New quantum obstructions to sliceness

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Abstract

It is well known that generic perturbations of the complex Frobenius algebra used to define Khovanov cohomology each give rise to Rasmussen's concordance invariant s. This gives a concordance homomorphism to the integers and a strong lower bound on the smooth slice genus of a knot. Similar behavior has been observed in $\mathfrak{sl}(n)$ Khovanov–Rozansky cohomology, where a perturbation gives rise to the concordance homomorphisms s_n for each $n \geqslant 2$, and where we have $s_2 = s$.

We demonstrate that s_n for $n \geqslant 3$ does not in fact arise generically, and that varying the chosen perturbation gives rise both to new concordance homomorphisms and to new sliceness obstructions that are not equivalent to concordance homomorphisms.

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1. Introduction

1.1. History

In [**9**] Khovanov and Rozansky gave a way of associating, for each $n \geqslant 2$, a finitely generated bigraded complex vector space to a knot K. It arises as the cohomology of a cochain complex

$$\cdots \longrightarrow C_{x^n}^{i-1,j}(D) \longrightarrow C_{x^n}^{i,j}(D) \longrightarrow C_{x^n}^{i+1,j}(D) \longrightarrow \cdots$$

defined from any diagram D of K which is invariant under Reidemeister moves up to cochain homotopy equivalence. We write this vector space as

$$H_{x^n}^{i,j}(K),$$

and refer to $i \in \mathbb{Z}$ as the *cohomological* grading and $j \in \mathbb{Z}$ as the *quantum* grading. This bigraded vector space exhibits as its graded Euler characteristic

$$\sum_{i,j}(-1)^i q^j \dim_{\mathbb{C}} H_{x^n}^{i,j}(K)$$

the Reshetikhin–Turaev polynomial of K associated to the fundamental irreducible representation of $\mathfrak{sl}(n)$.

The reason for the subscript x^n in the notation is that in the definition of H a choice is made of a polynomial $w \in \mathbb{C}[x]$. Khovanov and Rozansky took $w = x^{n+1}$ as their polynomial but what is important for the definition is really the first derivative of w, and that only up to multiplication by a non-zero complex number. We record this renormalized first derivative in the subscript.

In fact there is a cohomology theory associated to each degree n monic polynomial $\partial w \in \mathbb{C}[x]$ (we write ∂w to remind of us of the connection with the first derivative) which we write as

$$H^i_{\partial w}(K).$$

We refer to ∂w as the *potential* of the cohomology theory. Note that the cohomology theory $H^i_{\partial w}$ keeps a cohomological grading but does not necessarily retain a quantum grading. However, for any choice of ∂w, there is at least a *quantum filtration* on the cohomology:

$$\cdots \subseteq \mathcal{F}^{j-1} H^i_{\partial w}(K) \subseteq \mathcal{F}^j H^i_{\partial w}(K) \subseteq \mathcal{F}^{j+1} H^i_{\partial w}(K) \subseteq \cdots,$$

arising from a filtration on the cochain complex associated to a diagram

$$\cdots \longrightarrow \mathcal{F}^j C^{i-1}_{\partial w}(D) \longrightarrow \mathcal{F}^j C^i_{\partial w}(D) \longrightarrow \mathcal{F}^j C^{i+1}_{\partial w}(D) \longrightarrow \cdots.$$

The filtered cochain homotopy type of the cochain complex was shown to be an invariant of K, as described by Wu [26].

We write the bigraded vector space associated to the filtration as

$$\mathrm{Gr}^j H^i(K) = \mathcal{F}^j H^i_{\partial w}(K)/\mathcal{F}^{j-1} H^i_{\partial w}(K).$$

Gornik was the first to consider a choice of ∂w different from x^n; he took $\partial w = x^n - 1$. In [7], Gornik showed that, for any diagram D of a knot, $H_{x^n-1}(D)$ is of dimension n and is supported in cohomological degree 0 and furthermore he observed that there is spectral sequence with E_1 page isomorphic to $H^{i,j}_{x^n}(K)$ and abutting to $\mathrm{Gr}^j H^i_{x^n-1}(D)$. Given a diagram D, the E_0 page of the spectral sequence can in fact be identified with the standard Khovanov–Rozansky cochain complex

$$\mathcal{F}^j C^i_{\partial w}(D)/\mathcal{F}^{j-1} C^i_{\partial w}(D) \equiv C^{i,j}_{x^n}(D).$$

This work of Gornik's can be considered a generalization of Lee's result in Khovanov cohomology [13] which essentially proved this for the case $n = 2$ (in work that predated the definition of Khovanov–Rozansky cohomology).

In works by the second author [18] and by Wu [26], this result of Gornik's was generalized to the case where ∂w has n distinct roots. Furthermore, the quantum gradings on the E_∞ pages of the associated spectral sequences were shown to give rise to lower bounds on the smooth slice genus of a knot.

These results should be thought of as a generalization of Rasmussen's seminal work [24]. This derived from Khovanov cohomology a combinatorial knot invariant $s(K) \in \mathbb{Z}$ and an associated lower bound $|s(K)|$ on the slice genus sufficiently strong to reprove Milnor's conjecture on the slice genus of torus knots (our normalization of s differs from Rasmussen's). We summarize:

THEOREM 1.1 (Gornik, Lobb, Wu).   *Suppose $\partial w \in \mathbb{C}[x]$ is a degree n polynomial which is a product of distinct linear factors and K is a knot. Then there is a spectral sequence, itself a knot invariant, with E_1 page $H^{i,j}_{x^n}(K)$ and abutting to $\mathrm{Gr}^j H^i_{\partial w}(K)$.*

Furthermore, $\mathrm{Gr}^j H^i_{\partial w}(K)$ is supported in cohomological degree $i = 0$ and is of rank n. We can write $j_1(K) \leqslant j_2(K) \leqslant \cdots \leqslant j_n(K)$ so that $\mathrm{Gr}^j H^i_{\partial w}(K)$ is isomorphic to the direct sum of n 1-dimensional vector spaces supported in bidegrees $(0, j_r)$.

If K_0 and K_1 are two knots connected by a connected knot cobordism of genus g, then

$$|j_r(K_0) - j_r(K_1)| \leqslant 2(n-1)g \quad \text{for } 1 \leqslant r \leqslant n.$$

It follows from this and knowing the cohomology of the unknot that we must have

$$g_*(K) \geqslant \frac{1}{2(n-1)} |j_r(K) - 2r + n + 1| \quad \text{for } 1 \leqslant r \leqslant n,$$

where we have written $g_*(K)$ for the slice genus of K.

The corresponding result in Khovanov cohomology, which can be thought of as the case $n = 2$ of Khovanov–Rozansky cohomology, admits a much neater formulation than that of Theorem 1.1. This is because of the work by Mackaay, Turner, and Vaz [21] who proved the following:

THEOREM 1.2 (Mackaay, Turner, Vaz). *Suppose we are in the situation of Theorem* 1.1 *with $n = 2$. Then we have that $j_1(K) = 2s(K) - 1$ and $j_2(K) = 2s(K) + 1$.*

It follows that, in the case $n = 2$, varying ∂w among quadratics with two distinct roots does not change the invariant $\mathrm{Gr}^j H^i_{\partial w}(K)$, which is always equivalent to Rasmussen's invariant $s(K)$.

For Gornik's prescient choice of ∂w the second author [20] showed that a similar 'neatness' result holds for general n.

THEOREM 1.3 (Lobb). *Taking $\partial w = x^n - 1$ we have that $j_r = 2(n-1)s_n(K) - n + 2r - 1$ for some knot invariant $s_n(K)$. Furthermore, s_n is a homomorphism from the smooth concordance group of knots to the integers $(1/(n-1))\mathbb{Z}$.*

As in the case $n = 2$, this Theorem 1.3 shows that $\mathrm{Gr}^j H^i_{x^n-1}(K)$ is bigraded isomorphic to the cohomology of the unknot but shifted in the quantum direction by an integer $2(n-1)s_n(K)$.

Taken with computations in [18, 26], Theorem 1.3 demonstrates that s_n is a *slice-torus* invariant (in that it is a concordance homomorphism and its absolute value provides a bound on the smooth slice genus which furthermore is tight for all torus knots). This establishes shared properties of s_n with Rasmussen's invariant $s = s_2$ and with the invariant τ arising from knot Floer homology. The first author showed that these invariants are not all equal [16], and in fact it seems probable that $\{\tau, s_2, s_3, \ldots\}$ is an infinite family of linearly independent invariants.

However, s_n do *not* comprise all slice genus bounds obtainable from separable potentials! In the light of Theorem 1.2 it might be guessed that the integers $j_r(K, \partial w)$ of Theorem 1.1 are in fact each equivalent to the single integer $s_n(K)$ in the sense of Theorem 1.3. This guess is wrong.

In fact we shall see that for $n \geqslant 3$, two different degree n separable potentials can induce different filtrations on the unreduced cohomology. These filtrations do give rise to slice genus lower bounds, but not in general to concordance homomorphisms (see Question 4.4). However, a separable potential ∂w and a choice of a root α of that potential gives a *reduced* cohomology theory from which one can extract a slice-torus concordance homomorphism. In this way we shall recover the classical s_n as well as a host of new invariants.

One may compare the results in this paper with the recent results due to Ozsváth, Stipsicz and Szabó [22] in which they determine that varying the filtration on Knot Floer homology gives rise to a number of different concordance homomorphisms. One may consider their family of homomorphisms to be obtained by varying the slope of a linear function, while ours are obtained by varying all coefficients of a degree n polynomial.

A relatively simple knot exhibiting interesting cohomologies for different choice of potential is the knot 10_{125}. We invite the reader to spend the next subsection exploring this knot.

FIGURE 1 (colour online). *The pretzel knot $P(2, -3, 5)$ prior to baking (thanks to Kate Horner and Lauren Scanlon for the image).*
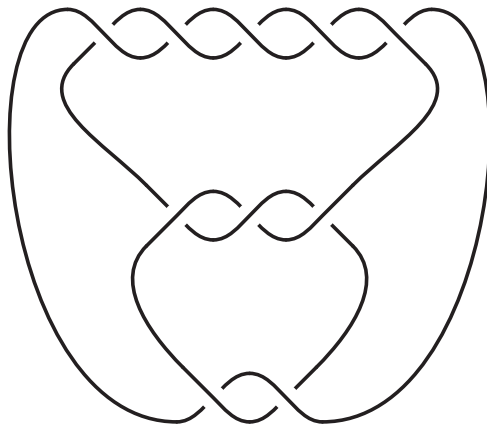

FIGURE 2. *A less appetizing diagram of the pretzel knot $P(2, -3, 5)$.*

1.2. *An appetizing example*

The pretzel knot $P(2, -3, 5)$ appears in the knot table as 10_{125}, and we shall refer to this knot as P for the remainder of this subsection (Figure 1 and 2).

In Tables 1 and 2 we give the reduced and unreduced Khovanov–Rozansky cohomologies of P for $n = 2$ and $n = 5$ (there is no particular reason to choose 5 over some other integer, but we just want to be explicit). We encourage the reader to get her hands dirty with a few spectral sequences starting from these cohomologies in order to appreciate something of the phenomena discussed in this paper.

Suppose, for example, that we want to apply Corollary 2.5 in order to compute $s_2(P)$ and $s_5(P)$ from the reduced cohomologies. We are looking for spectral sequences starting from E_1-pages the reduced cohomologies of Figures 1 and 2, and which have as their final pages 1-dimensional cohomologies supported in cohomological degree 0. The differentials on the page E_i increase the cohomological grading by 1 and decrease the quantum grading by $2i$.

In Table 1 we give the only possible spectral sequence from $E_1 = \widetilde{H}_{x^2}(K)$ to a 1-dimensional E_∞ page supported in cohomological degree 0, but in the other figure the reader will discover two such *a priori* possible E_∞ pages starting from $E_1 = \widetilde{H}_{x^5}(K)$.

There is better luck to be had in using the unreduced spectral sequences of Theorem 1.3. In the unreduced case the final page is again supported in cohomological degree 0, but now it is

TABLE 1. *Unreduced and reduced (the latter printed in yellow boxes) x^2-cohomology of the $(2, -3, 5)$-pretzel knot. Non-trivial differentials on the second page of the spectral sequence associated to reduced cohomology with a separable potential are drawn as arrows.*

q \ t	-3	-2	-1	0	1	2	3	4	5
-9									1
-8									1
-7									
-6								1	
-5							1	1	
-4							1		
-3						1			
-2						2			
-1						1			
0					1				
1				2	1				
2				2					
3				1					
4			1						
5		1	1						
6		1							
7									
8	1								
9	1								

of dimension n (so 2 or 5 in the cases under consideration). Furthermore, the only non-trivial differentials in the spectral sequence decrease the quantum grading by multiples of $2n$.

From this we can observe that $s_2(P) = 1$ and $s_5(P) = \frac{1}{4}$. The question arises: how far is it accidental that we were unable to compute $s_5(P)$ merely from looking at $\widetilde{H}_{x^5}(K)$? It turns out that this failure was inevitable once we determined that $s_5(P)$ is nonintegral, as we shall see later in Section 2.3.

We ask the reader to return to the unreduced cohomology of Table 2. Now look for spectral sequences from this E_1 page in which all nontrivial differentials decrease the quantum grading by multiples of $2(n-1) = 8$, and in which the final page is again of dimension $n = 5$ supported in cohomological degree 0. Whichever spectral sequence of this kind one finds, the final page never has the appearance of a shifted unknot as in Theorem 1.3. Such a spectral sequence would arise from the potential $\partial w = x^5 - x$ (demonstrating, for example, the nonvalidity of the Theorem 1.3 for this new choice of separable potential).

Finally, consider again the reduced cohomology of Table 2, and look for a spectral sequence in which all non-trivial differentials decrease the quantum grading by multiples of $2(n-1) = 8$ and the final page is of dimension 1 and is supported in cohomological degree 0. There is exactly one such spectral sequence for the knot in question.

In general, given a choice of degree n separable potential ∂w and a root α of that potential, there is a corresponding spectral sequence from reduced $\mathfrak{sl}(n)$ cohomology to a 1-dimensional final page supported in cohomological degree 0. In this particular case, the spectral sequence corresponds to the separable potential $x^5 - x$ and the choice of root $x = 0$.

Furthermore, the surviving quantum degree, written as $2(n-1)\widetilde{s}_{x^5-x,0}(K)$, gives a slice-torus knot invariant $\widetilde{s}_{x^5-x,0}$ generalizing s_5. Note that for the knot in question

TABLE 2. *Unreduced and reduced (the latter printed in yellow boxes) x^5-cohomology of the $(2, -3, 5)$-pretzel knot.*

q \ t	−3	−2	−1	0	1	2	3	4	5
−18									1
−16									1
−14							1		[1] 1
−12							1		1
−10					1		[1] 1		
−8					1		1	1	
−6					[1] 1	1		[1] 1	
−4						2		1	
−2				2		[2] 2		1	
0				[1] 3		2			
2		1		[2] 3		1			
4		1		3					
6		[1] 1			1	[1] 1			
8		1	1			1			
10			[1] 1			1			
12	1		1						
14	[1] 1		1						
16	1								
18	1								

we have

$$\widetilde{s}_{x^5-x,0}(P) = 0 \neq \tfrac{1}{4} = s_5(P).$$

We shall revisit the knot P in Section 2.3 where we shall shine more light on the concrete phenomena observed above.

1.3. Summary

In Section 2, we give the definitions and prove the basic properties of the slice genus lower bounds coming from separable potentials; in particular, it is shown that not only unreduced, but also reduced Khovanov–Rozansky cohomologies induce lower slice genus bounds, which are actually more well behaved than the unreduced bounds: they are all concordance homomorphisms (in particular, slice-torus invariants). We close by reanalyzing the example of the pretzel knot $P = P(2, -3, 5)$ in the light of the properties we have established. We expect that these results generalize to slice genus bounds for multi-component links, in the appropriate sense; but for the sake of simplicity we restrict ourselves to knots.

Section 3 introduces the notion of KR-equivalent potentials: potentials inducing homotopy equivalent filtered cochain complexes for all links. We show that there are at most countably many KR-equivalence classes, and that one of them is generic. By analyzing the cohomology of the trefoil, we establish that there are at least $n - 1$ KR-equivalence classes.

Section 4 exhibits further characteristics of the sliceness obstructions, which are much more complex than one would have reasonably guessed from what was previously known.

Section 5 discusses the simple form of the cochain complexes of bipartite knots, and the program khoca (**k**not **ho**mology **ca**lculator) that calculates their Khovanov–Rozansky cohomologies.

1.4. *Conventions*

For the most part we shall follow the conventions of [**9**]. These amount to choosing the degree of the variable x to be 2, and deciding in which cohomological degrees the complex associated to a positive crossing $\times$ is supported (in degrees 0 and 1). These choices have the consequence that the cohomology of the positive trefoil is supported in nonnegative cohomological degrees but in *negative* quantum degrees. This negative quantum support is in contrast to the situation of the normalization of standard Khovanov cohomology. Since we encounter Khovanov cohomology only as the case $n = 2$ of Khovanov–Rozansky cohomology, we are going to be normalizing the Rasmussen invariant $s = s_2$ so that it is negative on the positive trefoil.

2. *The slice genus lower bounds from separable potentials*

2.1. *Reduced cohomology and slice-torus invariants*

Given a knot K with marked diagram D, the Khovanov–Rozansky cohomology $H_{\partial w}(K)$ has the structure of a module over the ring $\mathbb{C}[x]/\partial w$. In fact, it is the cohomology of a cochain complex $C_{\partial w}(D)$ of free modules over $\mathbb{C}[x]/\partial w$.

This statement is best visualized by cutting the diagram D open at a point marked with the decoration x and thus presenting D as a $(1,1)$-tangle. Using Murakami-Ohtsuki-Yamada (MOY) moves, each cochain group can then be identified with finite sums of quantum-shifted matrix factorizations corresponding to the crossingless $(1,1)$-tangle. Closing all of these trivial tangles gives the complex associated to the uncut diagram D, and each circle now appearing corresponds to a copy of $\mathbb{C}[x]/\partial w$.

This module structure seems at first sight as if it may have some dependence on the choices of diagram and of marked point. However, if T is a tangle with endpoints labeled $x_1, x_2, \ldots, x_r$, then the Khovanov–Rozansky functor gives a complex of (vectors of) matrix factorizations over the ring $\mathbb{C}[x_1, x_2, \ldots, x_r]$. Reidemeister moves on T give homotopy equivalent complexes via homotopy equivalences respecting the ground ring. As a consequence, the $\mathbb{C}[x]/\partial w$-module structure on the Khovanov–Rozansky cohomology is invariant under Reidemeister moves performed on the $(1,1)$-tangle. Finally, it is an exercise for the reader to see that if D and D' are two Reidemeister-equivalent diagrams (each with a marked point on corresponding link components), then D and D' can be connected by a sequence of Reidemeister moves that take place away from the marked points and which take the marked point of D to that of D'.

In the case of standard Khovanov–Rozansky cohomology with $\partial w = x^n$, the action of x on the cochain complex preserves the cohomological grading and raises the quantum grading by 2. For explicitness we make a definition.

DEFINITION 2.1. We define the reduced Khovanov–Rozansky cohomology $\widetilde{H}_{x^n}(K)$ of a knot K to be the cohomology of the cochain complex $(x^{n-1})C_{x^n}(D)[1 - n]$, where the closed brackets denote a shift in quantum filtration.

The reduced Khovanov–Rozansky cohomology $\widetilde{H}_{x^n}$ has as its graded Euler characteristic the Reshetikhin–Turaev $\mathfrak{sl}(n)$ polynomial normalized so that the unknot is assigned the polynomial $1 \in \mathbb{Z}[q^{\pm 1}]$.

REMARK 2.2. We note that in the literature the reduced Khovanov cohomology, for example, is often defined in such a way that its graded Euler characteristic is the Jones polynomial with a surprising normalization: the unknot is assigned polynomial $q^{-1} \in \mathbb{Z}[q^{\pm 1}]$. We consider our convention to be possibly a little more natural.

We now wish to give a good definition for a reduced Khovanov–Rozansky cohomology of a knot K using a separable potential ∂w, that is, a potential that is the product of distinct linear factors:

$$\partial w = \prod_{i=1}^{n}(x - \alpha_i).$$

For any marked diagram D of K, $H_{\partial w}(K)$ is the cohomology of a cochain complex $C_{\partial w}(D)$ of free $(\mathbb{C}[x]/\partial w)$ modules, inducing a $(\mathbb{C}[x]/\partial w)$-module structure on $H_{\partial w}(K)$. In fact, we know that $H_{\partial w}(K)$ is n-dimensional and that the action of x on $H_{\partial w}(K)$ splits the cohomology into n 1-dimensional eigenspaces with eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$. In other words, $H_{\partial w}(K)$ is a free rank 1 module over the ring $\mathbb{C}[x]/\partial w$. The reader should note, however, that the quantum filtration of $H_{\partial w}(K)$ need not correspond to an overall shift of the usual filtration on $\mathbb{C}[x]/\partial w$.

DEFINITION 2.3.   Suppose that α is a root of the degree n monic separable polynomial ∂w. We define $\widetilde{H}_{\partial w, \alpha}(K)$ (the $(\partial w, \alpha)$-reduced cohomology of the knot K with marked diagram D) to be the cohomology of the cochain complex

$$\widetilde{C}_{\partial w, \alpha}(D) := \left(\frac{\partial w}{x - \alpha} \right) C_{\partial w}(D)[1 - n],$$

where the square brackets denote a shift in the quantum filtration.

First note that $\widetilde{H}_{\partial w, \alpha}(K)$ is certainly a knot invariant, this follows from a similar, but not totally isomorphic, discussion to that appearing at the start of this section: the cochain complex $\widetilde{C}_{\partial w, \alpha}(D)$ is a quantum-shifted subcomplex of $C_{\partial w}(D)$. If D and D' are marked-Reidemeister-equivalent marked diagrams (and the Reidemeister moves take place away from the marked point), then $C_{\partial w}(D)$ and $C_{\partial w}(D')$ are cochain homotopy equivalent $\mathbb{C}[x]/\partial w$-cochain complexes (where the x corresponds to the marked point). The homotopy equivalences can then be restricted to the subcomplexes $\widetilde{C}_{\partial w, \alpha}(D)$ and $\widetilde{C}_{\partial w, \alpha}(D')$ with no modification (this is really an argument for a general ring R about subcomplexes of R-complexes given by the action of an ideal of R).

We note that one should *not* expect, in general, that $\widetilde{H}_{\partial w, \alpha}(K)$ is filtered-isomorphic to $(\partial w/(x - \alpha))H_{\partial w}(K)[1 - n]$ (which is also a knot invariant). Indeed, we shall see examples where it certainly differs.

The shift in the quantum degree is to ensure that $\widetilde{H}_{\partial w, \alpha}(U)$ has Poincaré polynomial 1 for U the unknot and any choice of $(\partial w, \alpha)$. We shall show

THEOREM 2.4.   *For any knot K and for each separable choice of $(\partial w, \alpha)$, the reduced cohomology $\widetilde{H}_{\partial w, \alpha}(K)$ is 1-dimensional. Furthermore, there exists a spectral sequence with E_1-page $\widetilde{H}_{x^n}^{i,j}(K)$ and E_∞-page $\mathrm{Gr}^j \widetilde{H}_{\partial w, \alpha}^i(K)$.*

Taking Gornik's choice of potential, we obtain a corollary.

COROLLARY 2.5.   *For any knot K there exists a spectral sequence with E_1-page $\widetilde{H}_{x^n}(K)$ such that the E_∞-page is 1-dimensional and has Poincaré polynomial $q^{2(n-1)s_n(K)} \in \mathbb{Z}[q, t]$.*

This corollary is not surprising, and may even be considered 'folklore', but as far as we know there is no proof in the literature.

Since the reduced cohomology for a separable potential is always 1-dimensional we can make another definition.

DEFINITION 2.6. For a knot K and $(\partial w, \alpha)$ as above, we define the $(\partial w, \alpha)$ *reduced slice genus bound* $\widetilde{s}_{\partial w, \alpha}(K) \in (1/(n-1))\mathbb{Z}$ to be $\frac{1}{2}(n-1)$ times the j-grading of the support of the 1-dimensional vector space $\mathrm{Gr}^j \widetilde{H}^0_{\partial w, \alpha}(K)$

We have taken this choice of normalization so that we have

$$\widetilde{s}_{\partial w, \alpha}(T(2,3)) = s_n(T(2,3)) = -1,$$

for any choice of n and of $(\partial w, \alpha)$ where we write $T(2,3)$ for the positive trefoil.

DEFINITION 2.7 (cf. [**16**, **17**]). Let $S : \mathcal{C} \to \mathbb{R}$ be a homomorphism from the smooth concordance group of oriented knots to the reals. We say that S is a *slice-torus invariant* if

(1) $g_*(K) \geqslant |S(K)|$ for all oriented knots K, where we write $g_*(K)$ for the smooth slice genus of K.
(2) $S(T(p,q)) = -(p-1)(q-1)/2$ for $T(p,q)$ the (p,q)-torus knot.

THEOREM 2.8. *Suppose that α is a root of the degree n monic separable polynomial ∂w. Then we have that $\widetilde{s}_{\partial w, \alpha}$ defines a map*

$$\widetilde{s}_{\partial w, \alpha} : \mathcal{C} \longrightarrow \frac{1}{2(n-1)}\mathbb{Z}$$

which is a slice-torus invariant.

Before proving Theorems 2.4 and 2.8, we remind the reader of the work of Gornik's [**7**] which established cocycle representatives for cohomology with a separable potential. (In fact Gornik considered only the potential $\partial w = x^n - 1$, but his arguments apply to all separable potentials without critical change.)

Fix any separable ∂w with roots $\alpha_1, \ldots, \alpha_n$, and let Γ be a MOY graph. The cohomology of Γ, which we shall write simply as $h_{\partial w}(\Gamma)$, is a filtered complex vector space. If Γ occurs as a resolution of some link diagram D, then $h_{\partial w}(\Gamma)$ appears on a corner of the Khovanov–Rozansky cube as a cochain group summand of $C_{\partial w}(D)$.

A basis for $h_{\partial w}(\Gamma)$ is given by all *admissible decorations* of Γ, that is, all decorations of the thin edges of Γ with roots of ∂w satisfying the admissibility condition. This condition is the requirement that at each thick edge two distinct roots decorate each entering thin edge, and the same two roots decorate the exiting thin edges.

If we let Γ vary over all resolutions of a diagram D, we thus then obtain a basis for each cochain group $C^i_{\partial w}(D)$. By considering how the Khovanov–Rozansky differential acts on the bases, Gornik showed that a basis for the cohomology $H_{\partial w}(D)$ is given by cocycles corresponding to decorations that arise in the following way. Starting with a decoration by roots of ∂w of the components of D, take the oriented resolution at crossings where the roots agree and the thick-edge resolution at crossings where the roots differ. For each such decoration of components of D by roots, this produces a resolution Γ and a cocycle in $h_{\partial w}(\Gamma) \subseteq C_{\partial w}(D)$ surviving to cohomology. In the case that D is a diagram of a (1-component) knot, it follows that Gornik's cocycle representatives live in the summand of the cohomological degree 0 cochain group corresponding to the oriented resolution of D.

Essentially, Gornik's argument proceeded by Gauss elimination, grouping other basis elements into canceling pairs and leaving only the generating cocycles described above.

LEMMA 2.9 (Gauss Elimination as in [**2**]).   *In an additive category with an isomorphism h, the cochain complex*

$$\cdots \longrightarrow P \xrightarrow{\binom{*}{g}} Q \oplus R \xrightarrow{\left(\begin{smallmatrix} h & i \\ j & k \end{smallmatrix}\right)} S \oplus T \xrightarrow{(* \ \ell)} U \longrightarrow \cdots$$

is homotopy equivalent to

$$\cdots \longrightarrow P \xrightarrow{\ g\ } R \xrightarrow{k - jh^{-1}i} T \xrightarrow{\ \ell\ } U \longrightarrow \cdots.$$

More explicitly, we describe the situation in the case of knots. Let D be any knot diagram, and let $O(D)$ be the oriented resolution of D. We define

$$h_{\partial w}(O(D)) := \left(\bigotimes_c \mathbb{C}[x_c]/\partial w(x_c)[1-n] \right) [(1-n)w(D)]$$

where the tensor product is taken over all components c of $O(D)$ and $w(D)$ stands for the writhe of D. Now $h_{\partial w}(O(D))$ is naturally a summand of the cochain group $C^0_{\partial w}(D)$. Let

$$\partial w = \prod_{i=1}^{n} (x - \alpha_i),$$

then for any knot diagram D there exist n linearly independent cocycles

$$g^D_{\alpha_i} \in h_{\partial w}(O(D)) \subseteq C^0_{\partial w}(D) \quad \text{for } 1 \leqslant i \leqslant n$$

given by

$$g^D_{\alpha_i} = \bigotimes_c \frac{1}{\partial w'(\alpha_i)} \left(\frac{\partial w(x_c)}{x_c - \alpha_i} \right).$$

THEOREM 2.10 (Gornik [**7**]).   *These cocycles $g^D_{\alpha_i}$ descend to give a basis for the cohomology $H^0_{\partial w}(D)$. Furthermore, there is a spectral sequence with E_1-page $H^{i,j}_{x^n}(D)$ abutting to $Gr^j H^i_{\partial w}(D)$.*

Gornik's proof of the existence of the spectral sequence relied on identifying the E_0-page of the spectral sequence associated to the filtered complex $C_{\partial w}(D)$ with the Khovanov–Rozansky cochain complex $C_{x^n}(D)$.

Now we have enough to prove the theorems stated earlier.

Proof of Theorem 2.4.   First note that Gornik's proof that the unreduced cohomology $H_{\partial w}(K)$ is n-dimensional also demonstrates that the reduced $\widetilde{H}_{\partial w, \alpha}$ cohomology is 1-dimensional. To see this, observe that the reduced cohomology is the cohomology of a subcomplex $\widetilde{C}_{\partial w, \alpha}(D)$ of $C_{\partial w}(D)[1-n]$, where D is a marked diagram for K. The subcomplex is spanned by exactly $1/n$ of Gornik's generators for $C_{\partial w}(D)[1-n]$ (those with the decoration α at the marked thin edge). These generators can be Gauss-eliminated following Gornik's recipe, leaving just one cocycle

$$g^D_\alpha \in \widetilde{C}_{\partial w, \alpha}(D) \subseteq C_{\partial w}(D)[1-n]$$

which generates the cohomology.

To show the existence of the spectral sequence we wish to see that the E_0-page associated to $\widetilde{C}_{\partial w, \alpha}(D)$ corresponds exactly to the complex $x^{n-1}C_{x^n}(D)$ which computes standard reduced cohomology. The invariance of the spectral sequence under choice of diagram is automatic: the spectral sequence in question is just that associated to the filtered complex $\widetilde{C}_{\partial w, \alpha}(D)$, whose filtered homotopy type we know is independent of the diagram.

We shall proceed by using the restriction of the correspondence between the E_0-page of the spectral sequence associated to the filtered complex $C_{\partial w}(D)$ with the Khovanov–Rozansky cochain complex $C_{x^n}(D)$.

Let Γ be a resolution of D, inheriting the marked point of D. Then $h_{x^n}(\Gamma)$ is a $\mathbb{C}[x]/x^n$-module, and $h_{\partial w}(\Gamma)$ is a $\mathbb{C}[x]/\partial w$-module. To see the module structures, perform MOY decompositions away from the marked point until you arrive at a single marked circle. This gives the module structures explicitly as

$$h_{x^n}(\Gamma) = \bigoplus_i (\mathbb{C}[x]/x^n)[a_i]$$

and

$$h_{\partial w}(\Gamma) = \bigoplus_i (\mathbb{C}[x]/\partial w)[a_i]$$

for a finite sequence of integers $\{a_i\}$ where the first decomposition is as graded modules, and the second as filtered modules.

The conclusion we want is almost clear from this, we just need to say a bit more about the module structures.

From the definition by matrix factorizations, $h_{\partial w}(\Gamma)$ is the cohomology of a 2-periodic complex of free $\mathbb{C}[x]$-modules where the modules are graded and the differentials are filtered. On the other hand, $h_{x^n}(\Gamma)$ arises as the cohomology of the 2-periodic complex with the same cochain groups but where just the top-degree components of the differential are retained.

For any $\psi \in h_{\partial w}(\Gamma)$, write $\psi' \in h_{x^n}(\Gamma)$ for the associated graded element. Then observe that if $q(x)$ is a polynomial in x with leading term x^r, we have $(q(x)\psi)' = x^r\psi'$ if and only if they are of the same grading.

Now it is clear that the associated graded vector space to $(\partial w/(x - \alpha))h_{\partial w}(\Gamma)$ is exactly $x^{n-1}h_{x^n}(\Gamma)$. $\qquad \square$

Now we prove the second theorem.

Proof of Theorem 2.8. Firstly we show that $|\widetilde{s}_{\partial w,\alpha}(K)|$ is a lower bound for the slice genus of K. We make use of the arguments already given in the unreduced case by Lobb and Wu, which we briefly summarize here.

To each Morse move (otherwise known as handle attachment), from a diagram D to a diagram D', there is associated a cochain map $C_{\partial w}(D) \to C_{\partial w}(D')$. This cochain map is filtered of degree $n - 1$ in the case of a 1-handle attachment, and of degree $1 - n$ for 0- and 2-handle attachments. Taking these together with the homotopy equivalences between Reidemeister equivalent cochain complexes gives a way to associate a filtered cochain map to a representation of a link cobordism. Specifically, given a movie M of a cobordism between diagrams D^0 and D^1, by composing the cochain maps we already have, we can thus associate a filtered cochain map

$$M_* : C_{\partial w}(D^0) \longrightarrow C_{\partial w}(D^1).$$

Adding up the contributions from the various maps, we observe that this map is filtered of degree $(1 - n)\chi(M)$ where we write $\chi(M)$ for the Euler characteristic of the surface represented by M.

Finally, one shows that if M is a movie of a connected cobordism between the two knot diagrams D^0 and D^1, then $M_*(g_{D^0_\alpha})$ is a non-zero multiple of $g_{D^1_\alpha}$ for α any root of ∂w. Hence, we see that M_* induces an isomorphism on cohomology.

The slice genus bound statements in Theorem 1.1 follow immediately.

For the reduced statement, we note that the cochain maps on the unreduced cochain complexes induced by handle moves and Reidemeister moves on marked diagrams respect

the $\mathbb{C}[x]/\partial x$ structure of the cochain groups. Hence by restriction they also induce maps on the reduced cochain complexes.

We now take a marked movie M (which can be thought of as describing a cobordism together with an embedded arc without critical points) between the marked diagrams D^0 and D^1. This induces a map

$$M_* : \widetilde{C}_{\partial w, \alpha}(D^0) \longrightarrow \widetilde{C}_{\partial w, \alpha}(D^1),$$

again of filtered degree $(1 - n)\chi(M)$. Note, furthermore, that the generator of the cohomology is preserved since this map is just a restriction of the unreduced map which does preserve the generator.

This dispenses with the question of whether $|\widetilde{s}_{\partial w, \alpha}(K)|$ is a lower bound for the slice genus of K. That this bound is tight for torus knots (and in fact for all positive knots) is immediate from consideration of a positive diagram.

Finally we show that $\widetilde{s}_{\partial w, \alpha}$ is a concordance homomorphism. From the 0-crossing diagram of the unknot U we know that $\widetilde{s}_{\partial w, \alpha}(U) = 0$, it therefore remains to show that $\widetilde{s}_{\partial w, \alpha}(K_1 \# K_2) = \widetilde{s}_{\partial w, \alpha}(K_1) + \widetilde{s}_{\partial w, \alpha}(K_2)$, where we write $\#$ for the connect-sum operation.

Let then D_1 and D_2 be two marked diagrams, and let $D = D_1 \# D_2$ be the marked diagram formed by the connect sum, with the connect sum taking place at the marked point.

We write Φ for the map

$$\Phi : C_{\partial w}(D_1) \otimes_{\mathbb{C}} C_{\partial w}(D_2) \longrightarrow C_{\partial w}(D)$$

induced by 1-handle addition to the diagram $D_1 \sqcup D_2$. Let Γ_1 and Γ_2 be MOY resolutions of D_1 and D_2, respectively, and let $\Gamma = \Gamma_1 \# \Gamma_2$ be the corresponding resolution of D.

By repeated MOY simplification away from the marked points, we can reduce $\Gamma_1 \sqcup \Gamma_2$ to the disjoint union of two marked circles U_1 and U_2. Performing the same MOY simplification to Γ, we can reduce to the marked circle $U_1 \# U_2$. Thus for $\{a_i\}$ being a finite sequence of integer shifts we see that Φ restricted to the cochain group summand $S := h_{\partial w}(\Gamma_1) \otimes_{\mathbb{C}} h_{\partial w}(\Gamma_2)$ is the map

$$\Phi|_S : \bigoplus_i (\mathbb{C}[x_1, x_2]/(\partial w(x_1), \partial w(x_2)))[a_i] \longrightarrow \bigoplus_i (\mathbb{C}[x]/\partial w(x))[a_i]$$

given by 'multiplication' or, in other words, the identification $x_1 = x_2 = x$.

Restricting Φ to the shifted subcomplex $\widetilde{C}_{\partial w, \alpha}(D_1) \otimes \widetilde{C}_{\partial w, \alpha}(D_2)$, the restriction to the corresponding summand

$$\widetilde{S} := \frac{\partial w(x_1)}{x_1 - \alpha} h_{\partial w}(\Gamma_1) \otimes \frac{\partial w(x_2)}{x_2 - \alpha} h_{\partial w}(\Gamma_2)[2 - 2n]$$

is projectively the map

$$\Phi|_{S'} : \bigoplus_i \frac{\partial w(x_1)}{x_1 - \alpha} \frac{\partial w(x_2)}{x_2 - \alpha}(\mathbb{C}[x_1, x_2]/(\partial w(x_1), \partial w(x_2)))[2 - 2n + a_i]$$

$$\longrightarrow \bigoplus_i \frac{\partial w(x)}{x - \alpha}(\mathbb{C}[x]/\partial w(x))[1 - n + a_i]$$

again given by multiplication.

This is a filtered degree 0 isomorphism of vector spaces with filtered degree 0 inverse, and hence Φ restricts to an isomorphism of filtered cochain complexes $\widetilde{C}_{\partial w, \alpha}(D_1) \otimes \widetilde{C}_{\partial w, \alpha}(D_2)$ and $\widetilde{C}_{\partial w, \alpha}(D)$, and we are done. □

To deduce Corollary 2.5 we first prove some results of independent interest that relate unreduced cohomology to the reduced concordance homomorphisms.

PROPOSITION 2.11. *Let D be a diagram of the knot K, let α be a root of ∂w, and let $g_\alpha \in C^0_{\partial w}(D)$ be the Gornik cocycle corresponding to α.*
Then the filtration degree q of $[g_\alpha] \in H^0_{\partial w}(D)$ satisfies

$$q \leqslant s_{\partial w, \alpha}(K) + n - 1.$$

Proof. The degree q is the smallest filtration degree among elements of the coset $g_\alpha + d(C^{-1}(D))$ (where we write d for the differential in the Khovanov–Rozansky cochain complex). But the reduced cohomology also has g_α as a cocycle representative. From this it follows that $s_{\partial w, \alpha}(K) + n - 1$ is the smallest filtration degree among elements of the coset $g_\alpha + [\partial w/(x - \alpha)]d(C^{-1}(D))$. This latter coset is a subset of the former coset, from which the result follows. $\square$

We can also compare the filtration on the entire unreduced cohomology with the collection of n slice genus bounds corresponding to each root of ∂w. The next proposition implies in particular that the bounds arising from unreduced cohomology and the unreduced bounds differ by at most 1.

PROPOSITION 2.12. *Let K be a knot. We have $H_{\partial w}(K) = q^{j_1(K)} + \cdots + q^{j_n(K)}$ with $j_1(K) \leqslant \cdots \leqslant j_n(K)$ as in Theorem 1.1. Sort the roots $\alpha_1, \ldots, \alpha_n$ of ∂w such that $\widetilde{s}_{\partial w, \alpha_1}(K) \leqslant \cdots \leqslant \widetilde{s}_{\partial w, \alpha_n}(K)$. Then we have*

$$|j_i(K) - 2(n-1)\widetilde{s}_{\partial w, \alpha_i}(K)| \leqslant n - 1.$$

Proof. Multiplication by $\partial w/(x - \alpha_i)$ gives a filtered map of degree $2n - 2$

$$C_{\partial w}(D) \longrightarrow \frac{\partial w}{x - \alpha_i} C_{\partial w}(D),$$

in other words a filtered map

$$C_{\partial w}(D) \longrightarrow \widetilde{C}_{\partial w, \alpha_i}(D)[1 - n].$$

Summing over i yields a filtered map

$$C_{\partial w}(D) \longrightarrow \bigoplus_{i=1}^{n} \widetilde{C}_{\partial w, \alpha_i}(D)[1 - n].$$

This induces a bijective filtered map on cohomology (the inverse is not necessarily filtered, too)

$$H_{\partial w}(K) \longrightarrow \bigoplus_{i=1}^{n} \widetilde{H}_{\partial w, \alpha_i}(K)[1 - n],$$

since it takes generating cocycles to generating cocycles. Hence we get for each $i \in \{1, \ldots, n\}$:

$$j_i(K) \geqslant 2(n-1)\widetilde{s}_{\partial w, \alpha_i}(K) + 1 - n$$

or equivalently

$$2(n-1)\widetilde{s}_{\partial w, \alpha_i}(K) - j_i(K) \leqslant n - 1.$$

To complete the proof, one could now resort to taking the mirror image of D. But in fact, it suffices to consider the inclusion map

$$\frac{\partial w}{x - \alpha_i} C_{\partial w}(D) \longrightarrow C_{\partial w}(D),$$

which is filtered, and sum over i to produce a bijective filtered map

$$\bigoplus_{i=1}^{n} \widetilde{C}_{\partial w, \alpha_i}(D)[n - 1] \longrightarrow C_{\partial w}(D),$$

which also gives a bijective filtered map on cohomology. Thus we have

$$2(n-1)\widetilde{s}_{\partial w,\alpha_i}(K) + n - 1 \geqslant j_i(K) \implies j_i(K) - 2(n-1)\widetilde{s}_{\partial w,\alpha_i}(K) \leqslant n - 1.$$

and the proof is complete. $\square$

Finally it is now a simple matter to deduce Corollary 2.5.

Proof of Corollary 2.5. The homomorphism $\tilde{s}_{x^n-1,\alpha}$ is independent of the choice of root α since there exist linear maps of $\mathbb{C}$ which cyclically permute the roots. By Proposition 2.12, it follows that $2(n-1)\tilde{s}_{x^n-1,\alpha}(K)$ is a value at distance at most $n-1$ from each of $j_1(K), \ldots, j_n(K)$. Since $j_n(K) - j_1(K) = 2(n-1)$, this determines $\tilde{s}_{x^n-1,\alpha}(K)$ completely and we have

$$\tilde{s}_{x^n-1,\alpha}(K) = \frac{j_n(K) - j_1(K)}{4(n-1)} = s_n(K).$$

$\square$

2.2. *Unreduced cohomology*

In this subsection we consider the unreduced theory which is in a sense richer than the reduced theory. We still obtain slice genus lower bounds, but in general we give up the property of defining a concordance homomorphism, although we shall be able to define a concordance quasi-homomorphism.

We fix a separable potential ∂w and recall the definition of the integers $j_i(K)$ from Theorem 1.1 describing the filtration on $H_{\partial w}(K)$. Now, since the complex associated to the mirror image of a diagram is the dual complex, the invariants are still well behaved with respect to the mirror image, that is to say:

PROPOSITION 2.13. *For all $i \in \{1, \ldots, n\}$ we have*

$$j_i(\overline{K}) = -j_{n-i}(K).$$

However, the filtration on the unreduced cohomology $H_{\partial w}(K_1 \# K_2)$ is not in general determined by those on $H_{\partial w}(K_1)$ and on $H_{\partial w}(K_2)$, see Question 4.7. Still, some bounds can be given:

PROPOSITION 2.14. *For knots K_1 and K_2 we have*

$$j_1(K_1) + j_1(K_2) + 1 - n \leqslant j_i(K_1 \# K_2) \leqslant j_n(K_1) + j_n(K_2) - 1 + n$$

Proof. The 1-handle cobordism $K_1 \sqcup K_2 \to K_1 \# K_2$ induces a surjection on unreduced cohomology (this is part of the proof that unreduced cohomology gives slice genus lower bounds) which is filtered of degree $n - 1$. Furthermore, we have the isomorphism as filtered vector spaces $H_{\partial w}(K_1 \sqcup K_2) = H_{\partial w}(K_1) \otimes H_{\partial w}(K_2)$.

Hence we have $j_i(K_1 \# K_2) + 1 - n \leqslant j_n(K_1 \# K_2) + 1 - n \leqslant j_n(K_1) + j_n(K_2)$.

The other inequality follows from the same argument applied to the mirrors of K_1 and K_2. $\square$

Such boundedness results suggest that one should at least be able to extract from unreduced cohomology *quasi-homomorphisms* from the knot concordance group to the reals. For example, one could make the definition

$$s_{\partial w}(K) := \frac{j_1(K) + \cdots + j_n(K)}{2n(n-1)}.$$

The absolute value of $s_{\partial w}$ certainly gives a lower bound on the slice genus which is tight for torus knots (since $s_{\partial w}$ is the average of n functions with these properties), and furthermore one has

PROPOSITION 2.15. *The function $s_{\partial w}$ is a quasi-homomorphism.*

Proof.

$$|s_{\partial w}(K_1 \# K_2) - s_{\partial w}(K_1) - s_{\partial w}(K_2)|$$
$$= \left| \frac{1}{2n(n-1)} \sum_{i=1}^{n} (j_i(K_1 \# K_2) - j_i(K_1) - j_i(K_2)) \right|$$
$$\leqslant \frac{1}{2n(n-1)} \left(3n(n-1) + \sum_{\substack{\alpha: \\ \partial w(\alpha)=0}} 2(n-1)|\tilde{s}_\alpha(K_1 \# K_2) - \tilde{s}_\alpha(K_1) - \tilde{s}_\alpha(K_2)| \right)$$
$$= \frac{3}{2}$$

directly from Proposition 2.12. □

We now give a definition which will enable us to be briefer in the sequel.

DEFINITION 2.16. We put a partial order on Laurent polynomials in q with non-negative integer coefficients by writing $F_1(q) \geqslant F_2(q)$ if and only if $F_1(q) - F_2(q)$ is expressible as a sum of polynomials

$$F_1(q) - F_2(q) = \sum_i q^{u_i} - q^{v_i}$$

where $u_i \geqslant v_i$ for all i.

LEMMA 2.17. *Let ∂w be a separable potential, K a knot, and P a positive knot. Then we have*

$$\mathrm{Gr}^j H^i_{\partial w}(K \# P) = \mathrm{Gr}^j H^i_{\partial w}(K)[2(n-1)s_n(P)],$$

where the square brackets denote a shift in the quantum grading. In other words, taking connect sum with a positive knot P has the effect of an overall shift equal to the genus of P.

Proof. Writing

$$F_L(q) = \sum_j \dim(\mathrm{Gr}^j H^0_{\partial w}(L))q^j$$

for the Poincaré polynomial of a knot L, we will be done if we can show that

$$F_{K \# P}(q) = q^{2(n-1)s_n(P)} F_K(q).$$

First note that since P is positive we have that $s_n(P)$ is non-positive and furthermore $-s_n(P)$ is equal to the slice genus of P. Hence there is a genus $-s_n(P)$ cobordism from K to $K \# P$, and we can conclude from Theorem 1.1 that

$$F_{K \# P}(q) \geqslant q^{2(n-1)s_n(P)} F_K(q).$$

To establish the reverse inequality, let D_K, D_P, and $D_{K \# P}$ be a diagram for K, a positive diagram for P, and the diagram of $K \# P$ formed by a 1-handle addition between D_K and D_P, respectively.

Now consider the cochain complex $C^i_{\partial w}(D_P)$. The cochain complex is supported in non-negative homological degrees, and the cohomology of the cochain complex is supported in degree 0. It follows that if $g \in C^0_{\partial w}(D_P)$ is a cocycle, then the filtration grading of $[g] \in H^0_{\partial w}(D_P)$ agrees with the filtration grading of g.

Let U_+ be a positive diagram of the unknot whose oriented resolution $O(U_+)$ has the same number of components as $O(D_P)$. We know that we have a filtered isomorphism of $\mathbb{C}[x]/\partial w(x)$-modules:

$$H^0_{\partial w}(U_+) \equiv (\mathbb{C}[x]/\partial w(x))[1-n].$$

The argument above tells us that we can identify the cohomology of D_P with that of U_+ up to an overall shift, hence we have

$$H^0_{\partial w}(D_P) \equiv (\mathbb{C}[x]/\partial w(x))[(n-1)(|O(D_P)| - w(D_P) - 1)]$$
$$\equiv (\mathbb{C}[x]/\partial w(x))[1 - n + 2(n-1)s_n(P)]$$

where we write $|O(D_P)|$ for the number of components of $O(D_P)$.

Under this identification, each generator $[g_i^{D_P}] \in H^0_{\partial w}(D_P)$ corresponds to a non-zero multiple of an α_i-eigenvector of the action of x, in other words, to a non-zero multiple of the element

$$\frac{\partial w(x)}{x - \alpha_i} \in \mathbb{C}[x]/\partial w(x)[1 - n + 2(n-1)s_n(P)].$$

Now note that we have

$$1 = \sum_i \frac{1}{\prod_{j \neq i}(\alpha_i - \alpha_j)} \frac{\partial w(x)}{x - \alpha_i} \in \mathbb{C}[x]/\partial w(x)[1 - n + 2(n-1)s_n(P)],$$

so that we see that there is a cocycle $h \in C^0(D_P)$ such that the filtration grading of $[h] \in H^0_{\partial w}(D_P)$ agrees with that of h and is $1 - n + 2(n-1)s_n(P)$. Furthermore, note that h is a linear combination of the generators $g_i^{D_P}$ with each coefficient non-zero.

There is a map

$$\Phi : H^0_{\partial w}(K) \otimes H^0_{\partial w}(P) \longrightarrow H^0_{\partial w}(K \# P)$$

induced by the 1-handle addition Morse move from $D_K \sqcup D_P$ to $D_{K\#P}$. This map is filtered of degree $n-1$. If $0 \neq k \in H^0_{\partial w}(K)$, then we have that $0 \neq \Phi(k \otimes [h]) \in H^0_{\partial w}(K\#P)$, since there exists a cocycle representative for k expressible as a linear combination of the generators $g_i^{D_K}$ and h is a linear combination of the generators $g_i^{D_P}$ with each coefficient non-zero.

Writing gr for the filtration grading, we see that

$$\mathrm{gr}(\Phi(k \otimes [h])) \leqslant \mathrm{gr}(k) + \mathrm{gr}([h]) + n - 1 = \mathrm{gr}(k) + 2(n-1)s_n(P),$$

and this completes the proof. $\qquad\square$

Along with the usual mirror argument establishing the corresponding result for connect sum with negative knots, this is enough to deduce that $j_i(K)$ share many properties of slice-torus invariants. Some of these properties are described in [19]. The arguments there are specific to the situation of Khovanov cohomology, but there are topological proofs due to unpublished work by Kawamura and [16] by the first author.

We summarize the structure of the topological arguments and how they apply in the situation of a separable potential ∂w. Given a diagram D, one constructs cobordisms to positive and negative diagrams D^+ and D^-, respectively. By Theorem 1.1, one establishes lower bounds on $j_i(D)$ from the first cobordism and upper bounds on $j_i(D)$ from the second. If one can make good choices for D, D^+ and D^-, then one can make the upper and lower bounds agree, determining each $j_i(D)$ completely.

In such a good situation, because the cohomologies $H_{\partial w}(D^+)$ and $H_{\partial w}(D^-)$ are just shifted copies of the cohomologies of the unknot, it follows that $H_{\partial w}(D)$ is also a shifted version of the unknot. Note that nowhere in this process have we relied on the particular choice of ∂w. Hence we have an isomorphism as filtered vector spaces $H_{\partial w}(D) \equiv H_{x^n-1}(D)$. Furthermore, in such a good situation, the same argument applies in the reduced case so that $\tilde{s}_{\partial w,\alpha} = s_n$ for any choice of root α.

Moreover, one can take the obvious cobordisms between diagrams $D'\#D$ and $D'\#D^+$ and between $D'\#D$ and $D'\#D^-$. In the case of a good situation as above, it follows from the resulting inequalities that $H_{\partial w}(D'\#D)$ is a shift of $H_{\partial w}(D')$ by $2(n-1)s_n(D)$.

We give some known classes of knots which have such a good situation:

THEOREM 2.18. *If ∂w is separable and α is a root of ∂w, K is a quasi-positive, quasi-negative, or homogeneous knot (included in these categories are positive, negative, and alternating knots, but not all quasi-alternating knots) and K' is a knot, then we have that*

(i) $\tilde{s}_{\partial w,\alpha}(K) = s_n(K)$,
(ii) $H_{\partial w}(K) \equiv H_{x^n-1}(K)$ *as filtered vector spaces*,
(iii) $H_{\partial w}(K'\#K) \equiv H_{\partial w}(K')[2(n-1)s_n(K)]$.

There is an observation exploited by Livingstone [17] that says if K^+ and K^- are knots related by a crossing change, then there is a genus 1 cobordism between K^+ and $K^-\#T_{2,3}$ where we write $T_{2,3}$ for the positive trefoil (this is specific example of the general observation that two intersection points of opposite sign in a connected knot cobordism can be exchanged for a single piece of genus). The resulting inequality gives immediately

PROPOSITION 2.19. *If K_+ and K_- are knots with diagrams that are related by a crossing change, then*

$$0 \leqslant j_i(K_-) - j_i(K_+) \leqslant 2(n-1).$$

2.3. *Appetizing example revisited*

We return to our example of Section 1.2, the pretzel knot $P = 10_{125}$, and reanalyze its cohomology in light of what we now know. We start with an easy proposition in which we do not require that our potential ∂w is separable.

DEFINITION 2.20. *A page E_i (where $i \geqslant 1$) of a spectral sequence E is called* significant *if it is not isomorphic to E_{i-1} as a doubly graded vector space. Otherwise, it is called* insignificant.

PROPOSITION 2.21. *Let $\partial w = \sum_i a_i x^i \in \mathbb{C}[x]$ be a potential of degree $n > 0$, and K a knot. Suppose that $\exists m \geqslant 1 : i \not\equiv n \pmod{m} \implies a_i = 0$. Then there is a $\mathbb{Z}/2m\mathbb{Z}$-grading on the cohomology $H_{\partial w}(K)$ which is respected by the spectral sequence. In particular, all pages E_i of the spectral sequence arising from the filtered homotopy class of complexes corresponding to K and the potential ∂w with $i \not\equiv 1 \pmod{m}$ are insignificant.*

Proof. If D is a diagram of K, note that the differential of $C_{\partial w}(D)$ preserves the filtration degree modulo $2m$, thus giving a $\mathbb{Z}/2m\mathbb{Z}$-grading on the cohomology. Splitting the complex along this cyclic grading, we see that the differentials of the spectral sequence must also respect the grading. The differential on the ith page has q-degree $2i$, and thus is 0 if m does not divide i. $\square$

We can use the cyclic grading on the cohomology to deduce consequences that are manifested in the example of the knot P and the spectral sequences that we analyzed in Section 1.2. For example:

THEOREM 2.22.   *The concordance homomorphism $\tilde{s}_{x^n-x,0}$ factors through the integers.*

Observe that if $s_n(K)$ is not an integer, then we therefore have $s_n(K) \neq \tilde{s}_{x^n-x,0}(K)$. We note that this proposition implies the existence of two distinct spectral sequences from $\widetilde{H}_{x^n}(K)$ abutting to 1-dimensional E_∞ pages supported in cohomological degree 0, specifically one way these spectral sequences differ is in the quantum grading of the support of their E_∞ pages.

In fact in this situation the unreduced cohomology must also change with the potential.

PROPOSITION 2.23.   *Suppose K is such that*

$$s_n(K) \in \frac{1}{n-1}\mathbb{Z} \setminus \mathbb{Z}.$$

Then as a filtered vector space we have

$$H_{x^n-x}(K) \not\equiv H_{x^n-1}(K).$$

Proof of Theorem 2.22 and Proposition 2.23.   First observe that the potential $x^n - x$ is of the form considered in Proposition 2.21, so that the cohomology is $\mathbb{Z}/2(n-1)\mathbb{Z}$-graded.

The roots of the potential are $0, \xi^0, \xi^1, \xi^2, \ldots, \xi^{n-2}$, where $\xi = e^{2\pi i/(n-1)}$. If we are given a diagram D, then corresponding to these roots are the Gornik cocycles which generate the cohomology—we shall write these as $g, g_0, g_1, \ldots, g_{n-2}$.

Each Gornik generator is an element of the cochain group summand S corresponding the oriented resolution of D. If the oriented resolution of D has c components, then, ignoring the overall quantum shift, the corresponding cochain group summand is the vector space

$$S = \frac{\mathbb{C}[x_1]}{x_1^n - x_1} \otimes \cdots \otimes \frac{\mathbb{C}[x_c]}{x_c^n - x_c}.$$

The generators g_i are given by

$$g_i = \frac{(x_1^n - x_1) \cdots (x_c^n - x_c)}{(x_1 - \xi^i) \cdots (x_c - \xi^i)}.$$

We claim that the vector space $G = \langle g_0, \ldots, g_{n-2} \rangle$ has a basis $h_0, h_1, \ldots, h_{n-2}$, where h_i is a homogeneous polynomial of degree $2i$ with respect to the cyclic grading $\mathbb{Z}/2(n-1)\mathbb{Z}$ inherited from the usual $\mathbb{Z}$-grading on $\mathbb{C}[x_1, \ldots, x_c]$. The proof is a straightforward check and an explicit argument is given *mutatis mutandis* in Lemma 2.4 of [20].

Putting back in the overall quantum shift, it follows that the associated $\mathbb{Z}/2(n-1)\mathbb{Z}$-graded vector space to the 1-codimensional subspace

$$\langle [g_0], \ldots, [g_{n-2}] \rangle \subset H_{x^n-x}(D)$$

is 1-dimensional in each even grading if n is odd and in each odd grading if n is even. Furthermore, using Corollary 3.4 we see that s_{x^n-x,ξ^i} is independent of choice of i. This implies via Proposition 2.12 that the associated graded vector space to the 1-codimensional subspace above has Poincaré polynomial $q^r(1 + q^2 + \cdots + q^{2n-4})$ for some integer r.

The generator g is homogeneous of grading $n - 1$ with respect to the $\mathbb{Z}/2(n-1)\mathbb{Z}$-grading on the cochain complex. By the definition of $\tilde{s}_{x^n-x,0}$, it follows immediately that $\tilde{s}_{x^n-x,0}$ is always integral.

Moreover, since the cyclic grading $n - 1$ is 2-dimensional in unreduced cohomology, we see that if the Poincaré polynomial of the associated graded vector space to $H_{x^n - x}(K)$ is of the form $q^{2r}(q^{1-n} + q^{3-n} + \cdots + q^{n-1})$ for some integer r, then we must have that $2(n-1)$ divides r.

On the other hand, the Poincaré polynomial of the associated graded vector space to $H_{x^n - 1}(K)$ is exactly $q^{2(n-1)s_n(K)}(q^{1-n} + q^{3-n} + \cdots + q^{n-1})$. Hence if $s_n(K)$ is not an integer, then we must have

$$H_{x^n - x}(K) \not\equiv H_{x^n - 1}(K).$$

<div align="right">□</div>

3. Comparing different potentials

3.1. The KR-equivalence classes

DEFINITION 3.1. We call two potentials ∂w and $\partial w'$ *KR-equivalent over a link L*, denoted by $\partial w \sim_L \partial w'$, if $C_{\partial w}$ and $C_{\partial w'}$ are cochain homotopy equivalent over $\mathbb{C}$. Furthermore, we call those potentials *KR-equivalent*, denoted by $\partial w \sim \partial w'$, if they are KR-equivalent over all links.

This section is devoted to investigating the space of KR-equivalence classes. In this paper we restrict ourselves to the unreduced case, but the reduced case is also interesting. In particular, note that unreduced cohomologies with KR-equivalent potentials are filtered isomorphic, but the corresponding reduced cohomologies need not be.

Throughout this section, we will frequently use the following graded rings:

$$R_n = \mathbb{C}[a_0, \ldots, a_{n-1}], \quad \deg a_i = 2(n - i),$$
$$R_n[x], \quad \deg x = 2,$$
$$R_n[x]/p, \quad p = x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

THEOREM 3.2 ([11], cf. also [28]). *There is an equivariant $\mathfrak{sl}_n$-cohomology theory as follows: to a marked diagram D of a link L, a finite-dimensional graded cochain complex $C_{U(n)}(D)$ of free $R_n[x]/p$-modules is associated, such that complexes of equivalent marked diagrams are homotopy equivalent over $R_n[x]/p$. We will denote its cohomology by $H_{U(n)}(L)$. Evaluating by $e : R_n \to \mathbb{C}$ gives a filtered cochain complex of free $\mathbb{C}[x]/\partial w$-modules with $\partial w = x^n + e(a_{n-1})x^{n-1} + \cdots + e(a_0)$, that is, the usual $\mathfrak{sl}_n$-complex with potential ∂w.*

Equivariant cohomology is in a sense a *universal* $\mathfrak{sl}_n$-homology, from which the reduced and unreduced cohomologies and spectral sequences for all potentials of degree n can be recovered.

PROPOSITION 3.3. *Suppose that D is knot diagram, ∂w_1 is a degree n potential, and that we define another degree n potential ∂w_2 by*

$$\partial w_2(x) = \frac{1}{a^n} \partial w_1(ax + b),$$

where $a, b \in \mathbb{C}$ with $a \neq 0$. Then:

(i) *There is a filtered cochain homotopy equivalence $\varphi : C_{\partial w_1}(D) \to C_{\partial w_2}(D)$.*
(ii) *$\varphi(x \cdot c) = (ax + b) \cdot \varphi(c)$.*

Proof. The first part is due to Wu [27, Proposition 1.4]. The second part follows immediately from the construction of φ. □

This implies that every potential is KR-equivalent to a potential whose x^{n-1}-coefficient is zero. Another corollary is the following:

COROLLARY 3.4.   *Let $\alpha_1, \ldots, \alpha_n$ be the roots of ∂w_1. Then*

$$\widetilde{s}_{\partial w_1(x), \alpha_i}(K) = \widetilde{s}_{\partial w_2, a^{-1}(\alpha_i - b)}(K).$$

So far, particular attention has been focused on potentials of the following kind:

DEFINITION 3.5.   *We call ∂w a Gornik potential if for some $\beta, \gamma \in \mathbb{C}$ with $\gamma \neq 0$ we have*

$$\partial w = (x - \beta)^n - \gamma.$$

From the previous proposition, we get:

COROLLARY 3.6.   *Suppose that K is a knot, ∂w is a Gornik potential and α is any root of ∂w, then we have*

(i) $H_{\partial w}(K) \equiv H_{x^n - 1}(K)$ *as a filtered vector space,*
(ii) $\widetilde{s}_{\partial w, \alpha}(K) = s_n(K).$

Let us give a geometric interpretation of the situation for separable potentials: a separable potential can be given by its set of roots in the complex plane. If two such sets are related by an affine symmetry of the plane, then their corresponding potentials are KR-equivalent. In particular, there is only one KR-equivalence class of potentials of degree 2, as was first proved in [**21**]—all potentials of degree 2 are Gornik. For higher degrees, however, the situation is more complicated. For the main result of this section, identify the set of polynomials of degree n with $\mathbb{C}^n$ and endow it with the Zariski topology.

THEOREM 3.7.   *For a fixed link L and a fixed n, there are only finitely many KR-equivalence classes of polynomials of degree n over L. One of these classes is generic in the sense that all other classes are finite unions of intersections of an Zariski-open set with a Zariski-closed set that is not $\mathbb{C}^n$.*

COROLLARY 3.8.   *Let $n \geqslant 1$ be fixed. There are at most countably many classes of KR-equivalence. One of these classes is generic in the sense that it contains a countable intersection of non-empty Zarisiki-open (and thus dense) sets.*

This notion of genericity is strong enough, for example, to imply that the complement of the generic class has measure zero. At the moment, it is not clear whether for a fixed n, there is in fact an infinity of KR-equivalence classes; in the next Section 3.2 we will see that there are at least $n - 1$.

In the proof, we use the strategy of *successive Gauss elimination* as described in [**8**] to compute the spectral sequence. Let us briefly explain this strategy: our additive category of choice is finite-dimensional filtered vector spaces over a field. Gauss elimination may be used to dispose of all isomorphisms of a cochain complex C_0, yielding a homotopy equivalent cochain complex C_0' whose differentials on the 0th page of cohomology are trivial, that is, $E_0(C_0') = E_1(C_0')$. So, it is possible to define a filtered complex C_1 as regrading of C_0' by shifting the filtration degree of the tth cohomology group down by t. We have $E_k(C_1) = E_{k+1}(C_0')$. Now repeat this procedure—let C_1' be homotopy equivalent to C_1 with trivial differentials on the first page, C_2 its regrading etc. At some point C_ℓ will have trivial differentials, and at that point all the pages of the spectral sequence have been computed.

On the one hand, this gives a practical algorithm to compute a spectral sequence; indeed it is this algorithm that we use in our program khoca, see Section 5.2. On the other hand, it establishes that doing so determines the filtered homotopy type (Figure 3):

	# Distinct roots	Significant pages
X^3	1	1
$X^3 + 1$	3	1,4
$X^3 + X^2$	2	1,3
$X^3 + X$	3	1,3,5

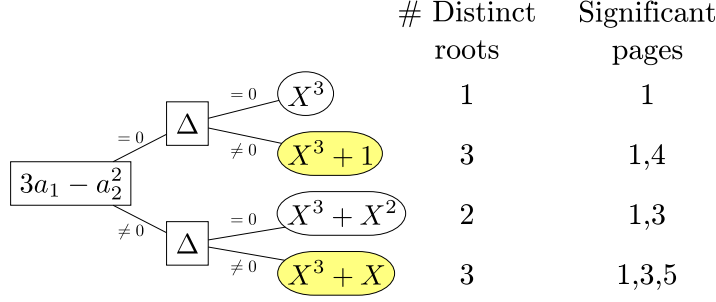$3a_1 - a_2^2$; Δ $(=0)$; Δ $(\neq 0)$

FIGURE 3 (colour online). *A tree illustrating the KR-equivalence classes of the trefoil for $n = 3$. Polynomials in rectangular boxes are those on whose vanishing the class depends, as in the proof of Theorem 3.7; Δ denotes the discriminant. The round boxes mark representative potentials of the corresponding KR-equivalence class. Separable potentials are in yellow boxes. The generic class is at the bottom.*

PROPOSITION 3.9. *Two finite-dimensional filtered cochain complexes over a field with respective spectral sequences E and E' are homotopy equivalent if and only if E_i and E'_i are isomorphic doubly graded vector spaces for all $i \geqslant 1$.*

Proof of Theorem 3.7. Forgetting x, the equivariant $\mathfrak{sl}_n$-cochain complex $C_{U(n)}$ is a complex of free graded R_n-modules of finite rank, where the a_i carry a non-negative degree. That is in fact all the information we need on $C_{U(n)}$; consider any cochain complex C with these properties and a subset $U \subset \mathbb{C}^n$. Then U is divided into equivalence classes of vectors $(v_0, \ldots, v_{n-1})$, on which evaluating by $a_i \mapsto v_i$ induces homotopy equivalent filtered cochain complexes. Let us prove for all pairs (C, U) that U decomposes as disjoint union of finitely many sets D_i, such that each equivalence class is the union of some of those sets, and such that for each D_i, one may select two sets $T_i, T'_i \subset R_n$ such that $D_i = U \cap Z(T_i) \cap Z(T'_i)^c$, where $Z(T) = \bigcap_{p \in T} p^{-1}(0)$. Moreover, for at most one i we have $Z(T_i) = \mathbb{C}^n$. This implies the statement of the Theorem 3.7.

We proceed by induction. The statement is obviously true for a complex with trivial differentials. Otherwise, assume that the statement holds for all pairs (C', U') such that either C' has smaller total dimension than C; or has equal total dimension, but fewer non-zero matrix entries. If all degree-preserving differentials of C are zero, regrade following the description of successive Gauss elimination above. Then pick a non-zero degree-preserving matrix entry p of C and consider $(C, U \cap p^{-1}(0))$. Without changing the equivalence classes, p may be replaced by zero, so by the induction hypothesis $U \cap p^{-1}(0) = \bigsqcup_{i=1}^{k} D_i$, where

$$D_i = U \cap p^{-1}(0) \cap Z(T_i) \cap Z(T'_i)^c = U \cap Z(T_i \cup \{p\}) \cap Z(T'_i)^c.$$

On the other hand, consider $(C, U \cap (p^{-1}(0))^c)$. The polynomial p does not vanish for any evaluation; so let us perform a Gauss elimination on p, and multiply the matrix with p afterwards. The ensuing complex C' has the same equivalence classes as C, because its evaluation at any point in $U \cap (p^{-1}(0))^c$ is homotopy equivalent to C. Moreover, C' has smaller total dimension, so by the induction hypothesis, $U \cap (p^{-1}(0))^c = \bigsqcup_{j=1}^{\ell} \widetilde{D}_j$, where

$$\widetilde{D}_j = U \cap (p^{-1}(0))^c \cap Z(\widetilde{T}_j) \cap Z(\widetilde{T}'_j)^c = U \cap Z(\widetilde{T}_j) \cap Z(\widetilde{T}'_j \cup \{p\})^c.$$

Note that $Z(T_i \cup \{p\})$ is a proper subset of $\mathbb{C}^n$ for all i, and at most one of the $Z(\widetilde{T}_j \cup \{p\})$ is not. So

$$U = \bigsqcup_i D_i \sqcup \bigsqcup_j \widetilde{D}_j$$

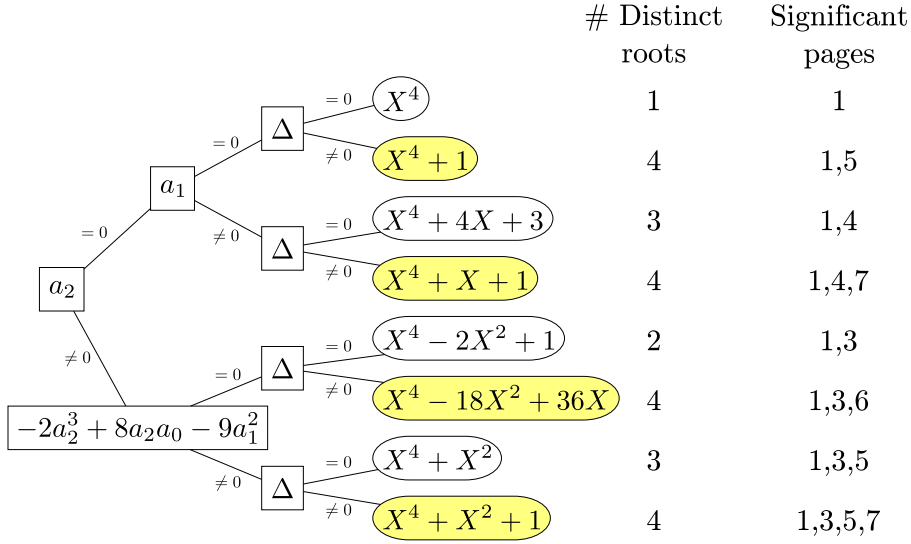|	# Distinct roots	Significant pages
X^4	1	1
$X^4 + 1$	4	1,5
$X^4 + 4X + 3$	3	1,4
$X^4 + X + 1$	4	1,4,7
$X^4 - 2X^2 + 1$	2	1,3
$X^4 - 18X^2 + 36X$	4	1,3,6
$X^4 + X^2$	3	1,3,5
$X^4 + X^2 + 1$	4	1,3,5,7

FIGURE 4 (colour online). *A tree as in Figure 3 for the trefoil and $n = 4$. For simplicity, we have set $a_3 = 0$ from the beginning; note that every polynomial is KR-equivalent to another with $a_3 = 0$ by Proposition 3.3.*

is the decomposition of U whose existence was to be proven (Figure 4). □

3.2. *A lower bound on the number of KR-equivalence classes*

THEOREM 3.10. (i) *There are at least $n - 1$ KR-equivalence classes of separable potentials of degree n.*

(ii) *Gornik potentials form an equivalence class, and for $n > 2$, it is not generic.*

We will prove this theorem by analyzing the equivariant $\mathfrak{sl}_n$-cohomology of the trefoil. It is a good exercise to compute it for general n using Theorem 5.5; but let us follow a different route here, which treats the cohomology theories more as a black box. The proof is split in several lemmas, and uses the following theorem, which was proved quite recently:

THEOREM 3.11 ([**25**]). *Let ∂w be a potential with distinct roots $\alpha_1, \ldots, \alpha_k$. For every link L, there is an isomorphism respecting the homological degree (but not the quantum degree)*

$$H_{\partial w}(L) \equiv \bigoplus_{i=1}^{k} H_{x^{(\mathrm{mult}_{\alpha_i} \partial w)}}(L).$$

Here we are writing $\mathrm{mult}_\alpha f$ to mean the multiplicity of the root α in the complex polynomial f.

LEMMA 3.12. *Let ∂w be a potential with distinct roots $\alpha_1, \ldots, \alpha_k$ and let $d \in \mathbb{C}[x]$. Denote the $\mathbb{C}$-endomorphism of $\mathbb{C}[x]/\partial w$ given by multiplication with d by M_d. Then*

$$\dim_{\mathbb{C}} \ker M_d = \sum_{i=1}^{k} \min\{\mathrm{mult}_{\alpha_i} \partial w, \mathrm{mult}_{\alpha_i} d\}.$$

Proof. This follows from the decomposition as a $\mathbb{C}[x]$-module

$$\mathbb{C}[x]/\partial w = \bigoplus_{i=1}^{k} \mathbb{C}[x]/(x-\alpha_i)^{\mathrm{mult}_{\alpha_i}\partial w}.$$

$\square$

LEMMA 3.13. *Let*

$$q = x^{n-1} + \sum_{i=0}^{n-2} b_i x^i \in R_n[x], \quad b_i \in R_n$$

be homogeneous of degree $2n-2$ with the following property: for all $e : R_n \to \mathbb{C}$, let $\alpha_1, \dots, \alpha_k$ be the roots of $e^*(p)$, then

$$n - k = \sum_{i=1}^{k} \min\{\mathrm{mult}_{\alpha_i} e^*(p), \mathrm{mult}_{\alpha_i} e^*(q)\}. \tag{$\dagger$}$$

Then the coefficient of a_{i+1} as monomial in b_i is not equal to 1.

Proof. Let $r \in \{1, \dots, n-1\}$. Assume that the statement were not true for b_{r-1}. This implies, in particular, that for $e : R_n \to \mathbb{C}$ given by $e^*(p) = x^n + x^r = x^r(x^{n-r}+1)$, we have $e(b_{r-1}) = 1$. We will show that this leads to a contradiction.

The polynomial $e^*(p)$ has $k = n - r + 1$ different roots, and so the left-hand side of ($\dagger$) equals $r - 1$. Every root but 0 has multiplicity 1 and thus contributes at most 1 to the sum on the right-hand side. So, the right-hand side is less or equal than $n - r + \min\{\mathrm{mult}_0 e^*(p), \mathrm{mult}_0 e^*(q)\}$, and thus

$$r - 1 \leqslant n - r + \min\{n - r, \mathrm{mult}_0 e^*(q)\}$$
$$\implies 2r - n - 1 \leqslant \min\{n - r, \mathrm{mult}_0 e^*(q)\}$$
$$\implies 2r - n - 1 \leqslant \mathrm{mult}_0 e^*(q).$$

Hence we have $e(b_i) = 0$ for all $i \leqslant 2r - n - 2$. For degree reasons,

$$e(b_i) \neq 0 \implies \deg a_r \mid \deg b_i$$
$$\implies (2n - 2r) \mid (2n - 2i - 2).$$

So, if $2r - n - 1 < 0$, then b_{r-1} is the only b_i with non-zero evaluation, and we have $e^*(q) = x^{r-1}(x^{n-r}+1)$, contradicting ($\dagger$).

If, on the other hand, $2r - n - 1 \geqslant 0$, then $e(b_{2r-n-1}) \neq 0$ is possible. In that case

$$e^*(q) = x^{n-1} + x^{r-1} + e(b_{2r-n-1})x^{2r-n-1}$$
$$= x^{2r-n-1}(x^{2n-2r} + x^{n-r} + e(b_{2r-n-1})).$$

We have $\mathrm{mult}_0 e^*(q) = 2r - n - 1$, which in turn implies that all other roots of $e^*(q)$ must be common roots with $e^*(p)$. Hence $x^{n-r} + 1$ divides $x^{2n-2r} + x^{n-r} + e(b_{2r-n-1})$, which contradicts $e(b_{2r-n-1}) \neq 0$. $\square$

REMARK 3.14. One may be tempted to think that the hypotheses of the previous lemma are in fact sufficient to show that

$$q = \frac{1}{n} \cdot \frac{\partial p}{\partial x}.$$

But this is not true, and indeed for $n = 3$ we have the following counterexample:

$$q = \frac{x^3 + 2a_2 x^2 + (4a_1 - a_2^2)}{3}.$$

LEMMA 3.15.   Let $\partial w = x^n + a_{n-2}x^{n-2} + \cdots + a_0 \in \mathbb{C}[x]$ with $\partial w \neq x^n$, and let E be the spectral sequence associated to $C_{\partial w}(K)$. Let ℓ be the smallest positive number such that $a_{n-\ell} \neq 0$ (that is, $2 \leqslant \ell \leqslant n$).

(i) The pages $E_2, \ldots, E_\ell$ are insignificant.
(ii) For $K = T_{2,3}$, the page $E_{1+\ell}$ is significant.

Proof.   For any knot K with a diagram D, $C_{U(n)}(D)$ is by Gauss elimination homotopy equivalent to a complex of free modules whose differentials have matrices all of whose entries are non-units in $R_n[x]/p$. We shall assume that we have performed such a Gauss elimination and we shall abuse notation and write the new complex as $C_{U(n)}(K)$.

Forgetting the action of x gives a chain complex $\overline{C}_{U(n)}(D)$ of free R_n-modules. One may continue Gaussian elimination as long as possible, arriving at a complex $\overline{C}_{U(n)}(K)$. Evaluating this chain complex by some $e : R_n \to \mathbb{C}$ gives a chain complex homotopy equivalent to $C_{e^*(p)}$. All non-vanishing matrix entries in $\overline{C}_{U(n)}(K)$ are homogeneous non-constant polynomials in R_n; so the degree of such an entry is at least 2ℓ. This implies part (i).

To obtain reduced $\mathfrak{sl}_n$-cohomology from $C_{U(n)}(K)$, one may evaluate by the map that sends all a_i and x to 0. So, $C_{U(n)}(K)$ has the same Poincaré polynomial as $\widetilde{H}_{x^n}(K)$. The reduced Homflypt cohomology of the trefoil has Poincaré polynomial

$$a^2 q^{-2} + t^2 a^2 q^2 + t^3 a^4.$$

This can be easily computed from the Homflypt polynomial and the signature, since the trefoil is a two-bridge knot and thus KR-thin. That also implies that Rasmussen's spectral sequences are all trivial, and hence the reduced $\mathfrak{sl}_n$-cohomology is obtained from the Homflypt cohomology simply by the regrading $a \mapsto q^n$. It has therefore Poincaré polynomial

$$q^{2n-2} + t^2 q^{2n+2} + t^3 q^{4n}.$$

Next, the differential between homological degree 2 and 3 is given by multiplication with a polynomial $d \in R_n[x]$ which is homogeneous of degree $2n - 2$. Using Theorem 5.5, one could compute by hand that $d = \partial p / \partial x$. Instead, we proceed as follows: let $e : R_n \to \mathbb{C}$ send all a_i to 0. Applying e to $C_{U(n)}(K)$ and forgetting the action of x will give give unreduced $\mathfrak{sl}_n$-cohomology. If $e(d)$ were 0, then we would have $H_{x^n}(K) = H_{x^n}(U) \otimes \widetilde{H}_{x^n}(K)$, where U is the unknot. But this is impossible since there is a spectral sequence induced by $C_{x^n-1}(K)$ from $H_{x^n}(K)$ which respects the quantum degree modulo $2n$ and whose limit is supported in cohomological degree 0. Therefore $e(d) \neq 0$, and hence $e(d)$ is a non-zero scalar multiple of x^{n-1}. This gives $\dim_{\mathbb{C}} H_{x^n}^2(K) = n - 1$, and so Theorem 3.11 implies that for all $e : R_n \to \mathbb{C}$, where $e^*(p)$ has distinct roots $\alpha_1, \ldots, \alpha_k$, we have

$$\dim_{\mathbb{C}} H_{e^*(p)}^2(K) = n - k.$$

On the other hand, Lemma 3.12 implies that

$$\dim_{\mathbb{C}} H_{e^*(p)}^2(K) = \sum_{i=1}^{k} \min\{\mathrm{mult}_{\alpha_i} \partial w, \mathrm{mult}_{\alpha_i} d\}.$$

So, the hypotheses of Lemma 3.13 are satisfied by d.

Now let us examine what happens when we pass to unreduced cohomology: this simply means forgetting the action of x, thus obtaining a cochain complex of vector spaces. With respect to the basis $(1, x, \ldots, x^{n-1})$ of $\mathbb{C}[x]/\partial w$, the differential between homological degree 2 and 3 is an $n \times n$-matrix M, whose (i, j)-th entry is the coefficient of x^{i-1} of the unique polynomial of degree at most $n - 1$ that equals $x^{j-1} \cdot d$ in $\mathbb{C}[x]/\partial w$. So, the first two columns of M can be

TABLE 3. *Number of KR-equivalence classes of separable potentials for the trefoil and a more complicated knot; for the former, classes can be determined precisely by continuing the calculation done in the proof of Lemma 3.15. For the latter, we calculated cohomologies with respect to a large batch of potentials with small coefficients.*

	# KR-equivalence classes of separable potentials...	
$\deg \partial w$	... of the trefoil	... of $P(5, -3, 2)^{\#2}$
2	1	1
3	2	$\geqslant 3$
4	4	$\geqslant 6$
5	8	$\geqslant 10$
n	$(n-1)! \geqslant \ldots \geqslant n-1$	$\geqslant n-1$

computed as (recall that w.l.o.g. we set $a_{n-1} = 0$)

$$\begin{pmatrix} b_0 & -a_0 & \\ b_1 & b_0 - a_1 & \\ \vdots & \vdots & \cdots \\ b_{n-3} & b_{n-4} - a_{n-3} & \\ 0 & b_{n-3} - a_{n-2} & \\ 1 & 0 & \end{pmatrix}$$

Applying Gauss elimination to the entry n at $(n, 1)$ gives an $(n-1) \times (n-1)$ matrix whose first column is

$$\begin{pmatrix} -a_0 & \\ b_0 - a_1 & \\ \vdots & \cdots \\ b_{n-4} - a_{n-3} & \\ b_{n-3} - a_{n-2} & \end{pmatrix}$$

We have already argued that all pages $E_2, \ldots, E_\ell$ of the spectral sequence are insignificant for degree reasons. Now because of Lemma 3.13, the differential on E_ℓ is non-trivial, and so $E_{1+\ell}$ is significant. □

Proof of Theorem 3.10. For (i), one can take, for example, $x^n + 1$ and $x^n + x^i + a_0$ for $i \in \{1, \ldots, n-2\}$ and some $a_0 \in \mathbb{C}$ such that the polynomial is separable. By Lemma 3.15, the cohomology of the trefoil associated to these potentials have spectral sequences with different significant pages, and are thus pairwise not KR-equivalent.

The second part (ii) follows from the fact that every potential is by Proposition 3.3 KR-equivalent to one with $a_{n-1} = 0$. To be KR-equivalent to a Gornik potential, the next significant page of the spectral sequence needs to be the $(n+1)$-st, and this can only be the case if $a_i = 0$ for all $i > 0$. □

REMARK 3.16. We picked the trefoil for ease of calculation, and to demonstrate that even over the simplest non-trivial knot there are at least $n-1$ different KR-equivalence classes. In fact, the numbers in Table 3 and a close look at the calculations suggest that the actual number of classes might rather be 2^{n-2}.

Note that there are non-KR-equivalent potentials that are KR-equivalent over the trefoil: for example, $x^3 - x \sim_{T_{3,2}} x^3 - x - 1$, but $x^3 - x \nsim_{P(5,-3-2)} x^3 - x - 1$. Hence the differentials of $C_{U(n)}(P(5, -3, -2))$ are not all equal to $\partial w'$. It would certainly be worthwhile to analyze

which forms $C_{U(n)}$ takes in general, or for certain classes of knots: for example, it could be the case that the equivariant cohomology of two-bridge knots decomposes as sum of $\mathbb{C}[x]/\partial w$ (in cohomological degree 0) and several summands of the form

$$\mathbb{C}[x]/\partial w \xrightarrow{\partial w'} \mathbb{C}[x]/\partial w.$$

Also, all separable potentials yield the same E_∞ page for the trefoil; but it seems a reasonable conjecture that there are knots (sufficiently complicated and certainly not positive) for which the different KR-equivalence classes actually yield different E_∞-pages.

4. *Further illuminating examples*

We have already seen through the example of Section 1.2 that the behavior of Khovanov–Rozansky with a separable potential can be quite unexpected, especially if one's intuition comes from Lee homology and Rasmussen's invariant. However, the structural results that we have proven in Section 2 constrain this behavior to some extent. There are some natural questions concerned with how unruly the invariants can be, and whether one might expect to be able to give much stronger constraints than we have hitherto done.

In this Section 4 we list some of these natural questions and indicate through (computational) examples where the answer lies.

QUESTION 4.1.   Are there knots whose sliceness is not obstructed by any of the reduced concordance homomorphisms, but is obstructed by some of the unreduced concordance invariants?

Let $K = P(9, -7, 6) \# P(-7, 5, -4)$. Then for all $n \geqslant 2$, we have $s_n(K) = 0$ [16], so none of the generalized Rasmussen concordance homomorphisms obstruct the sliceness of K. Neither do any of the reduced concordance homomorphisms we checked. However, khoca calculates the Poincaré-polynomial of $H_{x^3-x}(K)$ as $2 + q^2$, which shows that K is not slice.

QUESTION 4.2.   In Corollary 3.6, we have shown that all roots α of a Gornik potential ∂w give the same reduced concordance homomorphism $\widetilde{s}_{\partial w, \alpha}$. More generally, the symmetry of potentials such as $x^3 - x$, which is projectively invariant under $x \mapsto -x$, extends to their reduced concordance homomorphisms: we have $\widetilde{s}_{x^3-x,1} = \widetilde{s}_{x^3-x,-1}$ by Corollary 3.4. Is it actually true for every potential that all roots give the same reduced concordance homomorphism?

No, for example, we have that $\widetilde{s}_{x^5-x,0}(P(5, -3, 2)) = 0$, but $\widetilde{s}_{x^5-x,\alpha}(P(5, -3, 2)) = -1/4$ for $\alpha \in \{\pm 1, \pm i\}$, as can be computed with khoca.

QUESTION 4.3.   We have seen that unreduced cohomology does not always have a Poincaré polynomial of the form $q^{2(n-1)s} \cdot [n]$ with $s \in \mathbb{Z}$. What shapes does it take? For example, are the generators always in quantum degrees close to each other?

For $K_i = P(5, -3, 2)^{\#i}$, we have for all $i \geqslant 0 : \widetilde{s}_{x^5-x,0}(K_i) = 0$, but $\widetilde{s}_{x^5-x,\alpha}(K_i) = -i/4$: as i grows, so does the distance between the reduced concordance homomorphism of the root 0, and the other four. Since the distance between the $\widetilde{s}$ and the unreduced j_i is bounded above by Proposition 2.12, the shape of unreduced $(x^5 - x)$-cohomology of K_i is increasingly elongated with growing i. And indeed, khoca calculations for $i = 1, 2, 3, 4$ suggest that its Poincaré polynomial is $q^{-4} + q^{-1-2i} \cdot [4]$ for all $i \geqslant 1$.

QUESTION 4.4. Proposition 2.15 shows how to get a quasi-homomorphism from the smooth concordance group to the rationals using unreduced cohomology. For H_{x^n-1}, this is a homomorphism. Is there any way to define a homomorphism for other potentials?

There may be, but if we take for example $\partial w = x^3 + x + 1$, then it cannot be done in an obvious way, as the following proposition indicates:

PROPOSITION 4.5. Let $j_i(K)$ be defined as in the introduction with potential $x^3 + x + 1$. Suppose that φ is a function from the set $\{(x_1, x_2, x_3) \mid x_1 \leqslant x_2 \leqslant x_3\} \subset (2\mathbb{Z})^3$ to $\mathbb{R}$, such that $\varphi^* : K \mapsto \varphi(j_1(K), j_2(K), j_3(K))$ is a concordance homomorphism. Then φ^* takes all knots K with $H_{\partial w}(K) = q^{2(n-1)s} \cdot [n]$ to zero.

Proof. Let $K = P(7, -5, 4)^{\#2}$. By khoca calculations, we have

$$\forall i \in \{1, 2, 3\} : j_i(K) = j_i(K \# K) = -2.$$

Therefore, $\varphi^*(K \# K) = \varphi^*(K) \implies \varphi^*(K) = \varphi(-2, -2, -2) = 0$. By taking the mirror image, we get $\varphi(2, 2, 2) = 0$ as well. But we have $\varphi^*(K \# T_{-2,3}) = \varphi(2, 2, 2) = 0 \implies \varphi^*(T_{-2,3}) = 0 \implies \varphi(2, 4, 6) = 0$. Therefore, φ^* sends also any multiple of $T_{-2,3}$ to zero, and we have $\forall s \in \mathbb{Z} : \varphi(2 + 4s, 4 + 4s, 6 + 6s) = 0$. $\qquad \square$

This implies that if one can define such a homomorphism then it must be identically 0 on all quasi-positive and homogeneous knots, which would be very unusual behavior indeed. In fact, based on wider calculations of H_{x^3+x+1} which we do not report here, it seems very likely that any such homomorphism defined as in the proposition will be identically 0.

QUESTION 4.6. We have seen the effect on unreduced cohomology $H_{\partial w}(K)$ of taking the connected sum with homogeneous and quasi-positive knots K' in Theorem 2.18: the cohomology $H_{\partial w}(K \# K')$ is just a quantum shift of $H_{\partial w}(K)$. But perhaps it is not the quasi-positivity or homogeneity of K' that is important, but just the shape of the associated graded vector space to its cohomology (which is that of a shifted unknot). Is the result more generally true for knots K' with $H_{\partial w}(K') = q^{2(n-1)s} \cdot [n]$, that is, does $H_{\partial w}(K \# K') = H_{\partial w}(K)[2(n-1)s]$ hold?

As discussed in the answer to Question 4.3, we have

$$H_{x^5-x}(P(5, -3, 2)^{\#4}) = q^{-4} + q^{-9} \cdot [4] = q^{-8} \cdot [5],$$

but, for example,

$$H_{x^5-x}(P(5, -3, 2)^{\#4} \# P(5, -3, 2)^{\#4}) = q^{-4} + q^{-17} \cdot [4] \neq q^{-16} \cdot [5].$$

QUESTION 4.7. Is $H_{\partial w}(K \# K')$ determined by $H_{\partial w}(K)$ and $H_{\partial w}(K')$?

No — take $K, K' \in \{T(3, 2), P(5, -3, 2)^{\#4}\}$ (see the previous Question).

QUESTION 4.8. Is it possible that the reduced concordance homomorphisms arising from degree 3 polynomials are all just linear combinations of s_2 and s_3? (We thank Mikhail Khovanov for raising this question.)

No. We consider the reduced concordance invariant given by taking the root $x = 0$ of the potential $x^3 - x$. Then computing the invariants for the trefoil knot and the knot $P(5, -3, 2)$ one can deduce that if there is such a linear dependence it is of the form:

$$\widetilde{s}_{x^3-x,0} = 2s_3 - s_2.$$

Next, consider the pretzel knot $P(7, -5, 4)$. In [16] the first author showed that this knot satisfies $s_2(P(7, -5, 4)) = 1$ and $s_n(P(7, -5, 4)) = 0$ for any $n > 2$. We can compute the reduced $\mathfrak{sl}(3)$ cohomology (using for example [15]) and see that in cohomological degree 0 the cohomology is supported in quantum degree 0. Hence in particular $\widetilde{s}_{x^3-x,0}(P(7, -5, 4)) = 0$. This then shows that $\widetilde{s}_{x^3-x,0}$ is not in the span of s_2 and s_3.

5. *Computer calculations*

5.1. *Bipartite links*

In this section, we consider oriented links with *matched diagrams*, that is to say, diagrams obtained by gluing together copies of the basic 2-crossing tangle (and its mirror-image) as shown in Figure 5a. Such links are called *bipartite links*. If the orientations of the tangles are always as in Figure 5b (or its mirror image), we call the diagram *orientedly matched* and the link *orientedly bipartite*.

PROPOSITION 5.1. *An unoriented matched link diagram D admits an orientation that makes it orientedly matched. This orientation is unique up to overall reversals of orientations of disjoint diagram components of D.*

Proof. If D is a knot, then this is asserted without proof in [5]; and indeed, pick one of the basic tangles: then the two strands in the complement of the tangle pair up its four endpoints. A priori there are three different pairings possible; but pairing the upper two endpoints would give a two-component link, and pairing each endpoint with the one diametrically opposed would imply that the complement of the tangle has an odd number of crossings. So, the left endpoints are paired, which implies that the tangle is oriented in the matched sense.

Assume now that D has more than one component and is not split. Then one can rotate a subset S of the basic tangles constituting D by a quarter-turn, such that the result is a knot diagram D'. Note that the set of orientations of D that maked D orientedly matched are in 1–1 correspondence with the orientations of D' that make D' orientedly matched: the correspondence is given by rotating each tangle in S by a quarter-turn and reversing its orientation.

If D is split, then treat every component separately. $\square$

Matched diagrams were introduced in [23] in the context of the Homflypt polynomial. The authors conjectured that there were non-bipartite knots, a problem which remained open for 24 years, until it was solved by Duzhin and Shkolnikov [5], who showed that if a higher Alexander ideal of a bipartite knot contains the polynomial $1 + t$, then this ideal must be trivial. Thus various of 9- and 10-crossing knots are shown to be not bipartite, among them the $P(3, 3, 3)$-pretzel knot. In fact, this generalizes to $P(p, q, r)$-pretzel knots with p, q, r odd and $\lambda = \gcd(p, q, r) > 1$, because their second Alexander ideal is generated by λ and $1 + t$. If, on the other hand, p is even, then the $P(p, q, r)$-pretzel knot is bipartite, as we shall prove later on.

Our interest in bipartite links is motivated by Krasner's discovery [10] that the Khovanov–Rozansky cochain complexes of the basic oriented matched tangle (Figure 5b), and consequently of orientedly matched diagrams take a particularly simple form: they are homotopy equivalent to cochain complexes in the TQFT-subcategory — avoiding MOY-graphs and foams. In Theorem 5.5, Krasner's theorem is generalized to equivariant Khovanov–Rozansky cohomologies.

This observation has allowed us to write a computer program called `khoca` that computes Khovanov–Rozansky cohomologies of bipartite links. A description is given in the next section.

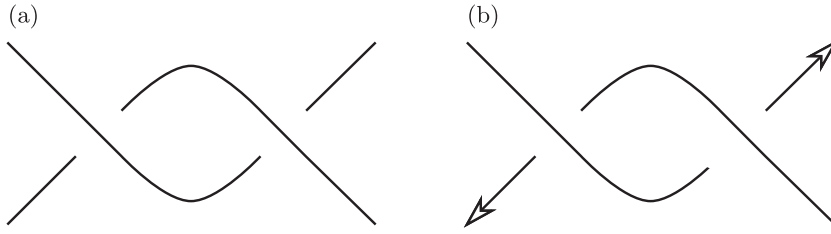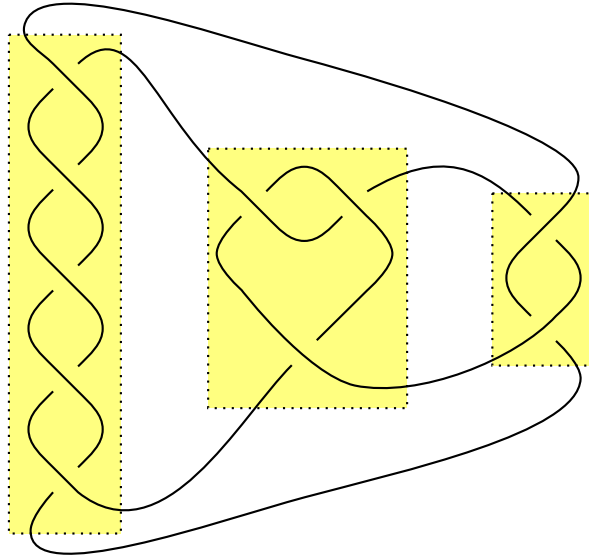FIGURE 5. *The basic matched tangle,* (a) *unoriented and* (b) *oriented.*

FIGURE 6 (colour online). *The Montesinos-knot* $(\frac{1}{5}, \frac{2}{3}, -\frac{1}{2})$ *(which is equivalent to* $M(\frac{1}{5}, -\frac{1}{3}, \frac{1}{2})$, *the* $(5, -3, 2)$-*pretzel knot).*

Duzhin and Shkolnikov prove that rational knots are bipartite; the following is a generalization, rendering precise a remark of Przytycki's that 'half of Montesinos knots should be bipartite'. A *Montesinos link* is a generalization of pretzel links, where the strands are replaced by *rational tangles*—see Figure 6 for an example. Rational tangles up to boundary-fixing isotopy are in one-to-one correspondence with $\mathbb{Q} \cup \{\infty\}$ [3]. The rational tangle with twists $a_1, \ldots, a_n$ corresponds to the value of the continued fraction

$$[a_1, \ldots, a_k] := a_k + \cfrac{1}{a_{k-1} + \cfrac{1}{\cdots + a_1}}.$$

So, a Montesinos link may be written as $L = M(p_1/q_1, \ldots, p_n/q_n)$, where $p_i \in \mathbb{Z}, q_i \in \mathbb{Z}^+, (p_i, q_i) = 1$. In this notation, for example, $M(3)$ is the trefoil and $M(1/a, 1/b, 1/c)$ the (a, b, c)-pretzel link. Clearly, without changing the isotopy type of L one may insert a 0 to or remove it from the list of fractions; and

$$M(p_1/q_1, \ldots, p_n/q_n) = M(\mp 1, p_1/q_1, \ldots, p_i/q_i \pm 1, \ldots, p_n/q_n).$$

THEOREM 5.2.   *Consider the unoriented Montesinos link $L = M(p_1/q_1, \ldots, p_n/q_n)$. If L has more than one component, then it is bipartite. If L is a knot and one of the denominators q_i is even, then L is bipartite.*

LEMMA 5.3 ([**4**, Lemma 2]).   *If either p or q is even, then p/q can be written as continued fraction $p/q = [a_1, \ldots, a_k]$ with all a_i even.*

Proof of Theorem 5.2.   Let $A = \{i \mid p_i, q_i \text{ odd}\}$. The Montesinos link L is isotopic to

$$M(\#A/1, p_1'/q_1, \ldots, p_n'/q_n),$$

where $p_i' = p_i - q_i$ if $i \in A$ an $p_i' = p_i$ otherwise. If L has more than one component, and none of the q_i is even, it follows that $\#A$ is even. If, on the other hand, one of the q_i is even, w.l.o.g. q_1, L' is isotopic to

$$M((p_1 + q_1 \cdot \#A)/q_1, p_2'/q_2, \ldots, p_n'/q_n).$$

So if one of the hypotheses of the Theorem 5.2 is satisfied, then L is isotopic to a Montesinos link who only contains rational tangles whose fractions have even numerator or denominator. But by Lemma 5.3, the corresponding rational tangles correspond to a continued fraction $[a_1, \ldots, a_k]$ with all a_i even, and can thus clearly be glued from copies of the basic unoriented matched tangle. □

LEMMA 5.4 ([**11**]).   *In the category of equivariant matrix factorizations the maps* (i), (ii) *and* (iii) *are filtered isomorphisms.*

$$\bigoplus_{i=0}^{n-1} q^{2i+1-n}\varnothing \xrightarrow{(\,c_0\ \cdots\ c_{n-1}\,)} \bigcirc, \tag{i}$$

where c_i is given by the composition of the following maps:

$$q^{2i+1-n}\varnothing \xrightarrow{\quad\iota\quad} q^{2i}\underset{x}{\bigcirc} \xrightarrow{\ x^i\ } \bigcirc.$$

$$\begin{array}{c} \left(\begin{smallmatrix} d_0 \\ \vdots \\ d_{n-2} \end{smallmatrix}\right) \\ \rightsquigarrow \mathbb{D} \xrightarrow{} \bigoplus_{i=0}^{n-2} q^{2i+2-n} \end{array} \tag{ii}$$

where d_i is given by the composition of the following maps:

$$\rightsquigarrow \mathbb{D} \xrightarrow{\ \chi_1\ } q^{-1}\underset{x}{\bigcirc} \xrightarrow{x^{n-2-i}} q^{2i+3-2n}\bigcirc \xrightarrow{\ \varepsilon\ } q^{2i+2-n}.$$

$$\left(\begin{array}{c} \left(\begin{smallmatrix} s_0 \\ \vdots \\ s_{n-3} \\ * \end{smallmatrix}\right) \\ \bowtie \xrightarrow{} \bigoplus_{i=0}^{n-3} q^{2i+3-n} \end{array} \right) \left(\oplus\ \asymp \right) \tag{iii}$$

where s_i is given by the composition of the following maps:

$$\bowtie \xrightarrow{\ \chi_1\chi_1'\ } q^{-2}\underset{x}{\bigcirc} \left(\xrightarrow{x^{n-3-i}} q^{2i+4-2n}\bigcirc \right) \left(\xrightarrow{\ \varepsilon\ } q^{2i+3-n} \right) \big($$

THEOREM 5.5. *The following filtered cochain complexes are homotopy equivalent:*

$$C_{U(n)}\left(\text{⟨diagram⟩}\right) \cong q^{1-n}\left.\right)\,\left(\xrightarrow{\;x_1-x_2\;} q^{-1-n}\right)\,\left(\xrightarrow{\text{saddle}} q^{-2n}\;\asymp\right).$$

Proof. We will compose a series of cochain homotopy equivalences to connect the two terms. By definition,

$$C_{U(n)}\left(\text{⟨diagram⟩}\right) = q^{2-2n}\left.\right)\bigcirc\left(\xrightarrow{\binom{d_1}{*}} \begin{matrix} q^{1-2n}\,\mathbf{D} \\ \oplus \\ q^{1-2n}\,\big) \end{matrix} \xrightarrow{(\,*\;e_1\,)} q^{-2n}\,\square\right).$$

Here and later, a star $(*)$ indicates a map that we do not need to know. To start, replace the circle in cohomological degree 0 and the first digon in cohomological degree 1 using the respective MOY-decompositions. This leads to d_1 being replaced by an $(n-1)\times n$-matrix d_2. For $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant n-2$, its $(i+1,j+1)$-entry is a map from $q^{3-3n+2j}\,)($ to $q^{3-3n+2i}\,)($ given by the following composition (dotted line):

$$q^{3-3n+2j}\left.\right)\,\left(\xrightarrow{\;\iota\;} q^{2-2n+2j}\underset{x_1}{\bigcirc}\left(\xrightarrow{\;x_1^j\;} q^{2-2n}\right)\bigcirc\left(\;\xrightarrow{d_1}\right.\right.$$

$$q^{3-3n+2i}\left.\right)\,\left(\xleftarrow{\;\varepsilon\;} q^{4-4n+2i}\bigcirc\left(\xleftarrow[x_2^{n-2-i}]{} q^{-2n}\underset{x_3}{\bigg)}\underset{x_2}{\bigcirc}\left(\xleftarrow{\;\chi_1\;} q^{1-2n}\,\mathbf{D}\right.\right.$$

We have $\chi_1 d_1 = (x_3 - x_2)$, and thus the whole map equals $\varepsilon x_1^{n-2-i+j}(x_3 - x_1)\iota$. This is clearly equal to a multiple of the identity of $)($ if $i = j$. On the other hand, if $i < j$, then this map is zero. Thus d_2 is an upper triangular matrix whose main diagonal consists of isomorphisms. Therefore the submatrix obtained by deleting the last column is invertible. So using Gauss elimination, the cochain complex is homotopy equivalent to

$$q^{1-n}\left.\right)\bigcirc\left(\xrightarrow{\;*\;} q^{1-2n}\right)\,\mathbf{C}\left(\xrightarrow{\;e_1\;} q^{-2n}\,\square\right).$$

To proceed, use MOY decompositions again, to replace the remaining digon and the square. For the digon, we will use the dual of the map given in Lemma 5.4 (ii). In this way, e_1 is replaced by a $(n-1)\times(n-1)$-matrix e_2. Let us ignore the last row and last column, and denote by e_2' the corresponding submatrix. For $i,j \in \{0,\dots n-2\}$, its $(i+1,j+1)$-entry is a map from $q^{3-3n+2j}\,)($ to $q^{3-3n+2i}\,)($, given by the following composition (dotted line):

$$q^{3-3n+2j}\left.\right)\,\left(\xrightarrow{\;\iota\;} q^{2-2n+2j}\underset{x_1}{\bigcirc}\left(\xrightarrow{\;x_1^j\;} q^{2-2n}\bigcirc\left(\xrightarrow{\;\chi_0\;} q^{1-2n}\right)\,\mathbf{C}\right.\right.$$

$$q^{3-3n+2i}\left.\right)\,\left(\xleftarrow{\;\varepsilon\;} q^{4-4n+2i}\bigcirc\left(\xleftarrow[x_2^{n-3-i}]{} q^{-2-2n}\underset{x_3}{\bigg)}\underset{x_2}{\bigcirc}\underset{x_4}{\left(\xleftarrow{\;\chi_1\chi_1'\;} q^{-2n}\,\square\right.}\right.$$

Because $\chi_1\chi_1'e_1\chi_0 = (x_3 - x_2)(x_2 - x_4)$, the whole map equals $\varepsilon x_1^{n-3-i+j}(x_3 - x_1)(x_1 - x_4)\iota$. As before, this is a non-zero multiple of the identity for $i = j$, and vanishes for $i < j$. Hence e_2' is an invertible submatrix of e_2. By Gauss elimination, our cochain complex is homotopy equivalent to

To determine the maps, note that the Hom-space $) (, \asymp)$ is one-dimensional in the q-degree in question. Thus e_3 is a multiple of the saddle. Now, close off the original tangle to the unknot. The cohomology of the unknot has support in cohomological degree 0; but if e_3 were 0, the above complex would have cohomology in cohomological degree 2 after closing off. Hence e_3 is a non-zero multiple of the saddle.

The Hom-space $) (,) ($ of the q-degree in question is two-dimensional, but only the subspace generated by $(x_1 - x_2)$ yields 0 when composed with the saddle. Closing off as before, we see that the dimension of the first cochain group is strictly greater than the dimension of the second. So, for the first cohomology group to vanish, d_3 needs to be non-zero. Thus d_3 is a non-zero multiple of $(x_1 - x_2)$.

A final isomorphism of cochain complexes may be used to do away with the non-zero factors. $\square$

REMARK 5.6. To our knowledge, there is no integral Khovanov-Rozansky cohomology theory yet that is defined for arbitrary tangles (not just pieces of braids as in [**12**]). But if such a theory can be defined based on the Khovanov–Rozansky cube of singular resolutions, it is likely to satisfy Theorem 5.5.

In particular, Theorem 5.5 gives a complex associated to a matched diagram that is defined over the integers. Hence one can compute a cohomology theory H over the integers for matched diagrams and thus make conjectural computations of the as-yet-undefined integral Khovanov–Rozansky cohomology.

5.2. *A computer program*

Although there is a variety of computer programs doing computations in Khovanov–Rozansky cohomologies, none of them can quickly calculate $\mathfrak{sl}_n$-cohomologies for small n of small knots. One reason is the difficulty of implementing the calculus of MOY-graphs and matrix factorisations (or some other formalism describing the differentials) on the tangle level, which is necessary for Bar–Natan's divide-and-conquer algorithm [**2**]. Theorem 5.5 shows that to compute the cohomology of *bipartite* knots, a computer program only needs to do calculations in the TQFT category of tangles and cobordisms. This category is much easier for a computer program. While not all knots are bipartite, and regrettably most torus knots appear not to be, there are still enough bipartite knots which are not two-bridge and have interesting Khovanov–Rozansky cohomologies, notably the odd-odd-even pretzel knots, which are our main source of examples.

Our program khoca calculates unreduced and reduced $\mathfrak{sl}_n$-cohomology (including all pages of the spectral sequence) of bipartite knots, for arbitrary potentials of arbitrary degree, over the complex numbers, integers and finite prime fields (beware: for $n \geqslant 4$, the results over integers and prime fields have not been proven correct, cf. Remark 5.6). Thanks to the divide-and-conquer algorithm (and implementation details such as sparse matrices and multiprocessing) it does so in reasonable time, for example, the calculation of $P(11, -9, 8)$ (a 28-crossing knot) over some random potential of degree 5 over the integers takes five minutes. Some of the

examples calculated with `khoca` can be found in Section 3.2. The program will shortly be made publicly available.

6. *Outlook*

Throughout the text, we have worked over the complex numbers. However, we expect our results to generalize to *cohomologies over finite fields*, yielding different slice genus lower bounds.

The *knight move conjecture* arose quickly after Khovanov cohomology [**1**, **6**], but is still open; phrased in the language of this article, it simply states that the spectral sequence of C_{x^2-1} (over the complex numbers) collapses on the third page (which is the first significant page after E_1). There is some weak evidence against the conjecture: no 'reason why it should be true' is known, and the lack of a counterexample could simply come from our limited ability to calculate cohomology of large knots. Moreover, generalizations of the conjecture fail: for example, the spectral sequences of $C_{x^2-1}(T(7,8))$ over $\mathbb{F}_7$, $C_{x^2-1}(T(6,7))$ over $\mathbb{F}_3$ or $C_{x^3-1}(T(5,6))$ over $\mathbb{F}_5$ collapse only on the *second* significant page after E_1 ([**2**], and calculations with `foamho` [**14**]). Nevertheless, it might be noteworthy that all small knots K that we considered displayed the following behavior: let E be the spectral sequence of $C_{\partial w}(K)$ (over $\mathbb{C}$), then $E_{2\deg\partial w-1} = E_\infty$.

There is a *new potential topological application* of the invariants: we have seen that the sliceness obstructions arising from unreduced cohomologies are not all equivalent to concordance homomorphisms. So, they could potentially be used to prove the non-sliceness of a knot that represents torsion in the concordance group, such as an amphichiral knot. We do not know, for example, of a reason why for some amphichiral knot K and some separable potential ∂w, we could not have

$$H_{\partial w}(K) = 3 \quad \text{where } \deg \partial w = 3,$$
$$\text{or} \quad H_{\partial w}(K) = 2q^{-1} + 2q \quad \text{where } \deg \partial w = 4.$$

Note that either of these is in accordance with Proposition 2.12 and 2.13, and obstructs sliceness since $H_{\partial w}(U) = [\deg \partial w]$. In contrast, invariants such as knot signatures or slice-torus invariants must necessarily vanish on such knots.

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