overline{X} \setminus U_0) \to \dots \tag{3.6}$$

The term on the right vanishes, because $\overline{X} \setminus U_0$ is biholomorphic to $\mathbb{C}^* \times X$ via $(u, x) \mapsto (u, f(x)/u, x)$, so

$$H^n(\overline{X} \setminus U_0) = (H^1(\mathbb{C}^*) \otimes H^{n-1}(X)) \oplus (H^0(\mathbb{C}^*) \otimes H^n(X)) = 0.$$

To evaluate the left-hand side, we use an idea due to M. Zaidenberg (see [Zai96]). Consider the normal bundle $\pi : N \to U_0$ of the closed submanifold U_0 in $\overline{X}$, with zero section $N_0 \cong U_0$. Fix a tubular neighborhood W of U_0 in $\overline{X}$ such that the pair (W, U_0) is diffeomorphic to (N, N_0). Then by excision, we have that

$$\tilde{H}^*(\overline{X}, \overline{X} \setminus U_0) \cong \tilde{H}^*(W, W \setminus U_0) \cong \tilde{H}^*(N, N \setminus N_0).$$

Let $t \in H^2(N, N \setminus N_0)$ be the Thom class of U_0 in $\overline{X}$, that is, the unique cohomology class taking value 1 on any oriented relative 2-cycle in $H_2(N, N \setminus N_0)$ defined by a fiber F of the normal bundle N (see e.g. [MS74, §9–10], for details). Then, by taking the cup-product of the pullback under π of a cohomology class with t, we obtain the Thom isomorphisms

$$H^i(U_0) \cong H^{i+2}(N, N \setminus N_0) \cong H^{i+2}(\overline{X}, \overline{X} \setminus U_0) \quad \forall i.$$

Since $U_0 \cong X_0 \times \mathbb{C}$, U_0 is homotopy equivalent to X_0, and we have $H^n(\overline{X}, \overline{X} \setminus U_0) \cong H^{n-2}(X_0)$. If $n \geq 3$, reduced cohomology coincides with standard cohomology, and therefore $H^n(\overline{X}) = 0$ by exactness of Equation 3.6. If $n = 2$, that sequence becomes

$$\cdots \to H^1(\overline{X} \setminus U_0) \to H^2(\overline{X}, \overline{X} \setminus U_0) \to H^2(\overline{X}) \to 0.$$

Let γ be an oriented 2-cycle in $\overline{X}$ whose boundary $\partial\gamma$ lies in $\overline{X} \setminus U_0$ (a disk transversal to U_0). A one-dimensional subspace of $H^1(\overline{X} \setminus U_0)$ is generated by a 1-cocycle taking value 1 on $\partial\gamma$, and this cocycle is sent via the coboundary operator (which is the first map in the above sequence) to a 2-cocycle taking value 1 on γ, i.e., to the Thom class t described previously, which is also a generator of a one-dimensional subspace of $H^2(\overline{X}, \overline{X} \setminus U_0)$. However, $H^1(\overline{X} \setminus U_0) \cong H^1(\mathbb{C}^* \times X) \cong \mathbb{C}$ and $H^2(\overline{X}, \overline{X} \setminus U_0) \cong H^0(U_0) \cong \mathbb{C}$, so the coboundary map is an isomorphism, and by exactness it follows that $H^2(\overline{X}) = 0$. $\square$

Next, we show how to lift a collection of semi-compatible fields to the suspension and span $\wedge^2 T\overline{X}$ with semi-compatible fields[1]. Recall that we denote $\mathrm{Aut}(X, \omega)$ the group of volume-preserving holomorphic automorphisms of X.

Theorem 3.9. *Let X be a Stein manifold with a finite collection S of semi-compatible pairs (α, β) of vector fields such that for some $x_0 \in X$*

$$\{\alpha(x_0) \wedge \beta(x_0); (\alpha, \beta) \in S\} \text{ spans } \wedge^2 (T_{x_0} X). \tag{3.7}$$

Assume that $\mathrm{Aut}(\overline{X})$ acts transitively on $\overline{X}$. Then there exists a finite collection $\overline{S}$ of semi-compatible pairs (A_j, B_j) on $\overline{X}$ with corresponding ideals I_j such that

$$\mathrm{Span}\{I_j(\bar{x}) A_j(\bar{x}) \wedge B_j(\bar{x})\}_j = \wedge^2 (T_{\bar{x}} \overline{X}) \quad \forall \bar{x} \in \overline{X}. \tag{3.8}$$

Moreover, if X has a volume form ω and the fields in S preserve it, and $\mathrm{Aut}_{\bar{\omega}}(\overline{X})$ acts transitively, then the fields in $\overline{S}$ can be chosen to preserve the form $\bar{\omega}$

Proof. We claim that it is sufficient to show that the conclusion holds for a single $\bar{x}_0 \in \overline{X}$. Indeed, let C be the analytic set of points $\bar{x} \in \overline{X}$ where Equation 3.8 does not hold, and decompose C into its (at most countably many) irreducible components C_i. For each i, let D_i be the set of automorphisms ϕ of $\overline{X}$ such that the image of $\overline{X} \setminus C_i$ under ϕ has a nonempty intersection with C_i. Clearly each D_i is open, and it is also dense: given $h \in \mathrm{Aut}(\overline{X})$ not in D_i, let $c \in C_i$, $d = h(c) \in C_i$ and $\gamma \in \mathrm{Aut}(\overline{X})$ mapping $\bar{x}_0$ to d. Now, since the assumption in Equation 3.8 implies that the tangent space at $\bar{x}_0$ is spanned by complete fields, there exists a complete field α from the collection $\overline{S}$ such that $\gamma_*(\alpha)$ is not tangent to C_i. If φ is the flow of $\gamma_*(\alpha)$, then $\varphi^t \circ h$ is an automorphism arbitrarily close to h mapping c out of C_i. By the Baire Category Theorem (see Section 1.3) there exists a $\psi \in \bigcap D_i$. By expanding $\overline{S}$ to $\overline{S} \cup \{(\psi_* \alpha, \psi_* \beta); (\alpha, \beta) \in \overline{S}\}$, we obtain a finite collection of semi-compatible fields which fail to satisfy Equation 3.8 in an exceptional variety of dimension strictly lower than that of C. The conclusion follows from the finite iteration of this procedure.

By the previous lemmas, if $(\alpha, \beta) \in S$ then (α_u, β_v) and (α_v, β_u) are semi-compatible pairs in $\overline{X}$. We let $\overline{S}$ consist of all those pairs. We will also add two pairs to $\overline{S}$, of the form $(\phi^* \alpha_u, \phi^* \beta_v)$, where ϕ is an automorphism of $\overline{X}$ (preserving

[1] A simpler algebraic case has been treated by J. Josi (Master thesis, 2013, unpublished)

a volume form, if necessary) to be specified later. We now select an appropriate $\bar{x}_0 = (u_0, v_0, x_0) \in \overline{X}$ by picking any element from the complement of finitely many analytic subsets which we now describe. The first analytic subset of $\overline{X}$ to avoid is the locus where any of the (finitely many) associated ideals I_j vanish. Note that Equation 3.7 is in fact satisfied everywhere on X except an analytic variety C: the second closed set in $\overline{X}$ to avoid is the preimage of C under the projection. Finally, we avoid $u = 0, v = 0$ and $d_{x_0} f = 0$. In short, we pick a $\bar{x}_0 = (u_0, v_0, x_0) \in \overline{X}$ with $u_0 \neq 0, v_0 \neq 0, d_{x_0} f \neq 0$, such that Equation 3.7 is satisfied at x_0, and such that none of the ideals $I_j(\bar{x}_0)$ vanish. Because of this last condition, it will suffice to show that $\{A(\bar{x}_0) \wedge B(\bar{x}_0); (A, B) \in \overline{S}\}$ spans $\wedge^2(T_{x_0} X)$.

Consider $\pi : \overline{X} \to X \times \mathbb{C}_u$, which at $\bar{x}_0$ induces an isomorphism $d_{\bar{x}_0}\pi : T_{\bar{x}_0}\overline{X} \to T_{x_0} X \times T_{u_0}\mathbb{C}$. Denote by $\partial_u = \frac{\partial}{\partial u}$ the basis of $T_{u_0}\mathbb{C}$, and consider

$$P : \wedge^2(T_{\bar{x}_0}\overline{X}) \to \wedge^2(T_{x_0} X \oplus \langle \partial_u \rangle) = \wedge^2(T_{x_0} X) \oplus (T_{x_0} X \otimes \langle \partial_u \rangle).$$

Since P is a linear isomorphism, it now suffices to show that the direct sum on the right-hand side equals $P(\Lambda)$, where

$$\Lambda = \mathrm{Span}\{A(\bar{x}_0) \wedge B(\bar{x}_0); (A, B) \in \overline{S}\}.$$

We will prove (i) that $\wedge^2(T_{x_0} X) \subseteq P(\Lambda)$, and (ii) that $T_{x_0} X \otimes \langle \partial_u \rangle \subseteq P(\Lambda)$.

Let us first show (i). Let $\alpha(x_0) \wedge \beta(x_0) \in \wedge^2(T_{x_0} X)$. Since $\{\alpha(x_0) \wedge \beta(x_0)\}_{(\alpha,\beta)\in S}$ spans $\wedge^2(T_{x_0} X)$, we can assume that (α, β) is a pair of vector fields lying in S (we will often omit to indicate the point x_0 at which these fields are evaluated). Then $(\alpha_u, \beta_v) \in \overline{S}$, so $P(\Lambda)$ contains

$$P(\alpha_u \wedge \beta_v) = P((v\tilde{\alpha} + \alpha(f)\partial_u) \wedge (u\tilde{\beta} + \beta(f)\partial_v)) = uv(\alpha \wedge \beta) - u\alpha(f)(\beta \wedge \partial_u). \quad (3.9)$$

At the point $\bar{x}_0$, we have assumed that u and v are both nonzero. If $\alpha(f)$ happens to vanish at x_0, then $\alpha(x_0) \wedge \beta(x_0)$ is in $P(\Lambda)$, as desired. Otherwise, consider the vector field $(u - u_0)\alpha_v$ on $\overline{X}$. Since α_v is complete and $(u - u_0)$ lies in the kernel of α_v, $(u - u_0)\alpha_v$ is a complete (and $\bar{\omega}$-divergence-free) vector field on $\overline{X}$. Quite generally one can compute, in local coordinates for example, that the flow at time 1 of the field $g\Theta$, where $\Theta \in \mathrm{CVF}(M)$ and $g \in \mathrm{Ker}(\Theta)$ with $g(p) = 0$, is a map ϕ whose derivative at $p \in M$ is given by:

$$w \mapsto w + d_p g(w)\Theta(p) \quad w \in T_p M.$$

Therefore, for a vector field $\mu \in \mathrm{VF}(M)$, we have

$$(\phi^{-1})^*(\mu)(p) = (d_p\phi)(\mu(p)) = \mu(p) + \mu(g)(p)\Theta(p). \quad (3.10)$$

Apply this in the case of $M = \overline{X}$, $p = \bar{x}_0$, $\Theta = \alpha_v$ and $g = u - u_0$. For the vector fields $\mu = \beta_v$, this equals β_v; for $\mu = \alpha_u$, it equals $\alpha_u + \alpha(f)\alpha_v$. Hence, if we add $((\phi^{-1})^*\alpha_u, (\phi^{-1})^*\beta_v)$ to $\overline{S}$, we obtain that $P(\Lambda)$ contains

$$P((\phi^{-1})^*\alpha_u \wedge (\phi^{-1})^*\beta_v - \alpha_u \wedge \beta_v) = P(\alpha(f)\alpha_v \wedge \beta_v) = \alpha(f)u^2(\alpha \wedge \beta).$$

We now show (ii). It will be useful to distinguish elements in $T_{x_0}X$ according to whether they belong to $K = \mathrm{Ker}(d_{x_0}f)$ or not. Since we have assumed $d_{x_0}f \neq 0$, $T_{x_0}X$ splits as $K \oplus V$, where V is a vector space of dimension 1, which may be spanned by some ξ satisfying $d_{x_0}f(\xi) = \xi(f) = 1$. The isomorphism is given by the unique decomposition $v = (v - v(f)\xi) + v(f)\xi$. This induces another splitting

$$\begin{aligned} &\wedge^2(T_{x_0}X) \to \wedge^2(K) \oplus (K \otimes V) \\ &\quad \alpha \wedge \beta \mapsto (\alpha - \alpha(f)\xi) \wedge (\beta - \beta(f)\xi) + (\alpha(f)\beta - \beta(f)\alpha) \wedge \xi. \end{aligned}$$

Since the left-hand side is generated by $\{\alpha \wedge \beta; (\alpha, \beta) \in S\}$, $K \otimes V$ is generated by $\{(\alpha(f)\beta - \beta(f)\alpha) \wedge \xi; (\alpha, \beta) \in S\}$, and therefore K by $\{\alpha(f)\beta - \beta(f)\alpha; (\alpha, \beta) \in S\}$. Consider Equation 3.9 and subtract $P(\alpha_v \wedge \beta_u) = uv(\alpha \wedge \beta) + u\beta(f)(\alpha \wedge \partial_u)$: recalling that $u_0 \neq 0$, we see that

$$\{u(\beta(f)\alpha - \alpha(f)\beta) \wedge \partial_u; (\alpha, \beta) \in S\} = K \otimes \langle \partial_u \rangle \subset P(\Lambda). \tag{3.11}$$

It remains to show that $V \otimes \langle \partial_u \rangle \subset P(\Lambda)$. By linearity, since V is of dimension 1, it suffices to find a single element in $P(\Lambda) \cap (V \otimes \langle \partial_u \rangle)$. In fact since we have already proven (i), it suffices to find an element in $P_2(\Lambda) \cap (V \otimes \langle \partial_u \rangle)$, where P_2 is the second component of the map P. If it were the case that for some pair $(\alpha, \beta) \in S$ both $\alpha(f)$ and $\beta(f)$ are nonzero at x_0, then by Equation 3.9 $-u\alpha(f)\beta \wedge \partial_u$ is such an intersection element. In the other case, there is at least a pair $(\alpha, \beta) \in S$ for which $\alpha(f)(x_0) = 0$ and both $\beta(f)(x_0) \neq 0$ and $\alpha(x_0) \neq 0$, for otherwise the spanning condition implied by Equation 3.11 would fail to be satisfied. As in the proof of (i), we will add to $\overline{S}$ the pair $(\phi^*(\alpha_u), \phi^*(\beta_v))$, where ϕ is the time 1 map of the flow of the complete (volume-preserving) field $\Theta = g(x)(u\partial_u - v\partial_v)$, and $g \in \mathcal{O}(X)$ vanishes at x_0. By Equation 3.10, we have that

$$\phi^*(\alpha_u) = \alpha_u + \alpha_u(g)\Theta = v\alpha + \alpha(f)\partial_u + v\alpha(g)(u\partial_u - v\partial_v)$$

which by assumption simplifies to

$$\phi^*(\alpha_u) = v\alpha + uv\alpha(g)\partial_u - v^2\alpha(g)\partial_v.$$

Similarly we have

$$\phi^*(\beta_v) = u\beta + u^2\beta(g)\partial_u + (\beta(f) - uv\beta(g))\partial_v.$$

Hence

$$P_2(\phi^*(\alpha_u) \wedge \phi^*(\beta_v)) = u^2 v\beta(g)\alpha \wedge \partial_u - u^2 v\alpha(g)\beta \wedge \partial_u.$$

By assumption, the first summand lies in $K \otimes \langle \partial_u \rangle$, which we have already shown to be contained in $P(\Lambda)$. Since $\beta(f) \neq 0$, the second summand, if nonzero, lies in $P_2(\Lambda) \cap (V \otimes \langle \partial_u \rangle)$. But it is clear that we may find a $g \in \mathcal{O}(X)$ such that $\alpha(g)(x_0) \neq 0$. □

Finally, we show how the transitivity requirement for the previous proposition can be inherited from the base space X. Recall from Section 1.4.1 that a Stein manifold X is *holomorphically (volume) flexible* if the complete (volume-preserving) vector fields span the tangent space $T_x X$ at every $x \in X$. Clearly, a manifold X is holomorphically (volume) flexible if one point $x \in X$ is, and $\mathrm{Aut}(X)$ (resp. $\mathrm{Aut}(X, \omega)$) acts transitively. Moreover, holomorphic (volume) flexibility implies the transitive action of $\mathrm{Aut}(X)$ (resp. $\mathrm{Aut}(X, \omega)$) on X.

Lemma 3.10. *If X is holomorphically flexible, then* $\mathrm{Aut}(\overline{X})$ *acts transitively. Moreover, if X is holomorphically volume flexible at a point $x \in X$ and* $\mathrm{Aut}(X, \omega)$ *acts transitively, then* $\mathrm{Aut}_{\bar{\omega}}(\overline{X})$ *acts transitively.*

Proof. For simplicity we prove the first statement: the second is proven in an exactly analogous manner. Let $\bar{x}_0 = (u_0, v_0, x_0) \in \overline{X}$ with $u_0 v_0 \neq 0$, and let us determine the orbit of $\bar{x}_0$ under $\mathrm{Aut}(\overline{X})$. Given $\Theta \in \mathrm{VF}(X)$, by Equation 3.5 we have, for each t, an automorphism of $\overline{X}$ of the form

$$(u, v, x) \mapsto (u + tg(x, tv), v, \phi^{tv}(x)). \tag{3.12}$$

The orbit of $\bar{x}_0$ must hence contain the hypersurface $\{v = v_0\} \subset \overline{X}$ (because $\mathrm{Aut}(X)$ acts transitively on X), and analogously, since $u_0 \neq 0$, the orbit contains $\{u = u_0\} \subset \overline{X}$. Let $(u_1, v_1, x_1) \in \overline{X}$ be another point with $u_1 v_1 \neq 0$. Note that the nonconstant function $f : X \to \mathbb{C}$ can omit at most one value ξ. Indeed, by flexibility there is a complete vector field which at x_0 points in a direction where f is not constant; precomposition with its flow map at x_0 gives an entire function which must omit at most one value. Of course ξ cannot be 0, and by definition neither $u_0 v_0$ nor $u_1 v_1$. Follow the orbit of $\bar{x}_0$ along the hypersurface $\{u = u_0\} \cap \overline{X}$ until (u_0, v_1, x'), then along $\{v = v_1\} \cap \overline{X}$ until (u_1, v_1, x_1) (if $\xi = u_0 v_1$ replace v_1 by $2v_1$). So the orbit contains all points $(u, v, x) \in \overline{X}$ with $uv \neq 0$ and by Equation 3.12 also those with either u or v nonzero. Consider now a point of the form $(0, 0, x_0) \in \overline{X}$. Since $x_0 \in X_0$ and X_0 is reduced, $d_{x_0} f \neq 0$, so there is a tangent vector evaluating to a nonzero number, which since X is flexible can be taken to be of the form $\Theta(x_0)$ for a complete field Θ. By lifting Θ we obtain an automorphism of $\overline{X}$ of the form $(u, v, x) \mapsto (g(0, x_0), 0, x_0)$. Since

$$g(0, x_0) = \lim_{t \to 0} \frac{f(\phi^t(x_0)) - f(\phi^0(x_0))}{t} = (f \circ \phi)'(0) = d_{x_0} f(\Theta(x_0)) \neq 0,$$

this automorphism moves $(0,0,x_0)$ to a point of nonzero u coordinate, and we are done. $\square$

In particular, by Theorem 1.25, the assumptions hold if X has the ω-VDP and is of dimension $n \geq 2$.

3.4 Examples & applications

The following theorem summarizes the previous discussion and gives conditions under which the suspension over a manifold has a VDP.

Theorem 3.11. *Let X be a Stein manifold of dimension $n \geq 2$ such that $H^n(X) = H^{n-1}(X) = 0$. Let ω be a volume form on X and suppose that $\mathrm{Aut}(X,\omega)$ acts transitively. Assume that there is a finite collection S of semi-compatible pairs (α,β) of volume-preserving vector fields such that for some $x_0 \in X$, $\{\alpha(x_0)\wedge\beta(x_0); (\alpha,\beta) \in S\}$ spans $\wedge^2 T_{x_0}X$. Let $f : X \to \mathbb{C}$ be a nonconstant holomorphic function with smooth reduced zero fiber X_0 and $\tilde{H}^{n-2}(X_0) = 0$. Then the suspension $\overline{X} \subset \mathbb{C}^2_{u,v} \times X$ of X along f has the VDP with respect to a natural volume form $\bar{\omega}$ satisfying $d(uv-f)\wedge\bar{\omega} = (du \wedge dv \wedge \omega)|_{\overline{X}}$.*

Proof. The spanning condition on $\wedge^2 TX$ implies holomorphic volume flexibility at x_0. So by Lemma 3.10, $\mathrm{Aut}_{\bar{\omega}}(\overline{X})$ acts transitively, and therefore Theorem 3.9 may be applied. By assumption and Proposition 3.8, the topological condition of Proposition 3.4 is also trivially satisfied. $\square$

Corollary 3.12. *Let $n \geq 1$ and $f \in \mathcal{O}(\mathbb{C}^n)$ be a nonconstant holomorphic function with smooth reduced zero fiber X_0, such that $\tilde{H}^{n-2}(X_0) = 0$ if $n \geq 2$. Then the hypersurface $\overline{\mathbb{C}^n_f} = \{uv = f(z_1,\ldots,z_n)\} \subset \mathbb{C}^{n+2}$ has the volume density property with respect to the form $\bar{\omega}$ satisfying $d(uv-f)\wedge\bar{\omega} = du \wedge dv \wedge dz_1 \wedge \cdots \wedge dz_n$.*

Proof. If $n \geq 2$ this follows immediately from the previous theorem, since in $\mathbb{C}^n$ the standards derivations ∂_{z_j} generate $\wedge^2 TX$. If $n = 1$, there are no semi-compatible pairs on $\mathbb{C}$, but it is possible to show the VDP directly. Given $\Theta \in \mathrm{VF}_\omega(\overline{\mathbb{C}_f})$ and a compact K of $\overline{\mathbb{C}_f}$, we must find a finite Lie combination of volume-preserving fields approximating Θ on K. Because of this approximation, we can reduce to the algebraic case, which is treated in [KK08b] by means of explicit calculation of Lie brackets of the known complete fields Θ_u, Θ_v, and $h(u\partial_u - v\partial_v)$. $\square$

Let $\phi : \mathbb{C}^{n-1} \to \mathbb{C}^n$ be a proper holomorphic embedding, and consider the closed subset $Z = \phi(\mathbb{C}^{n-1}) \subset \mathbb{C}^n$. It is a standard result that every multiplicative Cousin distribution in $\mathbb{C}^n$ is solvable, since $H^2(\mathbb{C}^n,\mathbb{Z}) = 0$. This implies that the divisor associated to Z is principal: in other words, there exists a holomorphic function f on $\mathbb{C}^n$ vanishing precisely on Z and such that $df \neq 0$ on Z. We may therefore consider the suspension $\overline{\mathbb{C}^n_f}$ of $\mathbb{C}^n$ along f, which according to the above corollary must have the volume density property. The significance of this lies in the existence of non-straightenable embeddings, see Section 1.2.3.

If the embedding ϕ is straightenable, it is clear that $\overline{\mathbb{C}^n_f}$ is trivially biholomorphic to $\mathbb{C}^{n+1}$, and a calculation shows that the form $\bar{\omega}$ is the standard one. So the result says something new only if ϕ is non-straightenable. Indeed, it is unknown whether $\overline{\mathbb{C}^n_f}$ is biholomorphic to $\mathbb{C}^{n+1}$. However, $\overline{\mathbb{C}^n_f} \times \mathbb{C}$ is biholomorphic to $\mathbb{C}^{n+2}$ (see [DK98]), and is therefore a potential counterexample to the holomorphic version of the important Zariski Cancellation Problem: if X is a complex manifold of dimension n and $X \times \mathbb{C}$ biholomorphic to $\mathbb{C}^{n+1}$, does it follow that X is biholomorphic to $\mathbb{C}^n$?

Moreover, $\overline{\mathbb{C}^n_f}$ is diffeomorphic to complex affine space. This is best shown in the algebraic language of modifications, as follows. Given a triple (X, D, C) consisting of a Stein manifold X, a smooth reduced analytic divisor D, and a proper closed complex submanifold C of D, it is possible construct the pseudo-affine modification of X along D with center C, denoted $\overline{X}$. It is the result of blowing up X along C and deleting the proper transform of D. We refer the interested reader to [KZ99] for a general discussion. In our situation we take $X = \mathbb{C}^n \times \mathbb{C}_u$, $D = \mathbb{C}^n \times \{0\}$, and $C = Z \times \{0\} = \phi(\mathbb{C}^{n-1}) \times \{0\}$: it can be shown that in this case $\overline{X}$ is biholomorphic to $\overline{\mathbb{C}^n_f}$ (see [KZ99]). We now invoke a general result giving sufficient conditions for a pseudo-affine modification to be diffeomorphic to affine space: since Z is contractible, Proposition 5.10 from [KK08b] is directly applicable, and therefore the following holds:

Corollary 3.13. *If $\phi : \mathbb{C}^{n-1} \to \mathbb{C}^n$ is a proper holomorphic embedding, then the suspension $\overline{\mathbb{C}^n_f}$ along the function f defining the subvariety $\phi(\mathbb{C}^{n-1})$, is diffeomorphic to $\mathbb{C}^{n+1}$ and has the volume density property with respect to a natural volume form $\bar{\omega}$. Moreover $\overline{\mathbb{C}^n_f} \times \mathbb{C}$ is biholomorphic to $\mathbb{C}^{n+2}$, and is therefore a potential counterexample to the Zariski Cancellation Problem if ϕ is not straightenable.*

Recall a conjecture of A. Tóth and D. Varolin [TV06] asking whether a complex manifold which is diffeomorphic to $\mathbb{C}^n$ and has the density property must be biholomorphic to $\mathbb{C}^n$. It is also unknown whether there are contractible Stein manifolds with the volume density property which are not biholomorphic to $\mathbb{C}^n$, and our construction provides a new potential counterexample. As pointed out in Section 3.1, this is to our knowledge the first non-algebraic one.

To conclude, we give another example of an application. Consider a proper holomorphic embedding $\mathbb{D} \hookrightarrow \mathbb{C}^2_{x,y}$ (that this exists is a classical theorem of K. Kasahara and T. Nishino, see however [Ste72]), and let f generate the ideal of functions vanishing on the embedded disk, as above. Then $M = \overline{\mathbb{C}^2_f} \subset \mathbb{C}^2_{u,v} \times \mathbb{C}^2_{x,y}$

admits a $\mathbb{C}^*$-action, namely

$$\lambda \mapsto (\lambda u, \lambda^{-1} v, x, y),$$

whose fixed point set is biholomorphic to $\mathbb{D}$. Therefore, the action cannot be linearizable, i.e., there is no holomorphic change of coordinates after which the action is linear. Recall the problem of linearization of holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^k$ (see e.g. [DK98]): for $k = 2$, every action is linearizable; there are counterexamples for $k \geq 4$; and the problem remains open for $k = 3$. If M is biholomorphic to $\mathbb{C}^3$, there would be a negative answer. Otherwise, it resolves in the negative the Tóth-Varolin conjecture mentioned above. By a result of Globevnik [Glo97], it is also possible to embed arbitrary small perturbations of a polydisc in $\mathbb{C}^n$ for any $n \geq 1$ into $\mathbb{C}^{n+1}$; by the same argument, we obtain for any $n \geq 3$, non-algebraic manifolds that are diffeomorphic to $\mathbb{C}^n$ with the volume density property. We can also do the same for a contractible affine surface $\mathcal{R}$ which is not homeomorphic to $\mathbb{R}^4$, but such that $\mathcal{R} \times \mathbb{C}$ is an "exotic" $\mathbb{C}^3$ (meaning an algebraic manifold diffeomorphic, but not algebraically isomorphic, to $\mathbb{C}^3$). This has been called a Ramanujam surface, in reference to the explicit construction of C.P. Ramanujam in [Ram71]. Since then, many other such surfaces have been found. Arguing as above, if we can embed $\mathcal{R}$ in $\mathbb{C}^3$, we consider the suspension along a defining function of the embedded surface. Since $\mathcal{R}$ is affine, it is of finite type, so again Proposition 5.10 from [KK08b] we obtain an example of a Stein manifold diffeomorphic to $\mathbb{C}^4$, with the VDP and a non-linearizable $\mathbb{C}^*$-action.

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Bern, 03.05.2016

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