Dirac-Krein systems on star graphs

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Abstract. We study the spectrum of a self-adjoint Dirac-Krein operator with potential on a compact star graph $\mathcal{G}$ with a finite number n of edges. This operator is defined by a Dirac-Krein differential expression with summable matrix potentials on each edge, by self-adjoint boundary conditions at the outer vertices, and by a self-adjoint matching condition at the common central vertex of $\mathcal{G}$. Special attention is paid to Robin matching conditions with parameter $\tau \in \mathbb{R} \cup \{\infty\}$. Choosing the decoupled operator with Dirichlet condition at the central vertex as a reference operator, we derive Krein's resolvent formula, introduce corresponding Weyl–Titchmarsh functions, study the multiplicities, dependence on τ, and interlacing properties of the eigenvalues, and prove a trace formula. Moreover, we show that, asymptotically for $R \to \infty$, the difference of the number of eigenvalues in the intervals $[0, R)$ and $[-R, 0)$ deviates from some integer κ_0, which we call dislocation index, at most by $n + 2$.

Mathematics Subject Classification (2010). 81Q10, 81Q35, 47A10, 47A75.

Keywords. Dirac operator, Dirac-Krein system, star graph, Krein's resolvent formula, trace formula, dislocation index.

1. Introduction

While there exists a vast literature on Schrödinger operators on metric graphs and their spectral properties (see e.g. [EKK$^+$08], [BK13], and the numerous lists of references therein), there are only a few papers on Dirac operators on metric graphs so far (see e.g. [BT90], [BH03], [Har08], [Pos09], [YPH11], [YH12], and [HW12]). This may be due to the fact that classical techniques for semi-bounded operators do not apply to Dirac operators and so mostly only special cases such as free Dirac operators or particular boundary and/or matching conditions are treated.

On the other hand, interest in and need for mathematical results on Dirac operators has rapidly grown in the last years due to novel applications and experiments. They include models for electronic properties of graphene

(see e.g. [Wol14]), propagation of electromagnetic waves in graphene-like photonic crystals (see e.g. [DKMOR15]), ultracold matter in optical lattices and "proof-of-principle quantum simulation of the one-dimensional Dirac equation using a single trapped ion-set" to confirm amazing features of relativistic quantum motion like Zitterbewegung and Klein paradox (see [WSK$^+$11], [GKZ$^+$10]).

In this paper, we establish some fundamental spectral properties of Dirac operators on star graphs. Since physically relevant vertex couplings are of local nature, star graphs play a role in quantum computing and vibrations of networks (see e.g. [Pou15], [PT15]) and may be viewed as building blocks for more complicated graph geometries (see e.g. [Piv15]); the latter also appear as shrinking limits of thin branched manifolds (see e.g. [EM14], [EP13]).

Here we study Dirac-Krein operators with potential on a compact star graph $\mathcal{G}$ with n edges e_j defined by differential expressions
$$-\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} p_j(x) & q_j(x) \\ q_j(x) & -p_j(x) \end{pmatrix}, \quad x \in e_j, \ j = 1, 2, \ldots, n,$$
on the edges with real-valued L^1-functions p_j, q_j, arbitrary symmetric boundary conditions at the outer vertices and a self-adjoint matching condition at the common central vertex. The main results may be summarized as follows. First we establish Krein's resolvent formula describing the resolvents of all self-adjoint extensions $\mathcal{T}$ of the corresponding minimal Dirac-Krein operator on $\mathcal{G}$ with respect to the fixed decoupled extension $\mathcal{T}_0$ given by Dirichlet conditions at the central vertex. The other two main results concern extensions $\mathcal{T}_\tau$ given by Robin matching conditions at the central vertex with parameter $\tau \in \mathbb{R} \cup \{\infty\}$. We establish a trace formula and we prove that the set of eigenvalues splits into two parts, simple eigenvalues that depend on the Robin parameter τ monotonically and possibly non-simple eigenvalues that are independent of τ. Moreover, we show that the eigenvalues for different Robin parameters τ_1 and τ_2 interlace. Thirdly we prove that, asymptotically for $R \to \infty$, the difference $d_R(\mathcal{T}_\tau)$ of the number of eigenvalues of $\mathcal{T}_\tau$ in intervals $[0, R)$ and $[-R, 0)$ differs from some fixed integer κ_0, which we call dislocation index, at most by $n + 2$; an even more precise estimate is given in terms of the boundary coefficients at the outer vertices.

A short synopsis of the paper is as follows. In Section 2 we introduce the L^2-spaces for the graph $\mathcal{G}$ and its edges $e_j = [v, v_j]$, the boundary conditions at the outer vertices v_j, the general form of the matching condition at the central vertex v, and the corresponding symmetric and self-adjoint operators. The boundary conditions $\cos \alpha_j \, f_j(v_j) + \sin \alpha_j \, \widehat{f}_j(v_j) = 0$ at the outer vertices are considered to be fixed in this paper, whereas the matching condition at v is varied; here f_j, $\widehat{f}_j$ are the two components of a vector function $\boldsymbol{f}_j = (f_j \ \widehat{f}_j)^{\mathrm{t}}$ on an edge e_j. Special attention is paid to matching condition of Robin type with parameter $\tau \in \mathbb{R} \cup \{\infty\}$, i.e.
$$f_1(v) = f_2(v) = \cdots = f_n(v), \quad f_1(v) = \tau\big(\widehat{f}_1(v) + \widehat{f}_2(v) + \cdots + \widehat{f}_n(v)\big).$$

In Section 3 we summarize results for a Dirac-Krein operator $\mathcal{T}_j$ on a single edge e_j (i.e. on an interval). Here the Weyl–Titchmarsh function

m_j is introduced and it is shown that it is a Nevanlinna function, i.e. it is holomorphic at least in $\mathbb{C} \setminus \mathbb{R}$, $m_j(\overline{z}) = \overline{m_j(z)}$, and $\operatorname{Im} m_j(z)/\operatorname{Im} z \geq 0$ for $z \in \mathbb{C} \setminus \mathbb{R}$. In this section we also derive a trace formula, i.e. an analytic expression for the trace of the resolvent difference $(\mathcal{T}_j - z_1)^{-1} - (\mathcal{T}_j - z_2)^{-1}$ in two points $z_1, z_2 \in \rho(\mathcal{T}_j)$.

The first main result of the paper is Theorem 4.4 in Section 4 where we describe the resolvents of the self-adjoint extensions $\mathcal{T}_{\mathcal{A},\mathcal{B}}$ of the minimal operator $\mathcal{S}$ on $\mathcal{G}$ through M.G. Krein's resolvent formula in terms of the boundary matrices in $\mathcal{A}(f_j(v))_1^n + \mathcal{B}(\widehat{f}_j(v))_1^n = 0$. It is the starting point for the study of the spectrum in the following sections. We also indicate how the Fourier expansion and the spectral representation for the operator $\mathcal{T}_{\mathcal{A},\mathcal{B}}$ follow (see Remark 4.6), but these questions are not considered in detail here.

Starting from Section 5, we concentrate on Robin matching conditions at the central vertex. We show that the zeros of the Weyl-Titchmarsh function

$$\boldsymbol{m}_\tau(z) := \frac{1}{\tau} + \sum_{j=1}^n m_j(z), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

give rise to the simple eigenvalues of $\mathcal{T}_\tau$, while possibly multiple eigenvalues occur if at least two m_j have a common pole. A corresponding result for symmetric relations on star graphs, but with $\tau = \infty$, was proved in [SW14]. We also study the monotonicity in τ of both types of eigenvalues, we prove an interlacing principle for the eigenvalues of the self-adjoint operators $\mathcal{T}_\tau$ for two different values of τ, and we derive a formula for the trace of the difference of the resolvents of the self-adjoint operators $\mathcal{T}_\tau$ and $\mathcal{T}_0$, i.e. for the perturbation determinant of $\mathcal{T}_\tau$ with respect to $\mathcal{T}_0$.

While most of the results of Sections 2-5 remain valid - with slight modifications - for more general canonical systems, this is not the case for the main theorem of Section 6 where we study the difference $d_R(\mathcal{T}_\tau)$ of the number of eigenvalues of $\mathcal{T}_\tau$ in intervals $[0, R)$ and $[-R, 0)$. We prove that there exists a number $\kappa_0 \in \mathbb{N}$, the dislocation index, such that, for $R > 0$ sufficiently large, $d_R(\mathcal{T}_\tau)$ differs from κ_0 at most by $n + 2$. More precisely,

$$-(n_\geq + 2) \leq d_R(\mathcal{T}_\tau) - \kappa_0 \leq n_\leq + 2$$

where $n_\geq$ and $n_\leq$ are the numbers of edges for which the parameters α_j in the boundary conditions at the outer vertices satisfy $\alpha_j \geq \frac{\pi}{2}$ and $\alpha_j \leq \frac{\pi}{2}$, respectively. The proof is based on the asymptotic behavior of the eigenvalues of Dirac-Krein operators on an interval established e.g. in [LM14a] (see also [LM15]) and on the interlacing principle proved in Section 5. Finally, we derive an analytic formula for the dislocation index κ_0 of $\mathcal{T}_\tau$ in Theorem 6.3 which is based on the trace formula for T_j in Section 3.

2. Symmetric vertex conditions

Consider a star graph $\mathcal{G}$ with central vertex v and edges $e_1, \ldots, e_n$ whose endpoints are denoted by $v_1, \ldots, v_n$. On each edge e_j we fix the direction to be outgoing from the central vertex v, i.e. we identify e_j with the interval

$[v, v_j]$. We denote by $\boldsymbol{L}^2(e_j) = L^2(e_j, \mathbb{C}) \oplus L^2(e_j, \mathbb{C})$ the Hilbert space of square-integrable 2-vector functions $\boldsymbol{f}_j = (f_j \; \widehat{f}_j)^{\mathrm{t}}$ on e_j with inner product

$$(\boldsymbol{f}_j, \boldsymbol{g}_j)_{\boldsymbol{L}^2(e_j)} := (f_j, g_j) + (\widehat{f}_j, \widehat{g}_j), \quad \boldsymbol{f}_j, \boldsymbol{g}_j \in \boldsymbol{L}^2(e_j),$$

and corresponding norm $\| \cdot \|_{\boldsymbol{L}^2(e_j)}$. Then the vector space $\mathcal{H} = \boldsymbol{L}^2(\mathcal{G}) :=$ $\boldsymbol{L}^2(e_1) \oplus \boldsymbol{L}^2(e_2) \oplus \cdots \oplus \boldsymbol{L}^2(e_n)$ of functions $\boldsymbol{f} = (\boldsymbol{f}_1 \; \boldsymbol{f}_2 \; \cdots \; \boldsymbol{f}_n)^{\mathrm{t}} =: (\boldsymbol{f}_j)_1^n$ becomes a Hilbert space if we equip it with inner product given by

$$(\boldsymbol{f}, \boldsymbol{g})_{\boldsymbol{L}^2(\mathcal{G})} := \sum_{j=1}^{n} \Big((f_j, g_j) + (\widehat{f}_j, \widehat{g}_j) \Big), \quad \boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{L}^2(\mathcal{G}),$$

and corresponding norm $\| \cdot \|_{\boldsymbol{L}^2(\mathcal{G})}$. The variables on e_j are denoted by x_j and those on $\mathcal{G}$ by $\boldsymbol{x} = (x_j)_1^n$, i.e. we identify $\boldsymbol{f}(\boldsymbol{x}) = (\boldsymbol{f}_j(x_j))_1^n$ for $\boldsymbol{x} = (x_j)_1^n$ with $x_j \in e_j$; in the particular case $x_j = v$, $j = 1, 2, \ldots, n$, we simply write $\boldsymbol{f}(v)$.

For fixed $j \in \{1, 2, \ldots, n\}$, we consider a so-called Dirac-Krein differential expression

$$H_j = -J \frac{\mathrm{d}}{\mathrm{d}x_j} + V_j \quad \text{on} \quad e_j = [v, v_j] \tag{2.1}$$

where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad V_j(x_j) := \begin{pmatrix} p_j(x_j) & q_j(x_j) \\ q_j(x_j) & -p_j(x_j) \end{pmatrix}, \quad x_j \in e_j, \tag{2.2}$$

with $p_j, q_j \in L^1(e_j, \mathbb{R})$, $j = 1, 2, \ldots, n$.

With the differential expression H_j in (2.1), which is regular at both end-points, we associate various operators in $\boldsymbol{L}^2(e_j)$. At the outer endpoint v_j of e_j we always fix the self-adjoint boundary condition

$$\cos \alpha_j \, f_j(v_j) + \sin \alpha_j \, \widehat{f}_j(v_j) = 0 \quad \text{with} \quad \alpha_j \in [0, \pi) \tag{2.3}$$

and introduce the corresponding maximal domain

$$\mathcal{D}_j := \{ \boldsymbol{f}_j \in \boldsymbol{L}^2(e_j) : \boldsymbol{f}_j \text{ abs. cont.}, \; H_j \boldsymbol{f}_j \in \boldsymbol{L}^2(e_j), \; \boldsymbol{f}_j \text{ satisfies (2.3)} \}.$$

The operators S_j and T_j in $\boldsymbol{L}^2(e_j)$ are defined by different boundary conditions at the central vertex v,

$$\begin{aligned} \operatorname{dom} S_j &:= \{ \boldsymbol{f}_j \in \mathcal{D}_j : f_j(v) = \widehat{f}_j(v) = 0 \}, & S_j \boldsymbol{f}_j &:= H_j \boldsymbol{f}_j, \\ \operatorname{dom} T_j &:= \{ \boldsymbol{f}_j \in \mathcal{D}_j : f_j(v) = 0 \}, & T_j \boldsymbol{f}_j &:= H_j \boldsymbol{f}_j. \end{aligned} \tag{2.4}$$

Then S_j is symmetric with defect index $(1, 1)$ and adjoint

$$\operatorname{dom} S_j^* = \mathcal{D}_j, \quad S_j^* \boldsymbol{f}_j = H_j \boldsymbol{f}_j,$$

and T_j is a self-adjoint extension of S_j,

$$S_j \subset T_j = T_j^* \subset S_j^*.$$

For $\boldsymbol{f}_j$, $\boldsymbol{g}_j \in \operatorname{dom} S_j^*$, using $J^* = -J$ we have

$$
\begin{aligned}
& (S_j^* \boldsymbol{f}_j, \boldsymbol{g}_j) - (\boldsymbol{f}_j, S_j^* \boldsymbol{g}_j) \\
&= \int_{e_j} \left(\left(-J \begin{pmatrix} f_j'(x_j) \\ \widehat{f}_j'(x_j) \end{pmatrix}, \begin{pmatrix} g_j(x_j) \\ \widehat{g}_j(x_j) \end{pmatrix} \right)_{\mathbb{C}^2} - \left(J \begin{pmatrix} f_j(x_j) \\ \widehat{f}_j(x_j) \end{pmatrix}, \begin{pmatrix} g_j'(x_j) \\ \widehat{g}_j{}'(x_j) \end{pmatrix} \right)_{\mathbb{C}^2} \right) \mathrm{d}x_j \\
&= -\left(\widehat{f}_j(v) \overline{g_j(v)} - f_j(v) \overline{\widehat{g}_j(v)} \right) \\
&= \left(J\boldsymbol{f}_j(v), \boldsymbol{g}_j(v) \right)_{\mathbb{C}^2}.
\end{aligned}
$$

On the graph $\mathcal{G}$ we study the self-adjoint extensions of the symmetric operator

$$
\mathcal{S} := S_1 \oplus S_2 \oplus \cdots \oplus S_n \tag{2.5}
$$

in $\boldsymbol{L}^2(\mathcal{G})$, i.e. the self-adjoint restrictions of the adjoint $\mathcal{S}^* = S_1^* \oplus S_2^* \oplus \cdots \oplus S_n^*$. To this end, it is more advantageous to reorder the $2n$ components of $\boldsymbol{L}^2(\mathcal{G})$. Thus, for $\boldsymbol{u} = \left(u_1 \ \widehat{u}_1 \ u_2 \ \widehat{u}_2 \ \cdots \ u_n \ \widehat{u}_n \right)^{\mathrm{t}} \in \mathbb{C}^{2n}$ we define $\boldsymbol{u}^\sharp \in \mathbb{C}^{2n}$ by

$$
\boldsymbol{u}^\sharp := \left(u_1 \ u_2 \ \cdots \ u_n \ \widehat{u}_1 \ \widehat{u}_2 \ \cdots \ \widehat{u}_n \right)^{\mathrm{t}} = \boldsymbol{P} \boldsymbol{u}, \tag{2.6}
$$

where $\boldsymbol{P} = \left(p_{ij} \right)_{i,j=1}^{2n} \in M_{2n}(\mathbb{C})$ is the permutation matrix given by

$$
p_{ij} := \begin{cases} 1, & j = 2i - 1, \\ 0, & j \in \{1, 2, \ldots, 2n\} \setminus \{2i-1\}, \\ 1, & j = 2(i-n), \\ 0, & j \in \{1, 2, \ldots, 2n\} \setminus \{2(i-n)\}, \end{cases} \quad \begin{array}{l} i = 1, 2, \ldots, n, \\[2ex] i = n+1, n+2, \ldots, 2n; \end{array}
$$

note that $\boldsymbol{P}^{-1} = \boldsymbol{P}^{\mathrm{t}}$. Correspondingly, for $\boldsymbol{f} \in \boldsymbol{L}^2(\mathcal{G})$ we set

$$
\boldsymbol{f}^\sharp(\boldsymbol{x}) := \boldsymbol{f}(\boldsymbol{x})^\sharp, \quad \boldsymbol{x} \in \mathcal{G}. \tag{2.7}
$$

Further, we denote the components of $\boldsymbol{u}^\sharp$ and $\boldsymbol{f}^\sharp$ with respect to the decompositions $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ and $\boldsymbol{L}^2(\mathcal{G}) = \boldsymbol{L}^2(e_j)^n \oplus \boldsymbol{L}^2(e_j)^n$, respectively, by

$$
\boldsymbol{u}^\sharp = \begin{pmatrix} \boldsymbol{u}_1^\sharp \\ \boldsymbol{u}_2^\sharp \end{pmatrix} := \begin{pmatrix} (u_i)_1^n \\ (\widehat{u}_i)_1^n \end{pmatrix}, \quad \boldsymbol{f}^\sharp = \begin{pmatrix} \boldsymbol{f}_1^\sharp \\ \boldsymbol{f}_2^\sharp \end{pmatrix} := \begin{pmatrix} (f_i)_1^n \\ (\widehat{f}_i)_1^n \end{pmatrix}. \tag{2.8}
$$

With this notation, for $\boldsymbol{f} = (\boldsymbol{f}_j)_1^n$, $\boldsymbol{g} = (\boldsymbol{g}_j)_1^n \in \operatorname{dom} \mathcal{S}^* = \bigoplus_1^n \operatorname{dom} S_j^*$, we obtain

$$
\begin{aligned}
(\mathcal{S}^* \boldsymbol{f}, \boldsymbol{g}) - (\boldsymbol{f}, \mathcal{S}^* \boldsymbol{g}) &= \sum_{j=1}^n (S_j^* \boldsymbol{f}_j, \boldsymbol{g}_j) - \sum_{j=1}^n (\boldsymbol{f}_j, S_j^* \boldsymbol{g}_j) = \sum_{j=1}^n \left(J \boldsymbol{f}_j(v), \boldsymbol{g}_j(v) \right)_{\mathbb{C}^2} \\
&= -\left((\boldsymbol{f}_2^\sharp(v), \boldsymbol{g}_1^\sharp(v))_{\mathbb{C}^n} - (\boldsymbol{f}_1^\sharp(v), \boldsymbol{g}_2^\sharp(v))_{\mathbb{C}^n} \right) \\
&= \left(\mathbf{J}_{2n} \boldsymbol{f}^\sharp(v), \boldsymbol{g}^\sharp(v) \right)_{\mathbb{C}^{2n}}
\end{aligned}
$$

with

$$
\mathbf{J}_{2n} := \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix} = \boldsymbol{P} \operatorname{diag}(\underbrace{J, \ldots, J}_{n \text{ times}}) \boldsymbol{P}^{\mathrm{t}},
$$

where $I_n, 0_n \in M_n(\mathbb{C})$ are the unit and zero matrix, respectively.

It is well-known that the self-adjoint extensions $\mathcal{T}$ of $\mathcal{S}$ in $\boldsymbol{L}^2(\mathcal{G})$ can be described as follows (comp. [AG93, Appendix II.3])). If we denote $\psi :$ $\operatorname{dom}\mathcal{S}^* \to \mathbb{C}^{2n}$, $\psi(\boldsymbol{f}) := \boldsymbol{f}^\sharp(v)$, then $\mathcal{S} \subset \mathcal{T} = \mathcal{T}^* \subset \mathcal{S}^*$ if and only if $\psi(\operatorname{dom}\mathcal{T})$ is a maximal $\mathbf{J}_{2n}$-neutral subspace of $\mathbb{C}^{2n}$. Every such subspace is of the form $\mathcal{L}_{\mathcal{A},\mathcal{B}} = \{\boldsymbol{u}^\sharp = \left(\boldsymbol{u}_1^\sharp\ \boldsymbol{u}_2^\sharp\right)^{\mathrm{t}} \in \mathbb{C}^n \oplus \mathbb{C}^n : \mathcal{A}\boldsymbol{u}_1^\sharp + \mathcal{B}\boldsymbol{u}_2^\sharp = 0\}$ with a pair of matrices $\mathcal{A}, \mathcal{B} \in M_n(\mathbb{C})$ such that

$$\operatorname{rank}\left(\mathcal{A}\ \mathcal{B}\right) = n, \quad \mathcal{A}\mathcal{B}^* = \mathcal{B}\mathcal{A}^*, \tag{2.9}$$

(see e.g. [KS99, Lemma 2.2]). This proves the following.

Proposition 2.1. *The self-adjoint extensions of $\mathcal{S}$ in $\boldsymbol{L}^2(\mathcal{G})$ are the restrictions $\mathcal{T}_{\mathcal{A},\mathcal{B}}$ of $\mathcal{S}^*$ by a vertex condition*

$$\mathcal{A}\,\boldsymbol{f}_1^\sharp(v) + \mathcal{B}\boldsymbol{f}_2^\sharp(v) = 0 \tag{2.10}$$

with $\mathcal{A}, \mathcal{B} \in M_n(\mathbb{C})$ satisfying (2.9).

Note that the case $\mathcal{A} = I_n$, $\mathcal{B} = 0_n$ corresponds to Dirichlet conditions $\boldsymbol{f}_1^\sharp(v) = 0$ at the central vertex and hence the corresponding self-adjoint extension decouples,

$$\mathcal{T}_0 := \mathcal{T}_{I_n,0_n} = T_1 \oplus T_2 \oplus \cdots \oplus T_n. \tag{2.11}$$

Particular attention will be paid to the self-adjoint extensions $\mathcal{T}_\tau$ given by Robin matching conditions at the central vertex v,

$$f_1(v) = f_2(v) = \cdots = f_n(v), \tag{2.12}$$

$$f_1(v) = \tau\big(\widehat{f}_1(v) + \widehat{f}_2(v) + \cdots + \widehat{f}_n(v)\big) \quad \text{with}\ \ \tau \in \mathbb{R} \cup \{\pm\infty\}. \tag{2.13}$$

Here, for $\tau \neq \pm\infty$, the matrices $\mathcal{A}, \mathcal{B}$ may be chosen as

$$\mathcal{A}_\tau := \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \mathcal{B}_\tau := \tau \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{2.14}$$

Note that, in accordance with the notation in (2.11), $\tau = 0$ amounts to Dirichlet conditions $\boldsymbol{f}_1^\sharp(v) = 0$ at v,

$$f_1(v) = f_2(v) = \cdots = f_n(v) = 0,$$

while $\tau = \infty$ and $\tau = -\infty$ both amount to

$$f_1(v) = f_2(v) = \cdots = f_n(v), \qquad \widehat{f}_1(v) + \widehat{f}_2(v) + \cdots + \widehat{f}_n(v) = 0 \tag{2.15}$$

with corresponding matrices $\mathcal{A}_{\pm\infty}$, $\mathcal{B}_{\pm\infty}$. We mention that, for Schrödinger operators, conditions of the form (2.15) with $\widehat{f}_j$ replaced by f'_j are called standard matching or Kirchhoff conditions; here these names may be less appropriate since the two components have equal roles and may be exchanged.

Later we need the following well-known lemma.

Lemma 2.2. *Let* $\mathcal{A}$, $\mathcal{B} \in M_n(\mathbb{C})$ *be such that*

$$\operatorname{rank}(\mathcal{A}\ \mathcal{B}) = n, \quad \mathcal{A}\mathcal{B}^* - \mathcal{B}\mathcal{A}^* = 0_n. \tag{2.16}$$

Then there exist $\mathcal{C}$, $\mathcal{D} \in M_n(\mathbb{C})$ *such that*

$$\operatorname{rank}(\mathcal{C}\ \mathcal{D}) = n, \quad \mathcal{C}\mathcal{D}^* - \mathcal{D}\mathcal{C}^* = 0_n, \tag{2.17}$$

and

$$\mathcal{B}\mathcal{C}^* - \mathcal{A}\mathcal{D}^* = I_n. \tag{2.18}$$

Proof. If we set $\mathcal{Q} := \mathcal{A}\mathcal{A}^* + \mathcal{B}\mathcal{B}^*$, it is easy to check that all conditions are satisfied e.g. for the matrices

$$\mathcal{C} := \mathcal{Q}^{-1}\mathcal{B}, \quad \mathcal{D} := -\mathcal{Q}^{-1}\mathcal{A}. \qquad \square$$

Clearly, the conditions $\mathcal{A}\mathcal{B}^* - \mathcal{B}\mathcal{A}^* = \mathcal{C}\mathcal{D}^* - \mathcal{D}\mathcal{C}^* = 0_n$ and $\mathcal{B}\mathcal{C}^* - \mathcal{A}\mathcal{D}^* = I_n$ in Lemma 2.2 mean that

$$\begin{pmatrix} \mathcal{C} & \mathcal{D} \\ \mathcal{A} & \mathcal{B} \end{pmatrix} \mathbf{J}_{2n} \begin{pmatrix} \mathcal{C}^* & \mathcal{A}^* \\ \mathcal{D}^* & \mathcal{B}^* \end{pmatrix} = \mathbf{J}_{2n},$$

which implies

$$\begin{pmatrix} \mathcal{C}^* & \mathcal{A}^* \\ \mathcal{D}^* & \mathcal{B}^* \end{pmatrix} \mathbf{J}_{2n} \begin{pmatrix} \mathcal{C} & \mathcal{D} \\ \mathcal{A} & \mathcal{B} \end{pmatrix} = \mathbf{J}_{2n}. \tag{2.19}$$

We remark that neither the choice of the matrices $\mathcal{A}$, $\mathcal{B}$ in (2.10) is unique nor is the choice of $\mathcal{C}$, $\mathcal{D}$ in Lemma 2.2 for given $\mathcal{A}$, $\mathcal{B}$. For example, it is easy to check that for the matrices $\mathcal{A}_\tau$, $\mathcal{B}_\tau$ in (2.14) for $\tau \neq \pm\infty$, instead of the matrices $\mathcal{C}$, $\mathcal{D}$ in the proof of Lemma 2.2 one may also choose

$$\mathcal{C}_\tau := \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \mathcal{D}_\tau := \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

3. Regular Dirac-Krein systems on an interval

In this section we collect some basic results for Dirac–Krein systems on an edge $e_j = [v, v_j]$ with $j \in \{1, 2, \ldots, n\}$ fixed; for simplicity, we write x for the variable on e_j here.

Assume p_j, $q_j \in L^1(e_j, \mathbb{R})$ and let J, V_j be as in (2.2). We consider the regular problem

$$-J\boldsymbol{f}_j'(x) + V_j(x)\boldsymbol{f}_j(x) - z\boldsymbol{f}_j(x) = \boldsymbol{g}_j(x), \quad \boldsymbol{g}_j \in \mathbf{L}^2(e_j), \quad x \in e_j, \ z \in \mathbb{C}, \tag{3.1}$$

for $\boldsymbol{f}_j = (f_j\ \widehat{f}_j)^{\mathrm{t}} \in \boldsymbol{L}^2(e_j) \oplus \boldsymbol{L}^2(e_j)$ subject to the boundary conditions

$$f_j(v) = 0, \quad \cos\alpha_j\, f_j(v_j) + \sin\alpha_j\, \widehat{f}_j(v_j) = 0 \ \text{ with } \ \alpha_j \in [0, \pi). \tag{3.2}$$

Using the self-adjoint operator T_j introduced in (2.4), the inhomogeneous problem (3.1), (3.2) can be written as

$$(T_j - z)\boldsymbol{f}_j = \boldsymbol{g}_j, \quad \boldsymbol{f}_j \in \operatorname{dom} T_j.$$

In the sequel we derive formulas for the resolvent of the self-adjoint operator T_j in $\boldsymbol{L}^2(e_j)$ defined in (2.4), as well as a trace formula for T_j.

3.1.

Denote by $Y_j(\cdot, z)$, $z \in \mathbb{C}$, the fundamental matrix of (3.1) with initial value I_2 at v, i.e. the unique 2×2–matrix function solving the initial value problem

$$- JY_j'(x, z) + V_j(x)Y_j(x, z) = zY_j(x, z), \quad x \in e_j, \quad Y_j(v, z) = I_2. \quad (3.3)$$

The function $Y_j(x, \cdot)$ is entire for each $x \in e_j$, satisfies $Y_j(\cdot, \overline{z}) = \overline{Y_j(\cdot, z)}$, and has the property

$$J - Y_j(x, \zeta)^* JY_j(x, z) = (z - \overline{\zeta}) \int_v^x Y_j(\xi, \zeta)^* Y_j(\xi, z) \, d\xi, \quad z, \zeta \in \mathbb{C}; \quad (3.4)$$

this follows easily from differentiating both sides and using (3.3). In particular, (3.4) for $\zeta = \overline{z}$ shows that

$$J = Y_j(x, \overline{z})^* JY_j(x, z) = Y_j(x, z)JY_j(x, \overline{z})^*, \quad x \in e_j, \ z \in \mathbb{C}. \quad (3.5)$$

We introduce the *Weyl–Titchmarsh function* m_j of the j-th edge, by the equation

$$\begin{pmatrix} \cos \alpha_j & \sin \alpha_j \end{pmatrix} Y_j(v_j, z) \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix} = 0, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.6)$$

Then m_j is meromorphic since, with $Y_j(\cdot, z) =: (y_{j,kl}(\cdot, z))_{k,l=1}^2$,

$$m_j(z) = \frac{y_{j,11}(v_j, z) \cos \alpha_j + y_{j,21}(v_j, z) \sin \alpha_j}{y_{j,12}(v_j, z) \cos \alpha_j + y_{j,22}(v_j, z) \sin \alpha_j} =: \frac{b_j(z)}{c_j(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.7)$$

The relation (3.6) can equivalently be written as

$$Y_j(v_j, z) \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix} = \begin{pmatrix} -\sin \alpha_j \\ \cos \alpha_j \end{pmatrix} \frac{1}{c_j(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}; \quad (3.8)$$

in fact, by (3.7) and (3.5),

$$c_j(z)Y_j(v_j, z) \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}$$
$$= Y_j(v_j, z) \begin{pmatrix} -c_j(z) \\ b_j(z) \end{pmatrix} = \overbrace{Y_j(v_j, z)JY_j(v_j, \overline{z})^*}^{=J} \begin{pmatrix} \cos \alpha_j \\ \sin \alpha_j \end{pmatrix} = \begin{pmatrix} -\sin \alpha_j \\ \cos \alpha_j \end{pmatrix}.$$

Proposition 3.1. *For $z \in \mathbb{C} \setminus \mathbb{R}$ and arbitrary $\boldsymbol{g}_j \in \mathbf{L}^2(e_j)$, the solution of the inhomogeneous problem (3.1), (3.2), i.e. the resolvent $(T_j - z)^{-1}$ is given by*

$$\big((T_j - z)^{-1} \boldsymbol{g}_j\big)(x) \quad (3.9)$$

$$= Y_j(x, z) \left(\begin{pmatrix} 0 & -1 \\ 0 & m_j(z) \end{pmatrix} \int_v^x Y_j(\xi, \overline{z})^* \boldsymbol{g}_j(\xi) d\xi + \begin{pmatrix} 0 & 0 \\ -1 & m_j(z) \end{pmatrix} \int_x^{v_j} Y_j(\xi, \overline{z})^* \boldsymbol{g}_j(\xi) d\xi \right)$$

for $x \in e_j = [v, v_j]$.

Proof. We sketch the proof for the convenience of the reader. A straightforward calculation using (3.3) and (3.5) shows that, for $\boldsymbol{f}_j = (T_j - z)^{-1}\boldsymbol{g}_j$,

$$-J\boldsymbol{f}_j'(x) + V_j(x)\boldsymbol{f}_j(x) - z\boldsymbol{f}_j(x)$$
$$= -JY_j(x,z)\left(\begin{pmatrix} 0 & -1 \\ 0 & m_j(z) \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & m_j(z) \end{pmatrix}\right) Y_j(x,\overline{z})^* \boldsymbol{g}_j(x)$$
$$= -JY_j(x,z)JY_j(x,\overline{z})^* \boldsymbol{g}_j(x) = \boldsymbol{g}_j(x).$$

The boundary condition $f_j(v) = 0$ is satisfied since the first integral in (3.9) vanishes at $x = v$ and the first component of the second term is 0. The boundary condition $\cos\alpha_j f_j(v_j) + \sin\alpha_j \widehat{f}_j(v_j) = 0$ is satisfied since, by (3.6), it is satisfied by the first term and the second integral in (3.9) vanishes at $x = v_j$. $\square$

In the following scalar- and matrix-valued Nevanlinna functions will play an important role, so we recall the notion for convenience. An $M_n(\mathbb{C})$-valued function $\mathcal{M}$ is a *matrix Nevanlinna function*, or Nevanlinna function if $n = 1$, if it is defined and locally holomorphic at least on $\mathbb{C} \setminus \mathbb{R}$ with

$$\mathcal{M}(\overline{z}) = \mathcal{M}(z)^*, \quad \frac{\operatorname{Im}\mathcal{M}(z)}{\operatorname{Im}z} \geq 0, \quad z \in \mathbb{C} \setminus \mathbb{R}; \tag{3.10}$$

the maximal domain of $\mathcal{M}$ consists of $\mathbb{C} \setminus \mathbb{R}$ together with all real points into which $\mathcal{M}$ can be continued analytically. A matrix Nevanlinna function admits an integral representation

$$\mathcal{M}(z) = \mathcal{M}_0 + \mathcal{M}_1 z + \int_{\mathbb{R}}\left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\right) d\Sigma(\lambda), \quad z \in \mathbb{C} \setminus \operatorname{supp}\Sigma,$$

where Σ is a non-decreasing function on $\mathbb{R}$ whose values are symmetric matrices in $M_n(\mathbb{C})$ such that $\int_{\mathbb{R}} \frac{d\Sigma(\lambda)}{1 + \lambda^2} < \infty$, $\operatorname{supp}\Sigma$ denotes the set of points of increase of Σ, and $\mathcal{M}_0, \mathcal{M}_1 \in M_n(\mathbb{C})$ are symmetric matrices with $\mathcal{M}_1 \geq 0$. The matrix Nevanlinna functions appearing in this paper are all meromorphic, i.e. the integral above is in fact a sum, and all their singularities are poles of first order with negative residue.

Further, we recall that a kernel $N : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ with $\mathcal{D} \subset \mathbb{C}$ is called *positive definite* if for all $n \in \mathbb{N}$, $z_i \in \mathcal{D}$, $w_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$,

$$\sum_{i,k=1}^{n} N(z_i, z_k) w_i \overline{w_k} \geq 0.$$

In order to show that m_j in (3.7) is a Nevanlinna function, for points $z \in \mathbb{C}$ where m_j is holomorphic we define $\Gamma_{j,z} : \mathbb{C} \to \boldsymbol{L}^2(e_j)$ by

$$\Gamma_{j,z}c := Y_j(\cdot, z)\begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}c, \quad c \in \mathbb{C}.$$

Lemma 3.2. *The function m_j has the property that*

$$N_{m_j}(z, \zeta) := \frac{m_j(z) - \overline{m_j(\zeta)}}{z - \overline{\zeta}} = \Gamma_{j,\zeta}^* \Gamma_{j,z}, \quad z, \zeta \in \mathbb{C} \setminus \mathbb{R}, \; z \neq \overline{\zeta}. \tag{3.11}$$

Hence N_{m_j} is a positive definite kernel and m_j is a Nevanlinna function with $\operatorname{Im} m_j(z) > 0$ for $\operatorname{Im} z > 0$.

Proof. Using (3.8) and (3.4), we find

$$
\begin{aligned}
\frac{m_j(z) - \overline{m_j(\zeta)}}{z - \overline{\zeta}}
&= \frac{\left(-1 \ \ \overline{m_j(\zeta)}\right) J \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}}{z - \overline{\zeta}} \\
&= \frac{1}{\overline{c_j(\zeta)}} \left(-\sin\alpha_j \ \ \cos\alpha_j\right) \frac{Y_j(v_j, \zeta)^{-*} J Y_j(v_j, z)^{-1}}{z - \overline{\zeta}} \begin{pmatrix} -\sin\alpha_j \\ \cos\alpha_j \end{pmatrix} \frac{1}{c_j(z)} \\
&= \frac{1}{\overline{c_j(\zeta)}} \left(-\sin\alpha_j \ \ \cos\alpha_j\right) \frac{Y_j(v_j, \zeta)^{-*} J Y_j(v_j, z)^{-1} - J}{z - \overline{\zeta}} \begin{pmatrix} -\sin\alpha_j \\ \cos\alpha_j \end{pmatrix} \frac{1}{c_j(z)} \\
&= \left(-1 \ \ \overline{m_j(\zeta)}\right) \frac{J - Y_j(v_j, \zeta)^* J Y_j(v_j, z)}{z - \overline{\zeta}} \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix} \\
&= \left(-1 \ \ \overline{m_j(\zeta)}\right) \int_v^{v_j} Y_j(\xi, \zeta)^* Y_j(\xi, z) \, d\xi \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix} \\
&= \Gamma_{j,\zeta}^* \Gamma_{j,z}.
\end{aligned}
$$

Further, for all $n \in \mathbb{N}$, $z_i \in \mathbb{C} \setminus \mathbb{R}$, $w_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$,

$$
\sum_{i,k=1}^n \Gamma_{j,z_k}^* \Gamma_{j,z_i} w_i \overline{w_k} = \left(\sum_{i=1}^n \Gamma_{j,z_i} w_i, \sum_{i=1}^n \Gamma_{j,z_i} w_i \right)_{\boldsymbol{L}^2(e_j)} \geq 0.
$$

Since $\Gamma_{j,z} \neq 0$, the last claim follows from (3.11). $\qquad\square$

3.2.

The resolvent $(T_j - z)^{-1}$, $z \in \rho(T_j)$, is a Hilbert–Schmidt operator since it is an integral operator with square integrable kernel on $e_j \times e_j$,

$$
\left((T_j - z)^{-1} \boldsymbol{g}_j\right)(x) = \int_v^{v_j} G_j(x, \xi; z) \boldsymbol{g}_j(\xi) \, d\xi, \quad \boldsymbol{g}_j \in \boldsymbol{L}^2(e_j), \ x \in e_j,
$$

with Green's function

$$
G_j(x, \xi; z) := \begin{cases} Y_j(x, z) \begin{pmatrix} 0 & -1 \\ 0 & m_j(z) \end{pmatrix} Y_j(\xi, \overline{z})^*, & v \leq \xi < x \leq v_j, \\[12pt] Y_j(x, z) \begin{pmatrix} 0 & 0 \\ -1 & m_j(z) \end{pmatrix} Y_j(\xi, \overline{z})^*, & v \leq x < \xi \leq v_j. \end{cases} \tag{3.12}
$$

Consequently, for $z_1, z_2 \in \rho(T_j)$, the difference

$$
(T_j - z_1)^{-1} - (T_j - z_2)^{-1} = (z_1 - z_2)(T_j - z_1)^{-1}(T_j - z_2)^{-1} \tag{3.13}
$$

is a trace class operator. Since, by (3.12) and (3.5),

$$
G_j(x, x+0; z) - G_j(x, x-0; z) = Y_j(x, z)(-J)Y_j(x, \overline{z})^* = -J, \ \ x \in e_j, \tag{3.14}
$$

the resolvent difference in (3.13) is an integral operator with continuous kernel. Hence its trace can be calculated as (see e.g. [GK69])

$$\operatorname{tr}\left((T_j - z_1)^{-1} - (T_j - z_2)^{-1}\right) = \operatorname{tr}\int_v^{v_j}\left(G_j(x, x+0; z_1) - G_j(x, x+0; z_2)\right)\mathrm{d}x. \quad (3.15)$$

Proposition 3.3. *For $z \in \rho(T_j)$, let $Y_j(\cdot, z) = (y_{j,kl}(\cdot, z))_{k,l=1}^2$ and let*

$$c_j(z) = y_{j,12}(v_j, z)\cos\alpha_j + y_{j,22}(v_j, z)\sin\alpha_j$$

be the denominator in the Weyl–Titchmarsh function m_j in (3.7). Then

$$\operatorname{tr}\left((T_j - z_1)^{-1} - (T_j - z_2)^{-1}\right) = -\frac{\dot{c}_j(z_1)}{c_j(z_1)} + \frac{\dot{c}_j(z_2)}{c_j(z_2)}, \quad z_1, z_2 \in \rho(T_j). \quad (3.16)$$

In the proof of Proposition 3.3 we use the following lemma.

Lemma 3.4. *If we set $\dot{Y}_j(x, z) := \dfrac{\partial Y_j(x, z)}{\partial z}$, then*

$$\int_v^{v_j} Y_j(x, \overline{z})^* Y_j(x, z)\,\mathrm{d}x = -Y_j(v_j, \overline{z})^* J\dot{Y}_j(v_j, z), \quad z \in \mathbb{C}.$$

Proof. Taking partial derivatives with respect to z in (3.3) for $Y_j(x, z)$, we find that, for $z \in \mathbb{C}$, the function $\dot{Y}_j(x, z)$ satisfies the initial problem

$$-J\dot{Y}_j'(x, z) + V_j(x)\dot{Y}_j(x, z) - z\dot{Y}_j(x, z) = Y_j(x, z), \quad x \in e_j, \quad \dot{Y}_j(v, z) = 0.$$

A straightforward computation shows that

$$Y_j(x, \overline{z})^* Y_j(x, z) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(Y_j(x, \overline{z})^* J\dot{Y}_j(x, z)\right),$$

and integration from v to v_j yields the claim. $\qquad\square$

Proof of Proposition 3.3. Let $z \in \rho(T_j)$. Because of (3.15), we first consider

$$\begin{aligned}
\operatorname{tr}\int_v^{v_j} G_j(x, x+0; z)\,\mathrm{d}x &= \operatorname{tr}\int_v^{v_j} Y_j(x, z)\begin{pmatrix} 0 & 0 \\ -1 & m_j(z) \end{pmatrix}Y_j(x, \overline{z})^*\,\mathrm{d}x \\
&= \int_v^{v_j}\operatorname{tr}\left(Y_j(x, z)\begin{pmatrix} 0 & 0 \\ -1 & m_j(z) \end{pmatrix}Y_j(x, \overline{z})^*\right)\mathrm{d}x \\
&= \int_v^{v_j}\operatorname{tr}\left(Y_j(x, \overline{z})^*Y_j(x, z)\begin{pmatrix} 0 & 0 \\ -1 & m_j(z) \end{pmatrix}\right)\mathrm{d}x \\
&= \operatorname{tr}\int_v^{v_j} Y_j(x, \overline{z})^*Y_j(x, z)\begin{pmatrix} 0 & 0 \\ -1 & m_j(z) \end{pmatrix}\mathrm{d}x \\
&= \operatorname{tr}\left(\left(\int_v^{v_j} Y_j(x, \overline{z})^*Y_j(x, z)\,\mathrm{d}x\right)\begin{pmatrix} 0 & 0 \\ -1 & m_j(z) \end{pmatrix}\right).
\end{aligned}$$

By Lemma 3.4, (3.8), and the definition of c_j in (3.7), we conclude that

$$\operatorname{tr} \int_v^{v_j} G_j(x, x+0; z)\,\mathrm{d}x = -\operatorname{tr}\left(Y_j(v_j, \overline{z})^* J \dot{Y}_j(v_j, z) \begin{pmatrix} 0 & 0 \\ -1 & m_j(z) \end{pmatrix} \right)$$

$$= -\operatorname{tr}\left(\begin{pmatrix} 0 & 0 \\ -1 & m_j(z) \end{pmatrix} Y_j(v_j, \overline{z})^* J \dot{Y}_j(v_j, z) \right)$$

$$= -\frac{1}{c_j(z)} \operatorname{tr}\left(\begin{pmatrix} 0 & 0 \\ -\sin\alpha_j & \cos\alpha_j \end{pmatrix} J \dot{Y}_j(v_j, z) \right)$$

$$= -\frac{\dot{c}_j(z)}{c_j(z)}.$$

Now the claim follows from (3.15). □

Remark 3.5. The trace formula (3.16) can be used to find a formula for a "regularized trace" of $(T_j - z)^{-1}$. In fact, since $T_j(T_j^2 + I)^{-1} = \frac{1}{2}(T_j + \mathrm{i})^{-1} + \frac{1}{2}(T_j - \mathrm{i})^{-1}$ and $Y_j(\cdot, -\mathrm{i}) = \overline{Y_j(\cdot, \mathrm{i})}$, we have

$$\operatorname{tr}\left((T_j - z)^{-1} - T_j(T_j^2 + I)^{-1} \right) = -\frac{\dot{c}_j(z)}{c_j(z)} + \operatorname{Re}\frac{\dot{c}_j(\mathrm{i})}{c_j(\mathrm{i})}.$$

If we denote the eigenvalues of T_j, which are all simple, by $\mu_{j,k}$, $k \in \mathbb{Z}$, this relation becomes

$$\sum_{k=-\infty}^{\infty} \left(\frac{1}{\mu_{j,k} - z} - \frac{\mu_{j,k}}{1 + \mu_{j,k}^2} \right) = -\frac{\dot{c}_j(z)}{c_j(z)} + \operatorname{Re}\frac{\dot{c}_j(\mathrm{i})}{c_j(\mathrm{i})};$$

note that the right hand side is a Nevanlinna function since so is the left hand side.

4. Dirac-Krein systems on a star graph

In this section we study the Dirac–Krein systems on a star graph $\mathcal{G}$ introduced in Section 2. We prove a resolvent formula describing all canonical self-adjoint extensions of the symmetric operator $\mathcal{S} = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ with respect to the self-adjoint extension $\mathcal{T}_0 = \mathcal{T}_{I_n, 0_n} = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ given by Dirichlet conditions $\boldsymbol{f}_1^\sharp(v) = 0$ at the central vertex.

Since the operator $\mathcal{T}_0$ decouples into independent operators on the edges, so does its resolvent $(\mathcal{T}_0 - z)^{-1} = (T_1 - z)^{-1} \oplus (T_2 - z)^{-1} \oplus \cdots \oplus (T_n - z)^{-1}$. The corresponding fundamental matrix $\boldsymbol{Y}(\cdot, z)$ with $\boldsymbol{Y}(v, z) = I_{2n}$ is block-diagonal,

$$\boldsymbol{Y}(\cdot, z) := \operatorname{diag}\left(Y_1(\cdot, z), Y_2(\cdot, z), \ldots, Y_n(\cdot, z) \right), \quad z \in \mathbb{C}. \qquad (4.1)$$

For the extensions $\mathcal{T}_{\mathcal{A}, \mathcal{B}}$ with boundary condition $\mathcal{A}\boldsymbol{f}_1^\sharp(v) + \mathcal{B}\boldsymbol{f}_2^\sharp(v) = 0$ at v, it is more convenient to reorder the components of $\boldsymbol{Y}$, using the notation (2.6), (2.7).

In the following the matrix function $\mathcal{M}$ given by

$$\mathcal{M}(z) := \operatorname{diag}\left(m_1(z), m_2(z), \ldots, m_n(z) \right), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

plays a crucial role. Clearly, $\mathcal{M}$ is a matrix Nevanlinna function (see (3.10)) since all m_j are Nevanlinna functions by Lemma 3.2. The properties (2.9) of $\mathcal{A}$, $\mathcal{B}$ and the fact that $\operatorname{Im}\mathcal{M}(z)$ is strictly positive for $\operatorname{Im} z > 0$ imply that $\mathcal{B}\mathcal{M}(z) - \mathcal{A}$ is invertible for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Proposition 4.1. *Define* $\boldsymbol{Y}(\cdot, z)^{\sharp} := \boldsymbol{P}\boldsymbol{Y}(\cdot, z)\boldsymbol{P}^{\mathrm{t}}$. *Then, for* $z \in \mathbb{C} \setminus \mathbb{R}$ *and arbitrary* $\boldsymbol{g} = (\boldsymbol{g}_j)_1^n \in \boldsymbol{L}^2(\mathcal{G})$, *the resolvent* $(\mathcal{T}_0 - z)^{-1}$ *satisfies*

$$
\begin{aligned}
\left((\mathcal{T}_0 - z)^{-1}\boldsymbol{g}\right)^{\sharp}(\boldsymbol{x}) = \boldsymbol{Y}(\boldsymbol{x}, z)^{\sharp}\Bigg(&\begin{pmatrix} 0_n & -I_n \\ 0_n & \mathcal{M}(z) \end{pmatrix}\left(\int_v^{x_j} Y_j(\xi_j, \overline{z})^* \boldsymbol{g}_j(\xi_j)\mathrm{d}\xi_j\right)_1^{n\,\sharp} \\
&+ \begin{pmatrix} 0_n & 0_n \\ -I_n & \mathcal{M}(z) \end{pmatrix}\left(\int_{x_j}^{v_j} Y_j(\xi_j, \overline{z})^* \boldsymbol{g}_j(\xi_j)\mathrm{d}\xi_j\right)_1^{n\,\sharp}\Bigg)
\end{aligned} \tag{4.2}
$$

for $\boldsymbol{x} = (x_j)_1^n$, $x_j \in e_j = [v, v_j]$, *while the resolvent* $(\mathcal{T}_{\mathcal{A},\mathcal{B}} - z)^{-1}$ *satisfies*

$$
\begin{aligned}
\left((\mathcal{T}_{\mathcal{A},\mathcal{B}} - z)^{-1}\boldsymbol{g}\right)^{\sharp}(\boldsymbol{x}) = &\left((\mathcal{T}_0 - z)^{-1}\boldsymbol{g}\right)^{\sharp}(\boldsymbol{x}) - \boldsymbol{Y}(\boldsymbol{x}, z)^{\sharp}. \\
&\cdot \begin{pmatrix} -I_n \\ \mathcal{M}(z) \end{pmatrix}(\mathcal{B}\mathcal{M}(z) - \mathcal{A})^{-1}\mathcal{B}\begin{pmatrix} -I_n & \mathcal{M}(z) \end{pmatrix}\left(\int_v^{v_j} Y_j(\xi_j, \overline{z})^* \boldsymbol{g}_j(\xi_j)\mathrm{d}\xi_j\right)_1^{n\sharp}.
\end{aligned} \tag{4.3}
$$

Proof. The formula for $(\mathcal{T}_0 - z)^{-1}$ follows if we take the direct sum of the formulas for $(T_j - z)^{-1}$ in (3.9), apply the permutation matrix $\boldsymbol{P}$ from the left, insert $\boldsymbol{P}^{\mathrm{t}}\boldsymbol{P} = \boldsymbol{P}^{-1}\boldsymbol{P} = I_n$ as a factor everywhere in between, and use that e.g.

$$
\boldsymbol{P}\operatorname{diag}\left(\begin{pmatrix} 0 & -1 \\ 0 & m_1(z) \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -1 \\ 0 & m_n(z) \end{pmatrix}\right)\boldsymbol{P}^{\mathrm{t}} = \begin{pmatrix} 0_n & -I_n \\ 0_n & \mathcal{M}(z) \end{pmatrix}. \tag{4.4}
$$

To prove the claim for $(\mathcal{T}_{\mathcal{A},\mathcal{B}} - z)^{-1}$, we denote the right hand side of (4.3) by $\boldsymbol{f}^{\sharp} = (\boldsymbol{f}_1^{\sharp}\ \boldsymbol{f}_2^{\sharp})^{\mathrm{t}}$, $\boldsymbol{f}_1^{\sharp} = (f_j)_1^n$, $\boldsymbol{f}_2^{\sharp} = (\widehat{f}_j)_1^n$ (see (2.8)). Since $\boldsymbol{Y}(\cdot, z)$ solves the homogeneous differential equation on $\mathcal{G}$ and $(\mathcal{T}_0 - z)^{-1}\boldsymbol{g}$ satisfies the inhomogeneous differential equation, so does $\boldsymbol{f}$.

At each vertex v_j the function $\boldsymbol{f}^{\sharp}$ satisfies the boundary condition $\cos\alpha_j\, f_j(v_j) + \sin\alpha_j\, \widehat{f}_j(v_j) = 0$ since, by (3.6), it is satisfied by the first term of $\left((\mathcal{T}_0 - z)^{-1}\boldsymbol{g}\right)^{\sharp}(\boldsymbol{x})$ in (4.2) as well as by the last term in (4.3), and the last integral in (4.2) vanishes at $x_j = v_j$.

At the central vertex v the boundary condition is satisfied since the first integral in (4.2) vanishes at $x_j = v$, $\boldsymbol{Y}(v, z) = I_{2n}$, and hence

$$
\begin{aligned}
&\mathcal{A}\boldsymbol{f}_1^{\sharp}(v) + \mathcal{B}\boldsymbol{f}_2^{\sharp}(v) \\
&= \left(\begin{pmatrix} -\mathcal{B} & \mathcal{B}\mathcal{M}(z) \end{pmatrix} - (\mathcal{B}\mathcal{M}(z) - \mathcal{A})(\mathcal{B}\mathcal{M}(z) - \mathcal{A})^{-1}\mathcal{B}\begin{pmatrix} -I_n & \mathcal{M}(z) \end{pmatrix}\right)I(\boldsymbol{g}, z)^{\sharp} = 0,
\end{aligned}
$$

where we have set

$$
I(\boldsymbol{g}, z) := \left(\int_v^{v_j} Y_j(\xi_j, \overline{z})^* \boldsymbol{g}_j(\xi_j)\mathrm{d}\xi_j\right)_1^n, \quad \boldsymbol{g} \in \boldsymbol{L}^2(\mathcal{G}), \tag{4.5}
$$

and hence $\boldsymbol{f} \in \operatorname{dom}\mathcal{T}_{\mathcal{A},\mathcal{B}}$. $\qquad\square$

The following alternative formula for the resolvent $(T_{\mathcal{A},\mathcal{B}}-z)^{-1}$ gathers all singularities in one term; it is an immediate consequence of (4.3) if we multiply by $\boldsymbol{P}^{-1}=\boldsymbol{P}^{\mathrm{t}}$ from the left and use $\boldsymbol{P}^{\mathrm{t}}\boldsymbol{Y}(\cdot,\lambda)^{\sharp}=\boldsymbol{Y}(\cdot,\lambda)\boldsymbol{P}^{\mathrm{t}}$.

Corollary 4.2. *For $z\in\mathbb{C}\setminus\mathbb{R}$, $\boldsymbol{g}\in\mathbf{L}^{2}(\mathcal{G})$, $\boldsymbol{x}=(x_{j})_{1}^{n}$, $x_{j}\in e_{j}=[v,v_{j}]$, we have*

$$((T_{\mathcal{A},\mathcal{B}}-z)^{-1}\boldsymbol{g})(\boldsymbol{x})=\boldsymbol{Y}(\boldsymbol{x},z)\boldsymbol{P}^{\mathrm{t}}\!\left(\!\begin{pmatrix}0_n & -I_n\\ 0_n & 0_n\end{pmatrix}\!\boldsymbol{P}\!\left(\int_{v}^{x_j}Y_j(\xi_j,\overline{z})^*\boldsymbol{g}_j(\xi_j)\mathrm{d}\xi_j\right)_{1}^{n}\right.$$
$$+\begin{pmatrix}0_n & 0_n\\ -I_n & 0_n\end{pmatrix}\!\boldsymbol{P}\!\left.\left(\int_{x_j}^{v_j}Y_j(\xi_j,\overline{z})^*\boldsymbol{g}_j(\xi_j)\mathrm{d}\xi_j\right)_{1}^{n}\right)$$
$$+\boldsymbol{Y}(\boldsymbol{x},z)\boldsymbol{P}^{\mathrm{t}}\mathcal{P}(z)\boldsymbol{P}\!\left(\int_{v}^{v_j}Y_j(\xi_j,\overline{z})^*\boldsymbol{g}_j(\xi_j)\mathrm{d}\xi_j\right)_{1}^{n}$$

where $\mathcal{P}$ is the meromorphic $2n\times 2n$-matrix function given by

$$\mathcal{P}(z):=\begin{pmatrix}0_n & 0_n\\ 0_n & \mathcal{M}(z)\end{pmatrix}-\begin{pmatrix}-I_n\\ \mathcal{M}(z)\end{pmatrix}(\mathcal{B}\mathcal{M}(z)-\mathcal{A})^{-1}\mathcal{B}\begin{pmatrix}-I_n & \mathcal{M}(z)\end{pmatrix}.\quad(4.6)$$

Lemma 4.3. *For $z,\zeta\in\mathbb{C}\setminus\mathbb{R}$, $z\neq\overline{\zeta}$,*

$$\frac{\mathcal{P}(z)-\mathcal{P}(\zeta)^*}{z-\overline{\zeta}}$$
$$=\begin{pmatrix}\mathcal{B}^*\\ -\mathcal{A}^*\end{pmatrix}(\mathcal{M}(\zeta)^*\mathcal{B}^*-\mathcal{A}^*)^{-1}\frac{\mathcal{M}(z)-\mathcal{M}(\zeta)^*}{z-\overline{\zeta}}(\mathcal{B}\,\mathcal{M}(z)-\mathcal{A})^{-1}\begin{pmatrix}\mathcal{B} & -\mathcal{A}\end{pmatrix}.$$

Proof. First we note that for any $\mathcal{X}\in M_n(\mathbb{C})$ such that $\mathcal{B}\mathcal{X}-\mathcal{A}$ is invertible, by (2.9),

$$(\mathcal{B}\mathcal{X}-\mathcal{A})^{-1}\mathcal{B}=\mathcal{B}^*(\mathcal{X}\mathcal{B}^*-\mathcal{A}^*)^{-1}.\qquad(4.7)$$

Let $z,\zeta\in\mathbb{C}\setminus\mathbb{R}$, $z\neq\overline{\zeta}$. The equality of the left upper entries in the claimed matrix identity follows since, by (4.7),

$$-(\mathcal{B}\mathcal{M}(z)-\mathcal{A})^{-1}\mathcal{B}+\mathcal{B}^*(\mathcal{M}(\zeta)^*\mathcal{B}^*-\mathcal{A}^*)^{-1}$$
$$=-\mathcal{B}^*(\mathcal{M}(z)\mathcal{B}^*-\mathcal{A}^*)^{-1}+\mathcal{B}^*(\mathcal{M}(\zeta)^*\mathcal{B}^*-\mathcal{A}^*)^{-1}$$
$$=\mathcal{B}^*(\mathcal{M}(\zeta)^*\mathcal{B}^*-\mathcal{A}^*)^{-1}\big(\mathcal{M}(z)-\mathcal{M}(\zeta)^*\big)\mathcal{B}^*(\mathcal{M}(z)\mathcal{B}^*-\mathcal{A}^*)^{-1}$$
$$=\mathcal{B}^*(\mathcal{M}(\zeta)^*\mathcal{B}^*-\mathcal{A}^*)^{-1}\big(\mathcal{M}(z)-\mathcal{M}(\zeta)^*\big)(\mathcal{B}\mathcal{M}(z)-\mathcal{A})^{-1}\mathcal{B}.$$

The equality of the right lower entries follows since, again by (4.7),

$$\mathcal{M}(z)-\mathcal{M}(z)(\mathcal{B}\mathcal{M}(z)-\mathcal{A})^{-1}\mathcal{B}\mathcal{M}(z)$$
$$-\mathcal{M}(\zeta)^*+\mathcal{M}(\zeta)^*\mathcal{B}^*(\mathcal{M}(\zeta)^*\mathcal{B}^*-\mathcal{A}^*)^{-1}\mathcal{M}(\zeta)^*$$
$$=-\mathcal{M}(z)(\mathcal{B}\mathcal{M}(z)-\mathcal{A})^{-1}\mathcal{A}+\mathcal{M}(\zeta)^*(\mathcal{B}\mathcal{M}(\zeta)^*-\mathcal{A})^{-1}\mathcal{A}$$
$$=\big(-\mathcal{M}(z)+\mathcal{M}(\zeta)^*(\mathcal{B}\mathcal{M}(\zeta)^*-\mathcal{A})^{-1}(\mathcal{B}\mathcal{M}(z)-\mathcal{A})\big)(\mathcal{B}\mathcal{M}(z)-\mathcal{A})^{-1}\mathcal{A}$$
$$=\big(-I+\mathcal{M}(\zeta)^*(\mathcal{B}\mathcal{M}(\zeta)^*-\mathcal{A})^{-1}\mathcal{B}\big)(\mathcal{M}(z)-\mathcal{M}(\zeta)^*)(\mathcal{B}\mathcal{M}(z)-\mathcal{A})^{-1}\mathcal{A}$$
$$=\big(-I+\mathcal{M}(\zeta)^*\mathcal{B}^*(\mathcal{M}(\zeta)^*\mathcal{B}^*-\mathcal{A}^*)^{-1}\big)(\mathcal{M}(z)-\mathcal{M}(\zeta)^*)(\mathcal{B}\mathcal{M}(z)-\mathcal{A})^{-1}\mathcal{A}$$
$$=\mathcal{A}^*(\mathcal{M}(\zeta)^*\mathcal{B}^*-\mathcal{A}^*)^{-1}(\mathcal{M}(z)-\mathcal{M}(\zeta)^*)(\mathcal{B}\mathcal{M}(z)-\mathcal{A})^{-1}\mathcal{A}.$$

The proof of the equality of the off-diagonal entries is similar. $\qquad\square$

The following is our first main result.

Theorem 4.4. (i) *The self-adjoint extensions of the symmetric operator $\mathcal{S}$ in $\boldsymbol{L}^2(\mathcal{G})$ are the restrictions $\mathcal{T}_{\mathcal{A},\mathcal{B}}$ of the operator $\mathcal{S}^*$ given by vertex conditions*

$$\mathcal{A}\,\boldsymbol{f}_1^\sharp(v) + \mathcal{B}\boldsymbol{f}_2^\sharp(v) = 0,$$

with $\mathcal{A}, \mathcal{B} \in M_n(\mathbb{C})$ such that

$$\operatorname{rank}\begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix} = n, \quad \mathcal{A}\mathcal{B}^* = \mathcal{B}\mathcal{A}^*.$$

(ii) *The resolvent of $\mathcal{T}_{\mathcal{A},\mathcal{B}}$ satisfies*

$$(\mathcal{T}_{\mathcal{A},\mathcal{B}} - z)^{-1} = (\mathcal{T}_0 - z)^{-1} - \Gamma_z(\mathcal{B}\mathcal{M}(z) - \mathcal{A})^{-1}\mathcal{B}\,\Gamma_{\overline{z}}^* \qquad (4.8)$$

for $z \in \rho(\mathcal{T}_0) \cap \rho(\mathcal{T}_{\mathcal{A},\mathcal{B}})$ with $\mathcal{T}_0 = \mathcal{T}_{I_n,0_n}$ and $\Gamma_z : \mathbb{C}^n \to \boldsymbol{L}^2(\mathcal{G})$ given by

$$\Gamma_z c := \boldsymbol{Y}(\cdot, z)\,\boldsymbol{P}^{\mathrm{t}}\begin{pmatrix} -I_n \\ \mathcal{M}(z) \end{pmatrix} c, \quad c \in \mathbb{C}^n, \qquad (4.9)$$

where $\boldsymbol{P}$ is the permutation matrix in (2.6).

(iii) *The fractional linear transformation $\mathcal{W}_{\mathcal{A},\mathcal{B}} \in M_n(\mathbb{C})$ of $\mathcal{M}$,*

$$\mathcal{W}_{\mathcal{A},\mathcal{B}}(z) = \big(\mathcal{D}\mathcal{M}(z) - \mathcal{C}\big)\big(\mathcal{B}\mathcal{M}(z) - \mathcal{A}\big)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

with $\mathcal{C}, \mathcal{D} \in M_n(\mathbb{C})$ related to $\mathcal{A}, \mathcal{B}$ as in Lemma 2.2, is a meromorphic matrix Nevanlinna function and

$$(\mathcal{T}_{\mathcal{A},\mathcal{B}} - z)^{-1} = \boldsymbol{Y}(\cdot, z)\,\boldsymbol{P}^{\mathrm{t}}\begin{pmatrix} \mathcal{B}^* \\ -\mathcal{A}^* \end{pmatrix}\mathcal{W}_{\mathcal{A},\mathcal{B}}(z)\begin{pmatrix} \mathcal{B} & -\mathcal{A} \end{pmatrix}\boldsymbol{P}I(\cdot, z) + \mathcal{F}(z) \quad (4.10)$$

where $I(\cdot, z)$ is given by (4.5) and $\mathcal{F}$ is an entire function whose values are bounded linear operators in $\boldsymbol{L}^2(\mathcal{G})$.

Proof. Claim (i) was proved in Proposition 2.1. To prove (ii) we multiply (4.3) with $\boldsymbol{P}^{-1} = \boldsymbol{P}^{\mathrm{t}}$ from the left, note $\boldsymbol{P}^{\mathrm{t}}\boldsymbol{Y}(\cdot, \lambda)^\sharp = \boldsymbol{Y}(\cdot, \lambda)\boldsymbol{P}^{\mathrm{t}}$, and that, for $z \in \mathbb{C} \setminus \mathbb{R}$, the adjoint $\Gamma_z^* : \boldsymbol{L}^2(\mathcal{G}) \to \mathbb{C}^n$ of Γ_z in (4.9) is given by

$$\Gamma_z^*\boldsymbol{g} = \begin{pmatrix} -I_n & \mathcal{M}(\overline{z}) \end{pmatrix}\boldsymbol{P}I(\boldsymbol{g}, \overline{z}), \quad \boldsymbol{g} \in \boldsymbol{L}^2(\mathcal{G}). \qquad (4.11)$$

(iii) The choice of $\mathcal{C}, \mathcal{D}$ in Lemma 2.2 ensures that (2.19) holds. This implies that, for $z, \zeta \in \mathbb{C} \setminus \mathbb{R}$, $z \neq \overline{\zeta}$,

$$\frac{\mathcal{W}_{\mathcal{A},\mathcal{B}}(z) - \mathcal{W}_{\mathcal{A},\mathcal{B}}(\zeta)^*}{z - \overline{\zeta}} = (\mathcal{M}(\zeta)^*\mathcal{B}^* - \mathcal{A}^*)^{-1}\frac{\mathcal{M}(z) - \mathcal{M}(\zeta)^*}{z - \overline{\zeta}}(\mathcal{B}\mathcal{M}(z) - \mathcal{A})^{-1}$$

and hence $\mathcal{W}_{\mathcal{A},\mathcal{B}}$ is a matrix Nevanlinna function. Moreover, by Lemma 4.3,

$$\frac{\mathcal{P}(z) - \mathcal{P}(\zeta)^*}{z - \overline{\zeta}} = \begin{pmatrix} \mathcal{B}^* \\ -\mathcal{A}^* \end{pmatrix}\frac{\mathcal{W}_{\mathcal{A},\mathcal{B}}(z) - \mathcal{W}_{\mathcal{A},\mathcal{B}}(\zeta)^*}{z - \overline{\zeta}}\begin{pmatrix} \mathcal{B} & -\mathcal{A} \end{pmatrix}$$

and thus $\mathcal{P}(z) = \begin{pmatrix} \mathcal{B}^* \\ -\mathcal{A}^* \end{pmatrix}\mathcal{W}_{\mathcal{A},\mathcal{B}}(z)\begin{pmatrix} \mathcal{B} & -\mathcal{A} \end{pmatrix} + X$ with $X = X^* \in M_n(C)$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Now (4.10) follows from the representation of $(T_{\mathcal{A},\mathcal{B}} - z)^{-1}$ in Corollary 4.2 if we note that $z \mapsto \boldsymbol{Y}(\cdot, z)$ is entire. □

Remark 4.5. By (4.11) and (4.1) we have, for $z \in \mathbb{C} \setminus \mathbb{R}$ and $c = (c_j)_1^n \in \mathbb{C}^n$, $\boldsymbol{g} = (\boldsymbol{g}_j)_1^n \in \boldsymbol{L}^2(\mathcal{G})$,

$$\Gamma_z(c_j)_1^n = \Big(c_j \boldsymbol{y}_j(\cdot, z) \Big)_1^n, \quad \boldsymbol{y}_j(\cdot, z) := Y_j(\cdot, z) \binom{-1}{m_j(z)} \in \ker(S_j^* - z), \quad (4.12)$$

$$\Gamma_z^* \boldsymbol{g} = \Big(\int_v^{v_j} \boldsymbol{y}_j(\xi_j, \overline{z})^{\mathrm{t}} \boldsymbol{g}_j(\xi_j) \mathrm{d}\xi_j \Big)_1^n. \tag{4.13}$$

Hence $\operatorname{ran}\Gamma_z = \ker(\mathcal{S}^* - z) = \operatorname{ran}(\mathcal{S} - \overline{z})^\perp$ is the defect space of $\mathcal{S}$ at $\overline{z}$.

Note that, due to the non-uniqueness of the boundary matrices $\mathcal{A}$, $\mathcal{B}$, two parameter pairs $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}'$, $\mathcal{B}'$ generate the same self-adjoint extension if the linear relations $\mathcal{B}^{-1}\mathcal{A}$ and $(\mathcal{B}')^{-1}\mathcal{A}'$ coincide; in this case the subspaces defined by the corresponding boundary conditions coincide.

The relation (4.8) is a version of M.G. Krein's resolvent formula (comp. [KL71]). It describes the resolvents of all the self-adjoint extensions of $\mathcal{S}$ within $\boldsymbol{L}^2(\mathcal{G})$, with the decoupled extension $\mathcal{T}_0$ fixed, by means of pairs of $n \times n$ matrices $\mathcal{A}$, $\mathcal{B}$ satisfying (2.9). The same formula with z-dependent matrices $\mathcal{A}$, $\mathcal{B}$ describes all self-adjoint extensions of $\mathcal{S}$ with exit, i.e. in Hilbert spaces of which $\boldsymbol{L}^2(\mathcal{G})$ is a subspace.

Remark 4.6. The resolvent formula in (4.10) may be used to derive eigenfunction expansions for the Dirac-Krein operator $\mathcal{T}_{\mathcal{A},\mathcal{B}}$. The poles of the meromorphic matrix Nevanlinna function $\mathcal{W}_{\mathcal{A},\mathcal{B}}$ are the eigenvalues of $\mathcal{T}_{\mathcal{A},\mathcal{B}}$. Since the resolvent difference of $\mathcal{T}_0$ and $\mathcal{T}_{\mathcal{A},\mathcal{B}}$ has finite rank by (4.8) and the eigenvalues of each T_j, whence those of $\mathcal{T}_0 = \bigoplus_{j=1}^n T_j$, accumulate only at $-\infty$ and $+\infty$, so do the eigenvalues of $\mathcal{T}_{\mathcal{A},\mathcal{B}}$; we denote them by $(\lambda_k)_{k \in \mathbb{Z}}$ with $\lambda_k < \lambda_{k+1}$, $k \in \mathbb{Z}$. The principal part of $\mathcal{W}_{\mathcal{A},\mathcal{B}}(z)$ at λ_k is of the form $\dfrac{-W_{\mathcal{A},\mathcal{B},k}}{z - \lambda_k}$ with non-negative $W_{\mathcal{A},\mathcal{B},k} \in M_n(\mathbb{C}) \setminus \{0_n\}$ of rank equal to the multiplicity of λ_k. Then the functional calculus and the residue theorem applied to (4.10) imply that, for all bounded intervals $\Delta \subset \mathbb{R}$, $\boldsymbol{f} = (\boldsymbol{f}_j)_1^n \in \boldsymbol{L}^2(\mathcal{G})$, and $\boldsymbol{x} \in \mathcal{G}$,

$$\big(E(\Delta)\boldsymbol{f}\big)(\boldsymbol{x})$$
$$= \sum_{\lambda_k \in \Delta} \boldsymbol{Y}(\boldsymbol{x}, \lambda_k) \boldsymbol{P}^{\mathrm{t}} \binom{\mathcal{B}^*}{-\mathcal{A}^*} W_{\mathcal{A},\mathcal{B},k} \big(\mathcal{B} \ \ -\mathcal{A}\big) \boldsymbol{P} \Big(\int_v^{v_j} Y_j(\xi_j, \lambda_k)^* \boldsymbol{f}_j(\xi_j) \mathrm{d}\xi_j \Big)_1^{n\,\sharp}.$$

Letting $\Delta = (-R, R)$ and $R \to \infty$, we arrive at the eigenfunction expansion

$$\boldsymbol{f} = \sum_{k=-\infty}^{\infty} \boldsymbol{Y}(\cdot, \lambda_k) \boldsymbol{P}^{\mathrm{t}} \binom{\mathcal{B}^*}{-\mathcal{A}^*} W_{\mathcal{A},\mathcal{B},k} \big(\mathcal{B} \ \ -\mathcal{A}\big) \boldsymbol{P} \Big(\int_v^{v_j} Y_j(\xi_j, \lambda_k)^* \boldsymbol{f}_j(\xi_j) \mathrm{d}\xi_j \Big)_1^{n\,\sharp}$$

in $\boldsymbol{L}^2(\mathcal{G})$. Hence the mapping

$$\boldsymbol{f} \mapsto \big(\mathcal{B} \ \ -\mathcal{A}\big) \boldsymbol{P} \Big(\int_v^{v_j} Y_j(\xi_j, \lambda_k)^* \boldsymbol{f}_j(\xi_j) \mathrm{d}\xi_j \Big)_1^{n\,\sharp} = \big(\mathcal{B} \ \ -\mathcal{A}\big) \boldsymbol{P} I(\boldsymbol{f}, \lambda_k)^\sharp$$

can be considered as the Fourier transformation associated with $\mathcal{T}_{\mathcal{A},\mathcal{B}}$, and Parseval's relation takes the form

$$(\boldsymbol{f}, \boldsymbol{g})_{\boldsymbol{L}^2(\mathcal{G})} = \sum_{k=-\infty}^{\infty} I(\boldsymbol{g}, \lambda_k)^{\mathrm{t}} \boldsymbol{P}^{\mathrm{t}} \binom{\mathcal{B}^*}{-\mathcal{A}^*} W_{\mathcal{A},\mathcal{B},k} \big(\mathcal{B} \ \ -\mathcal{A}\big) \boldsymbol{P} I(\boldsymbol{f}, \lambda_k)^\sharp.$$

Unlike the single interval case $n = 1$ (see e.g. [KL14]), for star graphs with $n > 1$ edges the Fourier transformation is no longer scalar but n-dimensional and hence higher spectral multiplicities may occur, see Theorem 5.3 in the next section.

5. Eigenvalues and multiplicities for Robin matching conditions

In this subsection, for the self-adjoint extension induced by the Robin matching conditions (2.12), (2.13) at the central vertex, we derive a characteristic equation for the eigenvalues in terms of the Weyl–Titchmarsh functions m_j on the edges, investigate their multiplicities, and we prove a trace formula for the resolvent difference of $\mathcal{T}_\tau$ and $\mathcal{T}_0$.

To this end, we first specialize the resolvent formulas established in Theorem 4.4. For the Robin matching conditions (2.12), (2.13) with $\tau \neq \pm\infty$, we choose the matrices $\mathcal{A} = \mathcal{A}_\tau$, $\mathcal{B} = \mathcal{B}_\tau$ as in (2.14), and correspondingly $\mathcal{A}_{\pm\infty}$ and $\mathcal{B}_{\pm\infty}$ for $\tau = \pm\infty$. The associated self-adjoint operator is denoted by $\mathcal{T}_\tau := \mathcal{T}_{\mathcal{A}_\tau,\mathcal{B}_\tau}$. Note that for $\tau = 0$ the operator coincides with $\mathcal{T}_0 = \mathcal{T}_{I_n,0_n}$. Hence we assume that $\tau \neq 0$ in the following if not explicitly stated otherwise.

Straightforward calculations yield that

$$(\mathcal{B}_\tau \mathcal{M}(z) - \mathcal{A}_\tau)^{-1}\mathcal{B}_\tau = \frac{1}{\dfrac{1}{\tau} + \displaystyle\sum_{j=1}^{n} m_j(z)} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (1\ 1\ \cdots\ 1).$$

Since all m_j, $j = 1, 2, \ldots, n$, are Nevanlinna functions, so is the function

$$\boldsymbol{m}_\tau(z) := \frac{1}{\tau} + \sum_{j=1}^{n} m_j(z), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{5.1}$$

Using the above formula for $(\mathcal{B}_\tau \mathcal{M}(z) - \mathcal{A}_\tau)^{-1}\mathcal{B}_\tau$ and the formulas (4.12), (4.13) for Γ_z, $\Gamma_{\bar{z}}^*$, it is immediate from Theorem 4.4 that, for the Robin matching conditions (2.12), (2.13), Krein's resolvent formula (4.8) takes the following form.

Corollary 5.1. *For $z \in \mathbb{C} \setminus \mathbb{R}$, $\boldsymbol{g} = (g_j)_1^n \in \mathbf{L}^2(\mathcal{G})$ and $\boldsymbol{x} = (x_j)_1^n$, $x_j \in e_j = [v, v_j]$,*

$$\big((\mathcal{T}_\tau - z)^{-1}\boldsymbol{g}\big)(\boldsymbol{x})$$

$$= \big((\mathcal{T}_0 - z)^{-1}\boldsymbol{g}\big)(\boldsymbol{x}) - \frac{1}{\boldsymbol{m}_\tau(z)}\bigg(\sum_{\ell=1}^{n} \int_v^{v_\ell} \boldsymbol{y}_\ell(\xi_j, \bar{z})^{\mathrm{t}} \boldsymbol{g}_\ell(\xi_j)\,\mathrm{d}\xi_j\bigg)\big(\boldsymbol{y}_j(x_j, z)\big)_1^n \tag{5.2}$$

$$= \big((\mathcal{T}_0 - z)^{-1}\boldsymbol{g}\big)(\boldsymbol{x}) - \frac{1}{\boldsymbol{m}_\tau(z)}\big(\boldsymbol{y}(\cdot, \bar{z}), \boldsymbol{g}\big)_{\boldsymbol{L}^2(\mathcal{G})}\boldsymbol{y}(\boldsymbol{x}, z),$$

where $\boldsymbol{y}(\cdot, z) = (\boldsymbol{y}_j(\cdot, z))_1^n$, $\boldsymbol{y}_j(\cdot, z) = Y_j(\cdot, z)\begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}$, and

$$\big((\mathcal{T}_0 - z)^{-1}\boldsymbol{g}\big)(\boldsymbol{x}) = \Big(\big((T_j - z)^{-1}\boldsymbol{g}_j\big)(x_j)\Big)_1^n \tag{5.3}$$

with $(T_j - z)^{-1}\boldsymbol{g}_j$ given by (3.9).

In Theorem 5.3 below we show that the set of eigenvalues splits into two parts, simple eigenvalues that depend on τ and possibly non-simple eigenvalues that are independent of τ.

Lemma 5.2. *Let $z \in \mathbb{C}$ be not a pole of $\boldsymbol{m}_\tau$, i.e. not a pole of any m_j, $j = 1, 2$, $\ldots, n$. Then*

$$z \in \sigma_{\mathrm{p}}(\mathcal{T}_\tau) \iff \mathcal{B}_\tau \mathcal{M}(z) - \mathcal{A}_\tau \text{ is not invertible} \iff \boldsymbol{m}_\tau(z) = 0. \quad (5.4)$$

These eigenvalues of $\mathcal{T}_\tau$ are simple, with eigenfunction $\boldsymbol{y}(\cdot, z) = \left(\boldsymbol{y}_j(\cdot, z)\right)_1^n$ given by $\boldsymbol{y}_j(\cdot, z) = u_j Y_j(\cdot, z) \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}$ where $\boldsymbol{u} = (u_j)_1^n \neq 0$ is a solution of $\left(\mathcal{B}_\tau \mathcal{M}(z) - \mathcal{A}_\tau\right) \boldsymbol{u} = 0$.

Proof. Let $\mathcal{A}_\tau$, $\mathcal{B}_\tau$ be the boundary matrices for $\mathcal{T}_\tau$ defined in (2.14) and assume $z \in \mathbb{C}$ is not a pole of any m_j, $j = 1, 2, \ldots, n$; then z is not a pole of $\mathcal{M} = \mathrm{diag}\,(m_1, m_2, \ldots, m_n)$.

By Proposition 2.1, $z \in \sigma_{\mathrm{p}}(\mathcal{T}_\tau)$ with eigenfunction $\boldsymbol{y}(\cdot, z) = \left(\boldsymbol{y}_j(\cdot, z)\right)_1^n$ if and only if the following hold:

(i) $\boldsymbol{y}(\cdot, z) \in \ker(S^* - z)$, i.e. $\boldsymbol{y}_j(\cdot, z) = \gamma_j Y_j(\cdot, z) \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}$ with $\gamma_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$;

(ii) $\boldsymbol{y}(\cdot, z)$ satisfies the boundary condition $\mathcal{A}_\tau \boldsymbol{y}_1^\sharp(v, z) + \mathcal{B}_\tau \boldsymbol{y}_2^\sharp(v, z) = 0$.

"$\Longrightarrow$" in the first equivalence in (5.4): Since $\boldsymbol{Y}_j(v, z) = I_2$ and due to (4.4), condition (i) implies that $\boldsymbol{y}_2^\sharp(v, z) = -\mathcal{M}(z) \boldsymbol{y}_1^\sharp(v, z)$; in particular, $\boldsymbol{y}_1^\sharp(v, z) \neq 0$ because z is no pole of $\mathcal{M}$ and $\boldsymbol{y}(\cdot, z) \neq 0$. Hence, by the condition in (ii),

$$(\mathcal{B}_\tau \mathcal{M}(z) - \mathcal{A}_\tau) \boldsymbol{y}_1^\sharp(v, z) = 0 \quad (5.5)$$

has a non-zero solution.

"$\Longleftarrow$" in the first equivalence in (5.4): If (5.5) has a non-zero solution $\boldsymbol{y}_1^\sharp(v, z) \in \mathbb{C}^n$ with components $\boldsymbol{y}_1^\sharp(v, z)_j$, $j = 1, 2, \ldots, n$, then the function $\boldsymbol{y}(\cdot, z) = (\boldsymbol{y}_j(\cdot, z))_1^n$ with components $\boldsymbol{y}_j(\cdot, z) = \boldsymbol{y}_1^\sharp(v, z)_j Y_j(\cdot, z) \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}$, $j = 1, 2, \ldots, n$, satisfies (i) and (ii) by construction.

The second equivalence in (5.4), e.g. for $\tau \neq \infty$, is immediate from

$$\mathcal{B}_\tau \mathcal{M}(z) - \mathcal{A}_\tau = \begin{pmatrix} -1 & 1 & & & & & 0 \\ & -1 & 1 & & & & \\ & & -1 & 1 & & & \\ & & & \ddots & & \ddots & \\ 0 & & & & & -1 & 1 \\ 1 + \tau m_1(z) & \tau m_2(z) & \cdots\cdots & \cdots & & \tau m_{n-1}(z) & \tau m_n(z) \end{pmatrix}$$

and

$$\det(\mathcal{B}_\tau \mathcal{M}(z) - \mathcal{A}_\tau) = (-1)^{n+1}\left(1 + \tau \sum_{j=1}^n m_j(z)\right) = (-1)^{n+1} \tau \boldsymbol{m}_\tau(z).$$

Since the dimension of the kernel of the matrix $\mathcal{B}_\tau \mathcal{M}(z) - \mathcal{A}_\tau$ is at most 1, every zero of $\boldsymbol{m}_\tau$ is a simple eigenvalue of $\mathcal{T}_\tau$. $\qquad\square$

Apart from the simple eigenvalues characterized in Lemma 5.2, there may be other eigenvalues of possibly higher multiplicity which arise as poles of at least two of the Weyl–Titchmarsh functions m_j on the edges e_j.

Theorem 5.3. *For the Robin matching conditions (2.12), (2.13) with $\tau \neq 0$, the spectrum $\sigma_\mathrm{p}(\mathcal{T}_\tau)$ of $\mathcal{T}_\tau$ consists of two types of eigenvalues:*

i) *The solutions of the equation*

$$\sum_{j=1}^n m_j(z) = -\frac{1}{\tau} \tag{5.6}$$

are simple eigenvalues and each of them is strictly increasing in τ on the intervals $[-\infty, 0)$ and $(0, \infty]$;

ii) *The common poles of $\nu + 1 \in \{2, 3, \ldots, n\}$ functions m_j are eigenvalues of multiplicity $\nu \in \{1, 2, \ldots, n-1\}$ and are independent of τ.*

Further, for any τ_1, $\tau_2 \in \mathbb{R} \cup \{\infty\}$, the eigenvalues of $\mathcal{T}_{\tau_1}$ and of $\mathcal{T}_{\tau_2}$ interlace.

Proof. First assume that $\lambda_0 \in \mathbb{R}$ is not a pole of any m_j, $j = 1, 2, \ldots, n$. Then, by Lemma 5.2, $\lambda_0 \in \sigma_\mathrm{p}(\mathcal{T})$ if and only if λ_0 is a solution of (5.6), and in this case λ_0 is simple.

Now suppose that $\lambda_0 \in \mathbb{R}$ is a pole of $\nu + 1$ functions m_j with $\nu \in \{0, 1, \ldots, n-1\}$, without loss of generality, of m_1, m_2, $\ldots$, $m_{\nu+1}$. The residue γ_j of m_j at λ_0 satisfies $\gamma_j < 0$, $j = 1, 2, \ldots, \nu + 1$, and we set $\gamma_j := 0$, $j = \nu + 2, \nu + 3, \ldots, n$.

The orthogonal projection onto the possible eigenspace of $\mathcal{T}_\tau$ at λ_0 is given by $\lim_{z \to \lambda_0} (\lambda_0 - z)(\mathcal{T}_\tau - z)^{-1}$. Using (5.2) and (4.2), we find that, for $\boldsymbol{g} = (\boldsymbol{g}_j)_1^n \in \boldsymbol{L}^2(\mathcal{G})$ and $\boldsymbol{x} = (x_j)_1^n$, $x_j \in e_j = [v, v_j]$,

$$\lim_{z \to \lambda_0} \left((\lambda_0 - z)(\mathcal{T}_\tau - z)^{-1}\boldsymbol{g} \right)(\boldsymbol{x})$$

$$= \left(Y_j(x_j, \lambda_0) \begin{pmatrix} 0 & 0 \\ 0 & \gamma_j \end{pmatrix} \int_v^{v_j} Y_j(\xi_j; \lambda_0)^* \boldsymbol{g}_j(\xi_j)\, \mathrm{d}\xi_j \right)_1^n \tag{5.7}$$

$$- \lim_{z \to \lambda_0} \frac{\tau(\lambda_0 - z)^2}{\left(1 + \tau \sum_{j=1}^n m_j(z)\right)(\lambda_0 - z)} \left(\left(\sum_{\ell=1}^n \int_v^{v_\ell} \boldsymbol{y}_\ell(\xi_j, \overline{z})^\mathrm{t} \boldsymbol{g}_\ell(\xi_j)\, \mathrm{d}\xi_j \right) \boldsymbol{y}_j(x_j, z) \right)_1^n.$$

Since $\boldsymbol{y}_j(x_j, z) = Y_j(x_j, z)\begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}$, the limit $\lim_{z \to \lambda_0} (\lambda_0 - z)\boldsymbol{y}_j(x_j, z)$ equals $Y_j(x_j, \lambda_0)\begin{pmatrix} 0 \\ \gamma_j \end{pmatrix}$ for $j = 1, 2, \ldots, \nu+1$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $j = \nu + 2, \nu + 3, \ldots, n$.

So the limit on the right hand side of (5.7) equals

$$
-\frac{1}{\displaystyle\sum_{j=1}^{\nu+1}\gamma_j}\left(\left(\sum_{\ell=1}^{\nu+1}\int_v^{v_\ell}(0\ \gamma_\ell)\,Y_\ell(\xi_j,\lambda_0)^*\,\boldsymbol{g}_\ell(\xi_j)\,\mathrm{d}\xi_j\right)Y_j(x_j,\lambda_0)\begin{pmatrix}0\\\gamma_j\end{pmatrix}\right)_1^n
$$

$$
=: -\frac{1}{\displaystyle\sum_{j=1}^{\nu+1}\gamma_j}\left(\gamma_j\left(\sum_{\ell=1}^{\nu+1}\gamma_\ell\mu_\ell\right)\begin{pmatrix}y_{j,12}(x_j,\lambda_0)\\y_{j,22}(x_j,\lambda_0)\end{pmatrix}\right)_1^n
$$

with

$$
\mu_\ell := \int_v^{v_\ell}\big(y_{\ell,12}(\xi_j,\lambda_0)\ y_{\ell,22}(\xi_j,\lambda_0)\big)\begin{pmatrix}g_\ell(\xi_j)\\\widehat{g}_\ell(\xi_j)\end{pmatrix}\mathrm{d}\xi_j,\quad \ell = 1,2,\ldots,\nu+1.
$$

The components of the first term on the right hand side of (5.7) can be written as

$$
\gamma_j\int_v^{v_j}\big(y_{j,12}(\xi_j,\lambda_0)\ y_{j,22}(\xi_j,\lambda_0)\big)\begin{pmatrix}g_j(\xi_j)\\\widehat{g}_j(\xi_j)\end{pmatrix}\mathrm{d}\xi\begin{pmatrix}y_{j,12}(x_j,\lambda_0)\\y_{j,22}(x_j,\lambda_0)\end{pmatrix} = \gamma_j\mu_j\begin{pmatrix}y_{j,12}(x_j,\lambda_0)\\y_{j,22}(x_j,\lambda_0)\end{pmatrix}.
$$

Altogether, (5.7) becomes

$$
\lim_{z\to\lambda_0}(\lambda_0-z)\big((\mathcal{T}_\tau-z)^{-1}\boldsymbol{g}\big)(\boldsymbol{x}) = \left(\left[\gamma_j\mu_j-\frac{\gamma_j}{\displaystyle\sum_{\ell=1}^{\nu+1}\gamma_\ell}\sum_{\ell=1}^{\nu+1}\gamma_\ell\mu_\ell\right]\begin{pmatrix}y_{j,12}(x_j,\lambda_0)\\y_{j,22}(x_j,\lambda_0)\end{pmatrix}\right)_1^n.
$$

This form shows that the dimension of the eigenspace of $\mathcal{T}_\tau$ at λ_0 is $\leq \nu$ since $\gamma_j = 0$ for $j = \nu+2, \nu+3, \ldots, n$, and

$$
\sum_{j=1}^{\nu+1}\left(\gamma_j\mu_j-\frac{\gamma_j}{\displaystyle\sum_{\ell=1}^{\nu+1}\gamma_\ell}\sum_{\ell=1}^{\nu+1}\gamma_\ell\mu_\ell\right) = 0;
$$

in particular, if $\nu=0$, i.e. λ_0 is a pole of only one m_j, then λ_0 is no eigenvalue of $\mathcal{T}_\tau$.

That the dimension of $\ker(\mathcal{T}_\tau - \lambda_0)$ cannot be $< \nu$ follows from the fact that the rank of $(\mathcal{T}_\tau - z)^{-1} - (\mathcal{T}_0 - z)^{-1}$ is equal to 1 by (5.2) and the multiplicity of λ_0 as an eigenvalue of $\mathcal{T}_0$ equals $\nu + 1$ by assumption.

For $\tau_1, \tau_2 \neq 0$, the claimed interlacing property is immediate from the fact that the left hand side of (5.6) is a Nevanlinna function. If e.g. $\tau_2 = 0$, then the eigenvalues of $\mathcal{T}_{\tau_2} = \mathcal{T}_0 = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ are the poles of the left hand side of (5.6) and a common pole of k functions m_j is an eigenvalue of $\mathcal{T}_0$ of multiplicity k, and the claim follows from i) and ii). $\square$

Corollary 5.4. *For $\tau \neq 0$ two consecutive eigenvalues of $\mathcal{T}_\tau$ cannot be multiple.*

Remark 5.5. For the special case $\tau = \infty$ of standard matching conditions, Theorem 5.3 was proved in [SW14, Thm. 3.19 (I/II)] in the more general setting of symmetric relations with not necessarily discrete spectrum. An analogous result for star graphs of Stieltjes strings was proved in [PRT13, Cor. 2.6].

Theorem 5.6. *With* $\boldsymbol{m}_\tau(z) = \dfrac{1}{\tau} + \displaystyle\sum_{j=1}^{n} m_j(z)$ *as in* (5.1), *the trace formula*

$$\mathrm{tr}\left((\mathcal{T}_\tau - z)^{-1} - (\mathcal{T}_0 - z)^{-1}\right) = -\frac{\dot{\boldsymbol{m}}_\tau(z)}{\boldsymbol{m}_\tau(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

holds, i.e. $\boldsymbol{m}_\tau$ *is the perturbation determinant of* $\mathcal{T}_\tau$ *with respect to* $\mathcal{T}_0$.

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$. By (5.2) the resolvent difference $(\mathcal{T}_\tau - z)^{-1} - (\mathcal{T}_0 - z)^{-1}$ is an integral operator in $\boldsymbol{L}^2(\mathcal{G})$ of rank 1 with semi-separable kernel

$$K_\tau(\boldsymbol{x}, \boldsymbol{\xi}; z) := -\frac{1}{\boldsymbol{m}_\tau(z)} \boldsymbol{y}(\boldsymbol{x}, z)\boldsymbol{y}(\boldsymbol{\xi}, \overline{z})^{\mathrm{t}}, \quad \boldsymbol{x}, \boldsymbol{\xi} \in \mathcal{G},$$

hence its trace is given by (comp. e.g. [GGK90, Thm. IX.3.2])

$$\mathrm{tr}\left((\mathcal{T}_\tau - z)^{-1} - (\mathcal{T}_0 - z)^{-1}\right) = -\frac{1}{\boldsymbol{m}_\tau(z)} \sum_{j=1}^{n} \int_{v}^{v_j} \boldsymbol{y}_j(\xi_j, \overline{z})^{\mathrm{t}} \, \boldsymbol{y}_j(\xi_j, z) \, \mathrm{d}\xi_j. \quad (5.8)$$

It remains to be shown that the integral in (5.8) equals $\dot{m}_j(z)$. To this end, we write $m_j(z) = \dfrac{b_j(z)}{c_j(z)}$ as in (3.7). Then, using Lemma 3.4, (3.8), and (3.6), we find

$$\int_{v}^{v_j} \boldsymbol{y}_j(\xi_j, \overline{z})^{\mathrm{t}} \, \boldsymbol{y}_j(\xi_j, z) \, \mathrm{d}\xi_j = \left(-1 \; m_j(z)\right) \int_{v}^{v_j} Y_j(\xi_j, \overline{z})^* \, Y_j(\xi_j, z) \, \mathrm{d}\xi_j \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}$$

$$= -\left(-1 \; m_j(z)\right) Y_j(v_j, \overline{z})^* \, J\dot{Y}_j(v_j, z) \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}$$

$$= -\frac{1}{c_j(z)}\left(-\sin\alpha_j \; \cos\alpha_j\right) J\dot{Y}_j(v_j, z) \begin{pmatrix} -1 \\ m_j(z) \end{pmatrix}$$

$$= -\frac{1}{c_j(z)}\left(\cos\alpha_j \; \sin\alpha_j\right) \begin{pmatrix} \dot{y}_{j,11}(v_j, z) & \dot{y}_{j,12}(v_j, z) \\ \dot{y}_{j,21}(v_j, z) & \dot{y}_{j,22}(v_j, z) \end{pmatrix} \begin{pmatrix} -1 \\ \frac{b_j(z)}{c_j(z)} \end{pmatrix}$$

$$= -\frac{1}{c_j(z)}\left(\dot{b}_j(z) \; \dot{c}_j(z)\right) \begin{pmatrix} -1 \\ \frac{b_j(z)}{c_j(z)} \end{pmatrix}$$

$$= -\frac{-\dot{b}_j(z)c_j(z) + b_j(z)\dot{c}_j(z)}{c_j(z)^2}$$

$$= \dot{m}_j(z). \qquad \qquad \square$$

6. Dislocation of eigenvalues for Robin matching conditions

In this section we study the difference $d_R(\mathcal{T}_\tau)$ of the number of eigenvalues in intervals $[0, R)$ and $[-R, 0)$ for the Dirac-Krein operator $\mathcal{T}_\tau$. We prove that there exists a number $\kappa_0 \in \mathbb{Z}$ such that, for $R > 0$ sufficiently large, $d_R(\mathcal{T}_\tau)$ differs from κ_0 at most by $n + 2$; in fact, we even show a stronger result in terms of the boundary parameters α_j at the outer vertices.

6.1.

As a first tool, we invoke results on the asymptotics of the eigenvalues of Dirac-Krein operators on a compact interval with summable potentials (see e.g. [AHM05], [LM14a]). In view of the boundary conditions (3.2) imposed for the operators T_j on an edge $e_j = [v, v_j]$ for fixed $j \in \{1, 2, \ldots, n\}$, we need to apply the more general results in [LM14a] and the detailed version [LM15] (see also [LM14b]). To this end, we first have to transform the differential equation (3.1) into the form considered in [LM14a] by means of the unitary matrix W diagonalizing J,

$$W := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad W^*W = I_2, \quad W^*JW = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{6.1}$$

Proposition 6.1. *A function $\boldsymbol{f}_j = \left(f_j \; \widehat{f}_j\right)^{\mathrm{t}} \in \boldsymbol{L}^2(e_j) \oplus \boldsymbol{L}^2(e_j)$ is a solution of the homogeneous Dirac-Krein system (3.1) for $z \in \mathbb{C}$ with boundary conditions (3.2) if and only if the function $\boldsymbol{h}_j := W^*\boldsymbol{f}_j = \left(h_j \; \widehat{h}_j\right)^{\mathrm{t}} \in \boldsymbol{L}^2(e_j) \oplus \boldsymbol{L}^2(e_j)$ is a solution of*

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \boldsymbol{h}'_j(x) - \begin{pmatrix} 0 & r_j(x) \\ r_j(x) & 0 \end{pmatrix} \boldsymbol{h}_j(x) - \lambda \boldsymbol{h}_j(x) = 0, \quad x \in e_j, \tag{6.2}$$

$$h_j(v) + \widehat{h}_j(v) = 0, \quad e^{-i\alpha_j} h_j(v_j) + e^{i\alpha_j} \widehat{h}_j(v_j) = 0, \tag{6.3}$$

with $\lambda = -z$ where $r_j := p_j + iq_j \in L^1(e_j, \mathbb{C})$.

Proof. Let $x \in e_j$ and $z \in \mathbb{C}$. If we insert $\boldsymbol{f}_j = W\boldsymbol{h}_j$ into (3.1) and multiply the resulting equation by $-W^*$ from the left, we obtain that (3.1) is equivalent to

$$W^*JW\boldsymbol{h}'_j(x) - W^*V_j(x)W\boldsymbol{h}_j(x) + z\boldsymbol{h}_j(x) = 0.$$

The latter coincides with (6.2) because of (6.1) and

$$W^*V_j(x)W = \begin{pmatrix} 0 & p_j(x) + iq_j(x) \\ p_j(x) - iq_j(x) & 0 \end{pmatrix}, \quad x \in e_j.$$

The equivalence of (3.2), (6.3) is also immediate from $\boldsymbol{f}_j = W\boldsymbol{h}_j$. $\qquad\square$

In the following, let $l_j := v_j - v$ be the length of the edge $e_j = [v, v_j]$. For the problem (3.1), (3.2) with $V_j \equiv 0$, we denote the corresponding operator, fundamental matrix, eigenvalues, and denominator of the Weyl–Titchmarsh function by T_j^0, $Y_j^0(\cdot, z)$, $\mu_{j,k}^0$, $k \in \mathbb{Z}$, and $c_j^0(z)$, i.e.

$$Y_j^0(x, z) = \begin{pmatrix} \cos((x-v)z) & -\sin((x-v)z) \\ \sin((x-v)z) & \cos((x-v)z) \end{pmatrix},$$

$$\mu_{j,k}^0 = \frac{\alpha_j + k\pi}{l_j}, \quad k \in \mathbb{Z}, \tag{6.4}$$

$$c_j^0(z) = -\sin((v_j - v)z)\cos\alpha_j + \cos((v_j - v)z)\sin\alpha_j = \sin(\alpha_j - l_j z). \tag{6.5}$$

Proposition 6.2. *Let p_j, $q_j \in L^1(e_j)$ and $\alpha_j \in [0, \pi)$. Then the fundamental matrix $Y_j(\cdot, z)$ of (3.1) with $Y_j(v, z) = I_2$ (see (3.3)) and the eigenvalues*

$\mu_{j,k}$, $k \in \mathbb{Z}$, of the operator T_j, suitably enumerated, admit the asymptotic expansions

$$Y_j(x, z) = Y_j^0(x, z) + o_j(1), \quad |z| \to \infty, \qquad uniformly\ in\ x \in e_j, \qquad (6.6)$$

$$\mu_{j,k} = \mu_{j,k}^0 + o_j(1), \qquad |k| \to \infty; \qquad (6.7)$$

here the subscript j indicates that the lower order terms depend on j. Further,

$$c_j(z) = c_j^0(z) + o_j(1), \qquad |z| \to \infty. \qquad (6.8)$$

Proof. If we transform the independent variable $x \in e_j = [v, v_j]$ to $t \in [0, 1]$ by means of $x = v + l_j t$, we find that the system (6.2) is a special case of the more general Dirac-type system [LM14a, (1)] with $b_1 = -l_j$, $b_2 = l_j$, $Q_{12}(t) = p_j(v + l_j t) + i q_j(v + l_j t)$, $Q_{21} = p_j(v + l_j t) - i q_j(v + l_j t)$, $t \in [0, 1]$. Further, the boundary conditions (6.3) are the special case $a_{11} = a_{12} = 1$, $a_{13} = a_{14} = 0$ and $a_{21} = a_{22} = 0$, $a_{23} = e^{-i\alpha_j}$, $a_{24} = e^{i\alpha_j}$ of the boundary conditions [LM14a, (3)]; in particular, the boundary conditions are separated and hence strictly regular (see [LM14a, Rem. 1]). Moreover, the assumption p_j, $q_j \in L^1(e_j)$ implies that Q_{12}, $Q_{21} \in L^1(0, 1)$ and hence all assumptions of [LM14a, Prop. 1-3] are satisfied.

By Proposition 6.1, $Y_j(x, z)$, $x \in e_j$, is the fundamental matrix of the Dirac-Krein system (3.1) with $Y_j(v, z) = I_2$ if and only if $W^* Y_j(v + l_j t, -\lambda)W = \Phi_j(t, \lambda)$, $t \in [0, 1]$, is the fundamental matrix of the corresponding transformed system [LM14a, (1)] with $\Phi_j(0, \lambda) = I_2$. By [LM14a, Prop. 2], we have

$$\Phi_j(t, \lambda) = \begin{pmatrix} e^{-i l_j t \lambda} & 0 \\ 0 & e^{i l_j t \lambda} \end{pmatrix} + o_j(1), \quad |\lambda| \to \infty,$$

uniformly in $t \in [0, 1]$, which implies the asymptotic expansion in (6.6) if we note that $l_j t = x - v$. Further, it is not difficult to check that the unperturbed eigenvalues λ_k^0 in [LM14a, (32)] can be computed explicitly to be (note that $b_1 - b_2 = -2l_j$ therein) $\lambda_k^0 = -\mu_{j,k}^0$, $k \in \mathbb{Z}$. Now the asymptotic expansion (6.7) follows from [LM14a, (33)].

Finally, (6.8) is immediate from (6.6) and the definition of c_j, c_j^0 (see (3.7), (6.5), respectively). □

6.2.

Now we return to the Dirac-Krein operator $\mathcal{T}_\tau$ on the star graph $\mathcal{G}$ with potentials V_j on the edges e_j, boundary conditions (2.3) at the outer vertices v_j, $j = 1, 2, \ldots, n$, and Robin matching conditions (2.12), (2.13) at the central vertex v.

For a self-adjoint operator T with discrete spectrum, in particular for non-semi-bounded T, we introduce the following eigenvalue counting functions. If λ_1, $\lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, and $\lambda \in [0, \infty)$, $\mu \in (-\infty, 0)$, we set

$$N\big([\lambda_1, \lambda_2); T\big) := \#\big(\sigma(T) \cap [\lambda_1, \lambda_2)\big),$$

$$N_+(\lambda; T) := N([0, \lambda); T), \quad N_-(\mu; T) := N([\mu, 0); T),$$

and, if $R > 0$,
$$d_R(T) := N_+(R; T) - N_-(-R; T). \tag{6.9}$$

The next theorem is the main result of this section. It shows that, for all sufficiently large $R > 0$, the difference $d_R(\mathcal{T}_\tau)$ of the numbers of eigenvalues of $\mathcal{T}_\tau$ in $[0, R)$ and $[-R, 0)$ deviates from some fixed integer κ_0, which we call *dislocation index*, at most by $n + 2$. In fact, the deviation can be expressed even more precisely in terms of the boundary conditions at the outer vertices.

Theorem 6.3. *There exists a number $\kappa_0 \in \mathbb{Z}$ such that, for $R > 0$ sufficiently large,*
$$- (n_\geq + 2) \leq d_R(\mathcal{T}_\tau) - \kappa_0 \leq n_\leq + 2 \tag{6.10}$$
with
$$n_\geq := \#\left\{ j \in \{1, 2, \ldots, n\} : \alpha_j \geq \frac{\pi}{2} \right\}, \quad n_\leq := \#\left\{ j \in \{1, 2, \ldots, n\} : \alpha_j \leq \frac{\pi}{2} \right\},$$
where α_j are the parameters in the boundary conditions (2.3) at the outer vertices.

Remark 6.4. It is immediate from (6.10) that, for $R > 0$ sufficiently large,
$$|d_R(\mathcal{T}_\tau) - \kappa_0| \leq n + 2.$$

For the proof of Theorem 6.3, and to obtain a formula for the dislocation index κ_0, we need some auxiliary results for the Dirac-Krein operators T_j on the edges e_j for fixed $j \in \{1, 2, \ldots, n\}$.

By (6.4), the sequence of eigenvalues $(\mu_{j,k}^0)_{k \in \mathbb{Z}}$ of the operator T_j^0 with $V_j \equiv 0$ is given by $\mu_{j,k}^0 = \frac{\alpha_j + k\pi}{l_j}$, $k \in \mathbb{Z}$, and ordered such that
$$\cdots \leq \mu_{j,-2}^0 \leq \mu_{j,-1}^0 < 0 \leq \mu_{j,0}^0 \leq \mu_{j,1}^0 \leq \mu_{j,2}^0 \leq \cdots.$$
Correspondingly, the sequence of eigenvalues $(\mu_{j,k})_{k \in \mathbb{Z}}$ of T_j is enumerated such that $\mu_{j,k} = \mu_{j,k}^0 + o_j(1)$, $|k| \to \infty$ (see (6.7)).

Lemma 6.5. *For every $\delta \in (0, \frac{\pi}{2l_j})$, there exists a $K_\delta \in \mathbb{N}$ such that*
$$\kappa(T_j) := N_+\left(\mu_{j,k-1}^0 + \delta; T_j\right) - N_-\left(\mu_{j,-k}^0 - \delta; T_j\right) \tag{6.11}$$
is constant for all $k \geq K_\delta$, independent of δ, and even; in particular, $\kappa(T_j^0) = 0$.

Proof. Let $\delta \in (0, \frac{\pi}{2l_j})$ be arbitrary. It is not difficult to see that, due to the eigenvalue asymptotics (6.7), there exists $K_\delta \in \mathbb{N}_0$ such that, for all $k \in \mathbb{Z}$, $k_1, k_2 \in \mathbb{Z}$, $k_1 \leq k_2$, with $|k|, |k_1|, |k_2| \geq K_\delta$, we have $\mu_{j,k}^0 \pm \delta \notin \sigma(T_j)$ and
$$N\left(\left[\mu_{j,k}^0 - \delta, \mu_{j,k}^0 + \delta\right); T_j\right) = 1; \tag{6.12}$$
$$N\left(\left[\mu_{j,k}^0 + \delta, \mu_{j,k+1}^0 - \delta\right); T_j\right) = 0; \tag{6.13}$$
$$N\left(\left[\mu_{j,k_1}^0 - \delta, \mu_{j,k_2-1}^0 + \delta\right); T_j\right) = k_2 - k_1. \tag{6.14}$$
Then, for all $k \geq K_\delta$, by (6.13) and (6.12),
$$N_+\left(\mu_{j,k-1}^0 + \delta; T_j\right) = N_+\left(\mu_{j,K_\delta-1}^0 + \delta; T_j\right) + N\left(\left[\mu_{j,K_\delta}^0 - \delta, \mu_{j,k-1}^0 + \delta\right); T_j\right)$$
$$= N_+\left(\mu_{j,K_\delta-1}^0 + \delta; T_j\right) + k - K_\delta,$$

and, analogously,

$$N_-\big(\mu^0_{j,-k} - \delta; T_j\big) = N_-\big(\mu^0_{j,-K_\delta} - \delta; T_j\big) + k - K_\delta.$$

Hence for $k \geq K_\delta$, the right hand side of (6.11)

$$\kappa(T_j, \delta) := N_+\big(\mu^0_{j,k-1} + \delta; T_j\big) - N_-\big(\mu^0_{j,-k} - \delta; T_j\big)$$
$$= N_+\big(\mu^0_{j,K_\delta-1} + \delta; T_j\big) - N_-\big(\mu^0_{j,-K_\delta} - \delta; T_j\big) \qquad (6.15)$$

is independent of k. The sum of the two terms on the right hand side is even because, by (6.14),

$$N_+\big(\mu^0_{j,K_\delta-1} + \delta; T_j\big) + N_-\big(\mu^0_{j,-K_\delta} - \delta; T_j\big) = N\big(\big[\mu^0_{j,-K_\delta} - \delta, \mu^0_{j,K_\delta-1} + \delta\big); T_j\big)$$
$$= 2K_\delta \in 2\mathbb{Z}; \qquad (6.16)$$

therefore the difference must be even as well, i.e. $\kappa(T_j, \delta) \in 2\mathbb{Z}$.

It remains to be shown that $\kappa(T_j, \delta)$ does not depend on δ and hence $\kappa(T_j)$ in (6.11) is well-defined. To this end, let $\delta_1, \delta_2 \in (0, \frac{\pi}{2l_j})$, without loss of generality $\delta_1 < \delta_2$. Then we may choose $K_{\delta_2} \leq K_{\delta_1}$ and, by (6.12) and (6.13), we obtain

$$N_+\big(\mu^0_{j,K_{\delta_1}-1} + \delta_1; T_j\big)$$
$$= N_+\big(\mu^0_{j,K_{\delta_2}-1} + \delta_2; T_j\big) + N\big(\big[\mu^0_{j,K_{\delta_2}-1} + \delta_2, \mu^0_{j,K_{\delta_1}-1} + \delta_1\big); T_j\big)$$
$$= N_+\big(\mu^0_{j,K_{\delta_2}-1} + \delta_2; T_j\big) + N\big(\big[\mu^0_{j,K_{\delta_2}} - \delta_2, \mu^0_{j,K_{\delta_1}-1} + \delta_1\big); T_j\big)$$
$$= N_+\big(\mu^0_{j,K_{\delta_2}-1} + \delta_2; T_j\big) + K_{\delta_1} - K_{\delta_2},$$

and, analogously,

$$N_-\big(\mu^0_{j,-K_{\delta_1}} - \delta_1; T_j\big) = N_-\big(\mu^0_{j,-K_{\delta_2}} - \delta_2; T_j\big) + K_{\delta_1} - K_{\delta_2}.$$

Taking the difference and using (6.15), we conclude $\kappa(T_j, \delta_1) = \kappa(T_j, \delta_2)$. $\square$

Lemma 6.6. *There exists an $R_j > 0$ such that, for all $R > R_j$,*

$$d_R(T_j) - \kappa(T_j) \in \begin{cases} \{0, 1\} & \text{if } \alpha_j < \frac{\pi}{2}, \\ \{-1, 0, 1\} & \text{if } \alpha_j = \frac{\pi}{2}, \\ \{-1, 0\} & \text{if } \alpha_j > \frac{\pi}{2}. \end{cases} \qquad (6.17)$$

Proof. Let $\delta > 0$ be such that $\delta < \dfrac{|\frac{\pi}{2} - \alpha_j|}{l_j}$. Then $\delta < \frac{\pi}{2l_j}$ and we choose $K_\delta \in \mathbb{N}$ according to Lemma 6.5. If $\alpha_j < \frac{\pi}{2}$, then

$$\mu^0_{j,-k} + \delta - \big(-(\mu^0_{j,k-1} - \delta)\big) = \mu^0_{j,-k} + \mu^0_{j,k-1} + 2\delta = \frac{2\alpha_j - \pi}{l_j} + 2\delta < 0$$

for $k \in \mathbb{N}$. Hence the interval $(\mu_{j,k-1}^0 - \delta, \mu_{j,k-1}^0 + \delta)$, reflected at 0, lies to the right of the interval $(\mu_{j,-k}^0 - \delta, \mu_{j,-k}^0 + \delta)$ and thus, by (6.9) and (6.12), (6.13),

$$\begin{aligned}
\kappa(T_j) &= N_+(\mu_{j,k-1}^0 + \delta; T_j) - \left(N_-(-(\mu_{j,k-1}^0 + \delta); T_j)\right. \\
&\quad + N\left(\left.[\mu_{j,-k}^0 - \delta, -(\mu_{j,k-1}^0 + \delta)); T_j\right)\right) \\
&= d_{\mu_{j,k-1}^0 + \delta}(T_j) - 1.
\end{aligned}$$

This proves (6.17) for $R = R_j := \mu_{j,k-1}^0 + \delta$ if $\alpha_j < \frac{\pi}{2}$. For $R \geq R_j$, the claim follows if we note that

$$\begin{aligned}
d_R(T_j) - \kappa(T_j) &= d_R(T_j) - d_{\mu_{j,k-1}^0 + \delta}(T_j) - \left(\kappa(T_j) - d_{\mu_{j,k-1}^0 + \delta}(T_j)\right) \\
&= d_R(T_j) - d_{\mu_{j,k-1}^0 + \delta}(T_j) - (-1)
\end{aligned}$$

and, since $\alpha_j < \frac{\pi}{2}$,

$$d_R(T_j) - d_{\mu_{j,k-1}^0 + \delta}(T_j) \in \{-1, 0\}.$$

The proof for the case $\alpha_j > \frac{\pi}{2}$ is similar. For $\alpha_j = \frac{\pi}{2}$ we have $-\mu_{j,k-1}^0 = \mu_{j,-k}^0$, hence $\kappa(T_j) = d_{\mu_{j,k-1}^0 + \delta}(T_j)$. Now the claim follows since, for $R \geq R_j$,

$$d_R(T_j) - d_{\mu_{j,k-1}^0 + \delta}(T_j) \in \{-1, 0, 1\}. \qquad \square$$

Proof of Theorem 6.3. We show that the two-sided inequality (6.10) holds with

$$\kappa_0 = \sum_{j=1}^{n} \kappa(T_j), \tag{6.18}$$

where $\kappa(T_j)$ is defined as in (6.11). The decoupled operator $\mathcal{T}_0$, corresponding to Dirichlet conditions at the central vertex v, and its eigenvalues are given by

$$\mathcal{T}_0 = \bigoplus_{j=1}^{n} T_j, \quad \sigma(\mathcal{T}_0) = \bigcup_{j=1}^{n} \{\mu_{j,k}^0 : k \in \mathbb{Z}\},$$

hence $d_R(\mathcal{T}_0) = \sum_{j=1}^{n} d_R(T_j)$. By (6.17), we can estimate

$$d_R(T_j) - \kappa(T_j) \leq \begin{cases} 1 & \text{if } \alpha_j \leq \frac{\pi}{2}, \\ 0 & \text{if } \alpha_j > \frac{\pi}{2}. \end{cases}$$

Therefore, together with (6.18), we obtain

$$d_R(\mathcal{T}_0) - \kappa_0 = \sum_{j=1}^{n} \left(d_R(T_j) - \kappa(T_j)\right) \leq n_\leq. \tag{6.19}$$

Further, by Theorem 5.3, the eigenvalues of $\mathcal{T}_\tau$ and $\mathcal{T}_0$ interlace. From this we conclude, considering the eigenvalues near $-R$, R, and 0, that

$$|d_R(\mathcal{T}_\tau) - d_R(\mathcal{T}_0)| \leq 2,$$

which, together with (6.19), proves the upper bound in (6.10). The proof for the lower bound is analogous. $\qquad \square$

6.3.

In this final subsection we derive an analytic formula for the numbers $\kappa(T_j)$, and hence for the dislocation index κ_0 of $\mathcal{T}_\tau$ in Theorem 6.3 given by (6.18). It is based on the trace formula for T_j proved in Section 3.

Proposition 6.7. *For $z \in \rho(T_j)$, let $Y_j(\cdot, z) = (y_{j,kl}(\cdot, z))_{k,l=1}^2$ and let*

$$c_j(z) = y_{j,12}(v_j, z) \cos \alpha_j + y_{j,22}(v_j, z) \sin \alpha_j$$

be the denominator in the Weyl–Titchmarsh function m_j in (3.7). Further, set

$$\omega_{j,-} := \max \big(\sigma(T_j) \cap (-\infty, 0)\big), \quad \omega_{j,+} := \min \big(\sigma(T_j) \cap [0, \infty)\big).$$

Then, for every $\omega_j \in (\omega_{j,-}, \omega_{j,+})$,

$$
\begin{aligned}
\kappa(T_j) &= -\frac{1}{\pi} \int_{\mathbb{R}}' \frac{\dot{c}_j(\omega_j + \mathrm{i}s)}{c_j(\omega_j + \mathrm{i}s)} \mathrm{d}s \\
&= \frac{1}{\pi} \lim_{M \to \infty} \Big(\arg \big(c_j\big(\omega_j - \mathrm{i}M\big)\big) - \arg \big(c_j\big(\omega_j + \mathrm{i}M\big)\big) \Big),
\end{aligned}
\tag{6.20}
$$

where $'$ denotes the Cauchy principal value at ∞.

Proof. By the residue theorem, for $-\infty < \lambda_1 < \lambda_2 < \infty$ and $\mu \in \mathbb{R}$,

$$\frac{1}{2\pi} \int_{\mathbb{R}}' \Big(\frac{1}{\mu - \lambda_1 - \mathrm{i}s} - \frac{1}{\mu - \lambda_2 - \mathrm{i}s} \Big) \mathrm{d}s = \begin{cases} 1, & \mu \in (\lambda_1, \lambda_2), \\ 0, & \mu \notin [\lambda_1, \lambda_2]. \end{cases}$$

This and Proposition 3.3 imply that, if $\lambda_1, \lambda_2 \notin \sigma(T_j)$, then the number of eigenvalues of T_j in (λ_1, λ_2), counted with multiplicity, is

$$
\begin{aligned}
N([\lambda_1, \lambda_2); T_j) &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}}' \Big(\frac{1}{\mu_{j,k} - \lambda_1 - \mathrm{i}s} - \frac{1}{\mu_{j,k} - \lambda_2 - \mathrm{i}s} \Big) \mathrm{d}s \\
&= \frac{1}{2\pi} \int_{\mathbb{R}}' \mathrm{tr}\, \big((T_j - \lambda_1 - \mathrm{i}s)^{-1} - (T_j - \lambda_2 - \mathrm{i}s)^{-1} \big) \mathrm{d}s \quad (6.21) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}}' \Big(\frac{\dot{c}_j(\lambda_2 + \mathrm{i}s)}{c_j(\lambda_2 + \mathrm{i}s)} - \frac{\dot{c}_j(\lambda_1 + \mathrm{i}s)}{c_j(\lambda_1 + \mathrm{i}s)} \Big) \mathrm{d}s.
\end{aligned}
$$

By Lemma 6.5, for every $\delta \in (0, \frac{\pi}{2l_j})$ there exists $K_\delta \in \mathbb{N}$ such that for all $k \in \mathbb{N}_0$, $k \geq K_\delta$, we have $\mu_{j,k}^0 \pm \delta \notin \sigma(T_j)$ and

$$\kappa(T_j) = N_+ \big(\mu_{j,k-1}^0 + \delta; T_j \big) - N_- \big(\mu_{j,-k}^0 - \delta; T_j \big). \tag{6.22}$$

Since $(\omega_{j,-}, \omega_{j,+}) \cap \sigma(T_j) = \emptyset$ and $\omega_j \in (\omega_{j,-}, \omega_{j,+})$, it follows that

$$\kappa(T_j) = N \big([\omega_j, \mu_{j,k-1}^0 + \delta); T_j \big) - N \big([\mu_{j,-k}^0 - \delta, \omega_j); T_j \big).$$

Then, due to (6.21),

$$\kappa(T_j) = \frac{1}{2\pi} \int_{\mathbb{R}}' \Big(\frac{\dot{c}_j(\mu_{j,k-1}^0 + \delta + \mathrm{i}s)}{c_j(\mu_{j,k-1}^0 + \delta + \mathrm{i}s)} + \frac{\dot{c}_j(\mu_{j,-k}^0 - \delta + \mathrm{i}s)}{c_j(\mu_{j,-k}^0 - \delta + \mathrm{i}s)} - 2\frac{\dot{c}_j(\omega_j + \mathrm{i}s)}{c_j(\omega_j + \mathrm{i}s)} \Big) \mathrm{d}s.$$

It is not difficult to check that $\dfrac{\dot{c}_j}{c_j} = \dfrac{\dot{c}_j^0}{c_j^0} + \dfrac{\big(\frac{c_j}{c_j^0}\big)^{\!\cdot}}{\frac{c_j}{c_j^0}}$. Then

$$\kappa(T_j) = \frac{1}{2\pi} \int_{\mathbb{R}}' \Big(\frac{\dot{c}_j^0(\mu_{j,k-1}^0 + \delta + is)}{c_j^0(\mu_{j,k-1}^0 + \delta + is)} + \frac{\dot{c}_j^0(\mu_{j,-k}^0 - \delta + is)}{c_j^0(\mu_{j,-k}^0 - \delta + is)} \Big) ds$$

$$+ \frac{1}{2\pi} \int_{\mathbb{R}}' \Big(\frac{\big(\frac{c_j}{c_j^0}\big)^{\cdot}(\mu_{j,k-1}^0 + \delta + is)}{\frac{c_j}{c_j^0}(\mu_{j,k-1}^0 + \delta + is)} + \frac{\big(\frac{c_j}{c_j^0}\big)^{\cdot}(\mu_{j,-k}^0 - \delta + is)}{\frac{c_j}{c_j^0}(\mu_{j,-k}^0 - \delta + is)} \Big) ds$$

$$- \frac{1}{2\pi} \int_{\mathbb{R}}' 2\frac{\dot{c}_j(\omega_j + is)}{c_j(\omega_j + is)} ds.$$

We denote the first two integrals in the above formula by $I_{1,k}$ and $I_{2,k}$ and show that $I_{1,k} = 0$ for $k \geq K_\delta$ and $I_{2,k} = 0$ for sufficiently large $k \geq K_\delta$; then the first equality in (6.20) follows. By means of (6.21), we conclude that, for $k \geq K_\delta$,

$$I_{1,k} = N\Big(\Big[\frac{\alpha_j - \frac{\pi}{2}}{l_j}, \mu_{j,k-1}^0 + \delta \Big); T_j^0 \Big) - N\Big(\Big[\mu_{j,-k}^0 - \delta, \frac{\alpha_j - \frac{\pi}{2}}{l_j} \Big); T_j^0 \Big)$$

$$+ \frac{1}{\pi} \int_{\mathbb{R}}' \frac{\dot{c}_j^0(\frac{\alpha_j - \frac{\pi}{2}}{l_j} + is)}{c_j^0(\frac{\alpha_j - \frac{\pi}{2}}{l_j} + is)} ds.$$

Since the first two numbers are both equal to k and since $c_j^0(z) = \sin(\alpha_j - l_j z)$ by (6.8), we obtain

$$I_{1,k} = -l_j \frac{1}{\pi} \int_{\mathbb{R}}' \cot\Big(\frac{\pi}{2} + is \Big) ds = i\, l_j \frac{1}{\pi} \int_{\mathbb{R}}' \tanh s \, ds = 0, \quad k \geq K_\delta.$$

Because $Y_j(\cdot, \overline{z}) = \overline{Y_j(\cdot, z)}$, we have $\frac{c_j}{c_j^0}(\overline{z}) = \overline{\frac{c_j}{c_j^0}(z)}$ and hence

$$I_{2,k} = \frac{1}{2\pi} \lim_{M \to \infty} \Big(\int_{\mu_{j,k-1}^0 + \delta - iM}^{\mu_{j,k-1}^0 + \delta + iM} (-i) \, d_\theta \Big(\ln \Big| \frac{c_j}{c_j^0}(\theta) \Big| + i \arg\Big(\frac{c_j}{c_j^0}(\theta) \Big) \Big)$$

$$+ \int_{\mu_{j,-k}^0 - \delta - iM}^{\mu_{j,-k}^0 - \delta + iM} (-i) \, d_\theta \Big(\ln \Big| \frac{c_j}{c_j^0}(\theta) \Big| + i \arg\Big(\frac{c_j}{c_j^0}(\theta) \Big) \Big) \Big)$$

$$= \frac{1}{2\pi} \lim_{M \to \infty} \Big(\int_{\mu_{j,k-1}^0 + \delta - iM}^{\mu_{j,k-1}^0 + \delta + iM} d_\theta \arg\Big(\frac{c_j}{c_j^0}(\theta) \Big) + \int_{\mu_{j,-k}^0 - \delta - iM}^{\mu_{j,-k}^0 - \delta + iM} d_\theta \arg\Big(\frac{c_j}{c_j^0}(\theta) \Big) \Big)$$

$$= \frac{1}{2\pi} \lim_{M \to \infty} \Big(\int_{\mu_{j,k-1}^0 + \delta - iM}^{\mu_{j,k-1}^0 + \delta + iM} d_\theta \arg\Big(1 + \frac{c_j - c_j^0}{c_j^0}(\theta) \Big)$$

$$+ \int_{\mu_{j,-k}^0 - \delta - iM}^{\mu_{j,-k}^0 - \delta + iM} d_\theta \arg\Big(1 + \frac{c_j - c_j^0}{c_j^0}(\theta) \Big) \Big).$$

We denote the last two integrals by $I_{2,k}^+$ and $I_{2,k}^-$ and show that, for sufficiently large $k \geq K_\delta$, they tend to 0 for $M \to \infty$. If we use that $\mu_{j,k-1}^0$ is a zero of $c_j^0(z) = \sin(\alpha_j - l_j z)$, it is elementary to show that, for all $\theta = \mu_{j,k-1}^0 + \delta + it$, $t \in \mathbb{R}$,

$$|c_j^0(\theta)| \geq |\sin(l_j \delta)| \cosh t \geq |\sin(l_j \delta)| > 0.$$

Since $(c_j - c_j^0)(z) = o_j(z)$, $|z| \to \infty$, by (6.8), we have

$$\lim_{M \to \infty} \frac{c_j - c_j^0}{c_j^0} (\mu_{j,k-1}^0 + \delta \pm iM) = 0. \qquad (6.23)$$

Moreover, we can choose $k_0 \geq K_\delta$ so large that for all $\theta = \mu_{j,k_0-1}^0 + \delta + it$, $t \in \mathbb{R}$, we can estimate $|(c_j - c_j^0)(\theta)| < \sin(l_j\delta)/2$ and hence

$$\left| \frac{c_j - c_j^0}{c_j^0}(\theta) \right| < \frac{1}{2}. \qquad (6.24)$$

Now (6.23) and (6.24) imply that $I_{2,k_0}^+ = 0$. Analogously, we obtain $I_{2,k_0}^- = 0$ and hence $I_{2,k_0} = 0$.

The second equality in (6.20) follows from the first one using the symmetry property of $\frac{\dot{c}_j}{c_j}$ in the same way as in the reasoning for $I_{2,k}$ above. $\square$

Acknowledgment

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